Relations Between the Likelihood Ratios for 2-D Continuous and Discrete Time Stochastic Processes

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Abstract

In this paper we consider the likelihood ratio for two-dimensional processes. In order to detect this ratio, we need to compute the determinant of the covariance operator of the 'signal plus noise' observation process. In the continuous case, this is in general a difficult problem. For cyclic processes, using Fourier transforms we can compute the determinant for continuous and discrete processes. For the 2-D Poisson equation and its discretization, we show that the discretized determinant converges to the continuous one if the stepsize tends to zero.

1 Introduction

Consider a two dimensional continuous time signal process X_t defined on a rectangular $\mathcal{D} = [0, T_1] \times [0, T_2]$. Let Y_t be the observation process for the signal process. For purposes of signal detection and identification, one is interested in determination of the likelihood ratio. In [LueBa1] Luesink and Bagchi give an explicit formula for the likelihood ratio in the case where we consider the Gaussian 'signal plus noise' model, that is where Y is the sum of the Gaussian signal X and a Gaussian standard white noise. Problem in this expression is that one should detect the determinant of the covariance operator of the observation process which is an infinite product of eigenvalues. One way to circumvent this problem is using 2-D Krein factorization. We can rewrite the likelihood ratio expression in a way where we only need the 'one directional filter' for the signal. This result may be used in cases where certain causality appears, see [LueBa2].

Another class of processes we can handle are the cyclic processes. Using Fourier transforms one can detect orthogonal parts of the observation process. This immediately gives the eigenvalues of the covariance operator and hence the determinant value. The results pointed out above hold for processes defined on a continuous area $\mathcal{D}$. We may of course pose the same problem for the signal processes defined on a discrete area $\mathcal{N} = \{0..N_1\} \times \{0..N_2\}$. The class of discrete time processes we consider here are the Nearest Neighbor Models (see [Lev] for definition). Nearest Neighbor Models are often used, for instance in image processing [Bie] and geophysics [SmiFr]. The models also appear in a very natural way when we discretize second order partial differential equations. In [Lev], Levy shows that any discrete Gauss-Markov 2-D process may be written as a NNM. For discrete time 2-D processes defined on $\mathcal{N}$ we also obtained an expression for the likelihood ratio [Lue]. In this expression we need to compute the determinant of the covariance matrix of the observation process which has $(N_1 + 1)(N_2 + 1)$ rows and columns. In the paper [Lue] we showed that if we consider a cyclic stationary Gaussian NNM, we can determine the eigenvalues of the covariance matrix using Fourier matrices.

An interesting difference between the continuous and discrete time case is that we can always compute the determinant of a covariance matrix (however it may take many efforts if sizes are big), while determination of the covariance operator may cause big problems. Therefore we may look at the discretization of

continuous time processes and see whether the determinant of the discretized version tends to the determinant of the covariance operator. In this field, hardly any work is done. This paper is a first attempt to tackle this problem.

In the next section we will consider the Poisson equation on an area $\mathcal{D}$, cyclic in both directions (i.e. defined on a torus). We will show that the determinant of the covariance matrix of the discretized version tends to the determinant of the covariance operator if the discretization step sizes $T_1/(N_1 - 1)$ and $T_2/(N_2 - 1)$ tend to zero.

2 Poisson equation

Consider the Poisson equation, driven by Gaussian white noise, defined on the rectangular $\mathcal{D}$ with cyclic boundary conditions, that is

$$\nabla^2 X_t - a^2 X_t = \omega_t \quad (1)$$

$$X_{(0,t_2)} = X_{(T_1,t_2)}, X_{(t_1,0)} = X_{(t_1,T_2)},$$

where ω is a two dimensional white noise sample function in $L_2(\mathcal{D}; \mathbb{R})$, that is, for each $h \in L_2(\mathcal{D}; \mathbb{R})$, $[\omega, h]$ defines a zero mean Gaussian with covariance $[h, h]$. This implies that the signal $X \in L_2(\mathcal{D}; \mathbb{R})$. Using Fourier expansion in two dimensions, we obtain that

$$X_t = \frac{1}{T_1 T_2} \sum_{k \in \mathbb{Z}^2} f_k \exp(i2\pi(\frac{k_1 t_1}{T_1} + \frac{k_2 t_2}{T_2})) \quad (2)$$

with

$$f_k = \int_0^{T_1} \int_0^{T_2} X_t \exp(-i2\pi(\frac{k_1 t_1}{T_1} + \frac{k_2 t_2}{T_2})) dt_1 dt_2. \quad (3)$$

