A MIXED ELASTOPLASTIC / RIGID-PLASTIC MATERIAL MODEL

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Abstract. *A new integration algorithm is described for large strain plastic deformations. The algorithm degenerates to the Euler forward elastoplastic-plastic model for small strain increments and to the rigid-plastic model for large strain increments. The model benefits from the advantages of both models: accuracy and fast convergence over a large range of strain increments.*

1 INTRODUCTION

Constitutive relations for plastic deformations are usually based on rate equations. For metal plasticity Drucker's postulate states that the direction of plastic flow rate is perpendicular to the yield surface. The magnitude of the plastic flow is determined by a consistency relation, so that the current stress remains on the yield surface.

For use in an incremental procedure, the plastic strain rate must be integrated to yield a plastic strain increment. Nowadays several algorithms are available. Many algorithms are based on some kind of elastic predictor / plastic corrector scheme. The direction of plastic flow can be interpolated between the directions calculated at the beginning and at the end of the increment.^{1,2}

For large deformation analysis, the elastic part of the strain rate can be neglected. In that case the total strain increment equals the plastic strain increment and there is no need to determine the direction of the plastic flow. This rigid-plastic integration algorithm is very robust for large strain increments but lacks accuracy for small strain increments. This is a serious drawback, because even in simulations with large strain increments there are often areas with almost no deformation (dead zones).

Some researchers have tried to include some elasticity into a rigid-plastic or viscoplastic model.^{3,4} This is needed if after a forming process e.g. the elastic spring-back must be calculated or the residual stresses or if dead metal zones exist. In this paper a mixed elastoplastic / rigid-plastic material model is described that can be regarded as an elastoplastic model, that degenerates to the rigid-plastic model for large strain increments. It inherits the good convergence and accuracy of the elastic predictor / plastic corrector schemes for small strain increments and the robustness and sufficient accuracy of the rigid-plastic schemes at large strain increments.

In sections 2 and 3 the basic rigid-plastic and elastoplastic material models are recapitulated. In section 4 the new mixed elastoplastic / rigid-plastic model is described. Finally three examples are used to compare the performance of the three algorithms.

2 BASIC CONCEPTS

The description in this section only considers geometrically linear behavior (small rotations). The algorithm however is not restricted to small strains and rotations if a proper correction is made for the rotating material.^{5,6} Furthermore, in this paper, a restriction is made to fully isotropic behavior, associated plasticity and the Von Mises yield criterion.

The total strain rate tensor is decomposed into an elastic and a plastic part:

$$\dot{\epsilon} = \dot{\epsilon}^e + \dot{\epsilon}^p \quad (1)$$

Since the Von Mises yield function is independent of the hydrostatic pressure, stresses and strains are split into a deviatoric and dilatational part.

$$\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{3}\boldsymbol{\sigma} : \mathbf{II} = \boldsymbol{\sigma} + p\mathbf{I} \quad (2)$$

$$p = -\frac{1}{3}\boldsymbol{\sigma} : \mathbf{I} \quad (3)$$

and

$$\mathbf{e}^e = \boldsymbol{\varepsilon}^e - \frac{1}{3}\boldsymbol{\varepsilon}^e : \mathbf{II} = \boldsymbol{\varepsilon}^e - \frac{1}{3}\varepsilon_V \mathbf{I} \quad (4)$$

$$\varepsilon_V^e = \boldsymbol{\varepsilon}^e : \mathbf{I} \quad (5)$$

Since only volume preserving plasticity is considered, the superscript e can be omitted from the dilatational part. The relation between stresses and strains can now be described in terms of the dilatational part and the deviatoric part:

$$p = -K\boldsymbol{\varepsilon} : \mathbf{I} = -K\varepsilon_V \quad (6)$$

$$\mathbf{s} = 2G\mathbf{e}^e \quad (7)$$

Here K is the bulk modulus and G the shear modulus. The dilatational part does not contribute to the plastic deformations. For Von Mises plasticity we can concentrate on the deviatoric part.

In rate form (7) becomes:

$$\dot{\mathbf{s}} = 2G\dot{\mathbf{e}}^e = 2G(\dot{\mathbf{e}} - \dot{\mathbf{e}}^p) \quad (8)$$

Here the elastic material stiffness is considered to be constant, not depending on temperature or any other quantity.

A yield function ϕ can be defined that separates the elastic from the plastic domain. Plastic deformation occurs if $\phi = 0$ and $\dot{\phi} = 0$, elastic deformation occurs if $\phi < 0$ or $\dot{\phi} < 0$. The yield function can never be larger than 0.

The Von Mises yield function ϕ can be written as:

$$\phi = \sqrt{\frac{3}{2}\mathbf{s} : \mathbf{s}} - \sigma_{yield} \quad (9)$$

where σ_{yield} is the uniaxial yield stress. It is a function of the equivalent plastic strain κ defined by:

$$\kappa(t) = \int_0^t \dot{\kappa} d\tau \quad \text{and} \quad \dot{\kappa} = \sqrt{\frac{2}{3}\dot{\mathbf{e}}^p : \dot{\mathbf{e}}^p} \quad (10)$$

For associated plasticity, the plastic deformation is perpendicular to the yield surface:

$$\dot{\mathbf{e}}^p = \dot{\lambda} \frac{\partial \phi}{\partial \mathbf{s}} = \frac{3}{2} \dot{\lambda} \frac{\mathbf{s}}{\sigma_{yield}} \quad (11)$$

Here $\dot{\lambda}$ is a plastic multiplier that can be determined by the consistency condition $\dot{\phi} = 0$. From (10) and (11) we easily derive that $\dot{\kappa} = \dot{\lambda}$.

3 TIME INTEGRATION

The relations from the previous section are not readily applicable in a finite element analysis. In taking finite time steps, the rate formulation must be integrated to give a stress increment. The assumptions, used in different numerical integration algorithms are decisive for the accuracy and stability of the method.

The stress at the end of a time increment must be calculated based on the total strain at the beginning and at the end of the increment. If the stresses do not match the weighed equilibrium condition a new strain increment will be calculated at a global level. The task of the time integration is only to calculate the stress for a known strain increment. For fast convergence of the global equilibrium a stiffness matrix must be set up that is consistent with the integration algorithm.⁷

In an elastoplastic analysis the strain decomposition (1) and the elastic stress–strain relation (8) can be written in incremental form:

$$\Delta \mathbf{s} = 2G(\Delta \mathbf{e} - \Delta \mathbf{e}^p) \quad (12)$$

Here $\Delta \mathbf{e}^p$ can not be determined without assumptions about the plastic multiplier $\dot{\lambda}$ and the plastic strain rate direction $\frac{\partial \phi}{\partial \mathbf{s}}$

$$\Delta \mathbf{e}^p = \int_{t_0}^{t_1} \dot{\lambda} \frac{\partial \phi}{\partial \mathbf{s}} d\tau \quad (13)$$

Time t_0 represents the start and t_1 the end of a particular time increment.

For large strain increments the choice of the direction of $\Delta \mathbf{e}^p$ is bypassed by neglecting the elastic strain increment, leaving $\Delta \mathbf{e}^p = \Delta \mathbf{e}$. This leads to a so called rigid–plastic material model.

3.1 The rigid–plastic model

In a rigid–plastic model, also referred to as the flow formulation, the elastic deformations are totally ignored. That means that a calculated strain increment equals the plastic strain increment. The globally calculated displacements must however be constrained to a ‘zero-dilatation’ strain field. This can be attained by e.g. a Lagrange multiplier method or by a penalty method. In a modified rigid–plastic model the dilatational elastic strain is included in the model. This results in a penalty-like method with a specific penalty factor, representing the elastic volume change.⁸