For the noise term ω we obtain

$$\omega_t = \frac{1}{T_1 T_2} \sum_{k \in \mathbb{Z}^2} \epsilon_k \exp(i2\pi(\frac{k_1 t_1}{T_1} + \frac{k_2 t_2}{T_2})). \quad (4)$$

If we apply this expression to the Poisson equation we derive $-\left(\left(\frac{2\pi k_1}{T_1}\right)^2 + \left(\frac{2\pi k_2}{T_2}\right)^2 + a^2\right) f_k = \epsilon_k$. For simplicity of our equations we introduce

$$d_k = \left(\left(\frac{2\pi k_1}{T_1}\right)^2 + \left(\frac{2\pi k_2}{T_2}\right)^2 + a^2\right). \quad (5)$$

One easily proves that the components f_k are independent. Now, let Y be the observation process, defined by $Y_t = X_t + \eta_t$ where η is canonical Gaussian white noise independent of X in $L_2(\mathcal{D}; \mathbb{R})$. Let $R : L_2(\mathcal{D}; \mathbb{R}) \rightarrow L_2(\mathcal{D}; \mathbb{R})$ denote the Hilbert Schmidt covariance operator of X , that is $Rf(\underline{t}) = \int_0^{T_1} \int_0^{T_2} r(\underline{t}, \underline{s}) f(\underline{s}) d\underline{s}$ with $r(\underline{t}, \underline{s}) = \text{cov}(X_t, X_s)$. The cyclic Poisson equation ensures stationarity of the signal process, i.e. $r(\underline{t}, \underline{s}) = r(\underline{t} - \underline{s})$. Y has a corresponding covariance operator $I + R$. This is the covariance operator whose determinant we are interested in. The functions

$e_{\underline{k}}(\underline{t}) = \exp^{i2\pi\left(\frac{k_1 t_1}{N_1} + \frac{k_2 t_2}{N_2}\right)}$, $\underline{k} \in \mathbf{Z}^2$ form the complete set of eigenfunctions of $(I + R)$, with eigenvalues $\left(1 + \frac{1}{d_{\underline{k}}^2}\right)$, hence

$$\det(I + R) = \prod_{\underline{k} \in \mathbf{Z}} \left(1 + \frac{1}{d_{\underline{k}}^2}\right). \quad (6)$$

3 Discretization of the Poisson Equation

In this section we will discretize the Poisson equation into a cyclic Nearest Neighbor Model, with step sizes $h_1 = T_1/(N_1 - 1)$ in the t_1 -direction and $h_2 = T_2/(N_2 - 1)$ in the t_2 -direction. For the derivatives in the Poisson equation we use the discretization formula's

$$\frac{\partial^2 X_{\underline{t}}}{\partial t_1^2} = \frac{X_{t_1+h_1, t_2} + X_{t_1-h_1, t_2} - 2X_{t_1, t_2}}{h_1^2},$$

$$\frac{\partial^2 X_{\underline{t}}}{\partial t_2^2} = \frac{X_{t_1, t_2+h_2} + X_{t_1, t_2-h_2} - 2X_{t_1, t_2}}{h_2^2},$$

and with ϵ the correct discretization of ω , see [Bal], we obtain the NNM

$$\begin{aligned} & \left(\frac{2}{h_1^2} + \frac{2}{h_2^2} + a^2\right) X_{\underline{k}} \\ &= \frac{1}{h_1^2} X_{k_1+1, k_2} + \frac{1}{h_1^2} X_{k_1-1, k_2} + \frac{1}{h_2^2} X_{k_1, k_2+1} + \frac{1}{h_2^2} X_{k_1, k_2-1} + \epsilon_{\underline{k}}, \end{aligned} \quad (7)$$

with $\text{cov}(\epsilon_{\underline{k}}, \epsilon_{\underline{l}}) = \delta_{\underline{k}, \underline{l}}$. Following [Lue] we find that

$$\det(I + R_d) = \prod_{\underline{k} \in \mathcal{N}} \left(1 + \frac{1}{\bar{d}_{\underline{k}}^2}\right), \quad (8)$$

where $\bar{d}_{\underline{k}}^2 = \frac{4}{h_1^2} \sin^2\left(\frac{\pi k_1}{N_1}\right) + \frac{4}{h_2^2} \sin^2\left(\frac{\pi k_2}{N_2}\right) + a^2$. R_d is the covariance matrix of the vector containing all signal points, i.e. $\mathcal{X} = (X_{1,1}, X_{2,1}, \dots, X_{N_1-1, N_2-1})$. $(I + R_d)$ is the covariance matrix of the discretized observation process.