For a Von Mises material model, the plastic strain rate $\dot{\mathbf{e}}$ is determined by (11). This equation can readily be inverted and integrated to give:

$$\mathbf{s} = \frac{2}{3} \frac{\sigma_{yield,1}}{\Delta \kappa} \Delta \mathbf{e} \quad (14)$$

where $\sigma_{yield,1}$ is the yield stress at the end of the increment. The contribution of the pressure p to the stress tensor is calculated from the volumetric change (6):

$$p_1 \mathbf{I} = -K \Delta \epsilon : \mathbf{II} + p_0 \mathbf{I} \quad (15)$$

The total stress at the end of the increment can now be written as

$$\boldsymbol{\sigma}_1 = \left[\frac{2}{3} \frac{\sigma_{yield,1}}{\Delta\kappa} \mathbf{H} + \left(K - \frac{2}{9} \frac{\sigma_{yield,1}}{\Delta\kappa} \right) \mathbf{II} \right] : \Delta\boldsymbol{\varepsilon} - p_0 \mathbf{I} \quad (16)$$

where $\mathbf{H}$ is the fourth order unit tensor. When the equivalent plastic strain increment $\Delta\kappa$ is very small it is set to a fixed value, avoiding division by zero. In algorithmic terminology this is represented as:

$$\Delta\kappa := \max(\Delta\kappa, \delta) \quad (17)$$

A consistent tangential stiffness matrix is derived with the differential form of (14)

$$d\mathbf{s}_1 = \frac{2}{3} \sigma_{yield,1} \frac{1}{\Delta\kappa} d\mathbf{e} - \frac{2}{3} \sigma_{yield,1} \frac{\Delta\mathbf{e}}{(\Delta\kappa)^2} d\kappa + \frac{2}{3} \frac{\Delta\mathbf{e}}{\Delta\kappa} d\sigma_{yield,1} \quad (18)$$

After some work this yields:

$$d\mathbf{s}_1 = \left[\frac{2}{3} \frac{\sigma_{yield,1}}{\Delta\kappa} \mathbf{H} + \left(h - \frac{\sigma_{yield,1}}{\Delta\kappa} \right) \frac{\mathbf{s}_1 \mathbf{s}_1}{(\sigma_{yield,1})^2} \right] : d\mathbf{e} \quad (19)$$

with $h = \frac{d\sigma_{yield}}{d\kappa}$. With $d\boldsymbol{\sigma}_1 = d\mathbf{s}_1 + K \mathbf{II} d\boldsymbol{\varepsilon}$ and $d\mathbf{e} = (\mathbf{H} - \frac{1}{3} \mathbf{II}) : d\boldsymbol{\varepsilon}$ the consistent stiffness matrix is now defined as:

$$\mathbf{D} = \frac{2}{3} \frac{\sigma_{yield,1}}{\Delta\kappa} \mathbf{H} + \left(h - \frac{\sigma_{yield,1}}{\Delta\kappa} \right) \frac{\mathbf{s}_1 \mathbf{s}_1}{(\sigma_{yield,1})^2} + \left(K - \frac{2}{9} \frac{\sigma_{yield,1}}{\Delta\kappa} \right) \mathbf{II} \quad (20)$$

For the first iteration $\Delta\kappa$ from the previous increment is used.

3.2 Elastoplastic model

Elastic strain increments can not be ignored for small strains or if elastic spring-back or residual stresses must be calculated. In that case an elastoplastic model must be used.

If the stress at the end of the increment happens to be outside the elastic domain, a plastic strain is calculated in the direction of the deviatoric stress. The magnitude is determined by the consistency condition.

For the calculation of the stress increment we start of with the generalized trapezoidal rule as in Ortiz and Popov.¹ The plastic strain increment is considered to be in the direction between the normal to the yield surface at the start of the increment (at known stress $\mathbf{s}_0$) and the normal to the yield surface at the end of the increment (at yet unknown stress $\mathbf{s}_1$).

$$\mathbf{s}_1 = \mathbf{s}_0 + 2G(\Delta\mathbf{e} - \Delta\mathbf{e}^p) \quad (21)$$

$$t = t_0 \rightarrow \dot{\mathbf{e}}_0^p = \frac{3}{2} \dot{\kappa}_0 \frac{\mathbf{s}_0}{\sigma_{yield,0}} \quad (22)$$

$$t = t_1 \rightarrow \dot{\mathbf{e}}_1^p = \frac{3}{2} \dot{\kappa}_1 \frac{\mathbf{s}_1}{\sigma_{yield,1}} \quad (23)$$