4 Comparison of the discrete and continuous time result

Comparing equations [6] and [8] we want to know if the latter converges to the first if the discretization step sizes tend to zero. We show that it indeed does if the Poisson parameter $a > 0$. In order to do this, define the following areas, with $1/2 < \alpha < 2/3$:

$$\begin{aligned} \mathcal{C}_{\underline{N}} &= \{-N_i^\alpha \leq k_i < N_i^\alpha\} \\ \mathcal{D}_{\underline{N}} &= \{-N_i/2 \leq k_i < N_i/2\} \\ \mathcal{E}_{\underline{N}} &= \mathcal{D}_{\underline{N}} \setminus \mathcal{C}_{\underline{N}} \\ \mathcal{F}_{\underline{N}} &= \mathbf{Z}^2 \setminus \mathcal{D}_{\underline{N}}. \end{aligned}$$

With these definitions we find that

$$\begin{aligned} \log \left(\frac{\det(I + R_d)}{\det(I + R)} \right) &= \sum_{\underline{k} \in \mathcal{D}_{\underline{N}}} \log \left(1 + \frac{1}{\bar{d}_{\underline{k}}^2} \right) - \sum_{\underline{k} \in \mathbf{Z}^2} \log \left(1 + \frac{1}{d_{\underline{k}}^2} \right) \\ &= \sum_{\underline{k} \in \mathcal{C}_{\underline{N}}} \log \left(\frac{1 + \frac{1}{\bar{d}_{\underline{k}}^2}}{1 + \frac{1}{d_{\underline{k}}^2}} \right) + \sum_{\underline{k} \in \mathcal{E}_{\underline{N}}} \log \left(1 + \frac{1}{\bar{d}_{\underline{k}}^2} \right) \\ &\quad - \sum_{\underline{k} \in \mathcal{F}_{\underline{N}} \cup \mathcal{E}_{\underline{N}}} \log \left(1 + \frac{1}{d_{\underline{k}}^2} \right). \end{aligned} \quad (9)$$

We can show that each term on the right hand side tends to zero if N_1 and N_2 tend to infinity. The proof is omitted in this paper but will be included in authors' dissertation, published spring 1992. Convergence to zero implies that in the limit both determinants have the same value, $\lim_{N_1, N_2 \rightarrow \infty} \det(I + R_d) = \det(I + R)$.

5 Concluding Remarks

We showed that for the Poisson equation, the determinant of the covariance operator of the observation process can be approximated by the determinant of the covariance matrix of the corresponding discretized process. We did not consider the speed of convergence of the discretized determinant to the continuous one with respect to the step size. A small example tells us that for $a = 1, T_1 = T_2 = 10$ and $h_1 = h_2 = 0.1$ the approximation is already very good. We may expect that convergence is slower for a smaller choice of a and faster for a larger choice. The cyclic Poisson equation is a case for which we can indeed compute the continuous determinant, and compare the results. More research is needed in this area.

References

- [Bal] A.V. Balakrishnan (1973). *Stochastic Differential Systems I*, Lecture Notes in Economics and Mathematical Systems 84, Springer Verlag, New York.
- [Bie] J. Biemond (1982). *Image Restoration; a Linear Stochastic Filtering Approach*. Report IT-82-02 Delft University of Technology, The Netherlands.
- [Lev] B.C. Levy (1989). Noncausal Estimation for Discrete Gauss-Markov Random Fields. *Realization and Modeling in Systems Theory*, Proc. MTNS 1989 Vol.1 (Eds. M.A. Kaashoek, J.H. van Schuppen and A.C.M. Ran), Birkhäuser Verlag, Basel Switzerland.
- [Lue] R. Luesink (1990). The Likelihood Ratio for Multi Dimensional Gauss-Markov Random Fields. *Memorandum nr. 861*, Department of Applied mathematics, University of Twente, The Netherlands.
- [LueBa1] R. Luesink and A. Bagchi (1991). Krein Factorization of Covariance Operators of 2-Parameter Random Fields and Application to the Likelihood Ratio. *IEEE Trans. on Information Theory* 37, pp. 53-59.
- [LueBa2] R. Luesink and A. Bagchi (1991). The Likelihood Ratio for Two-Dimensional Stochastic Processes. To appear in *Proceedings of the ECC*, July 2-5, 1991.
- [SmiFr] L. Smith and R.A. Freeze (1977). Stochastic Analysis of Steady State Groundwater Flow in a Bounded Domain. 2. Two-dimensional Simulations. *Water Resources Research* 15, pp 1543-1559.