The weighing of the start and the end directions is done by introducing a factor α . Suppose $\dot{\kappa}_0/\sigma_{yield,0} \equiv \dot{\kappa}_1/\sigma_{yield,1}$ then:

$$\begin{aligned}
 \Delta \mathbf{e}^p &= \int_{t_0}^{t_1} \dot{\mathbf{e}}^p d\tau \\
 &= (1 - \alpha) \dot{\mathbf{e}}_0^p \Delta t + \alpha \dot{\mathbf{e}}_1^p \Delta t \\
 &= (1 - \alpha) \Delta \mathbf{e}_0^p + \alpha \Delta \mathbf{e}_1^p \\
 &= \frac{3}{2} \frac{\Delta \kappa}{\sigma_{yield}} [(1 - \alpha) \mathbf{s}_0 + \alpha \mathbf{s}_1]
 \end{aligned} \tag{24}$$

then we have with (21):

$$\mathbf{s}_1 = \mathbf{s}_0 \left[1 - \frac{3G}{\sigma_{yield}} \Delta \kappa (1 - \alpha) \right] + 2G \Delta \mathbf{e} - \frac{3G}{\sigma_{yield}} \Delta \kappa \alpha \mathbf{s}_1 \tag{25}$$

Ortiz and Popov studied the accuracy and stability of integration algorithms for a fixed value of α .¹

The choice of $\alpha = 1$ leads to the well known elastic predictor, radial return method. In that case the elastic predictor calculates a deviatoric trial stress $\mathbf{s}^*$

$$\mathbf{s}^* = \mathbf{s}_0 + 2G \Delta \mathbf{e} \tag{26}$$

If $\phi(\mathbf{s}^*, \kappa) > 0$ a plastic corrector is calculated based on the direction $\frac{\partial \phi}{\partial \mathbf{s}}$ at the location of $\mathbf{s}^*$. In a Von Mises material model, the direction of plastic strain will remain constant during radial return and only $\Delta \lambda$ is variable. Equation (25) can then be rewritten as

$$\left(1 + \frac{3G}{\sigma_{yield}} \Delta \kappa \right) \mathbf{s} = \mathbf{s}^* \tag{27}$$

The value of $\Delta \kappa$ is derived from the consistency relation $\phi(\boldsymbol{\sigma}, \kappa) = 0$ leading to

$$\sigma_{yield} + 3G \Delta \kappa = \sqrt{\frac{3}{2} \mathbf{s}^* : \mathbf{s}^*} \tag{28}$$

Here σ_{yield} is a function of the equivalent plastic strain κ , therefore (28) is a scalar nonlinear relation, that can be solved e.g. by a Newton–Raphson method.

The hydrostatic part of the stress tensor is directly related to the dilatational strain (6).

$$p = -K \boldsymbol{\varepsilon} : \mathbf{I}$$

The final stress is calculated from (6) and (27).

$$\boldsymbol{\sigma} = \mathbf{s} - p \mathbf{I} = \frac{\mathbf{s}^*}{1 + \frac{3G}{\sigma_{yield}} \Delta \kappa} + K \mathbf{II} : \boldsymbol{\varepsilon} \tag{29}$$

The consistent tangential stiffness matrix is derived from the differential form of (27). After some work this yields:

$$d\mathbf{s} = \frac{1}{1 + \frac{3G}{\sigma_{yield}}\Delta\kappa} \left[2G\mathbf{H} - \left(1 - \frac{\Delta\kappa}{\sigma_{yield}}h \right) \frac{3G\mathbf{s}\mathbf{s}}{(1 + \frac{h}{3G})\sigma_{yield}^2} \right] : d\mathbf{e} \quad (30)$$

With $\boldsymbol{\sigma} = \mathbf{s} + K\mathbf{II}\boldsymbol{\epsilon}$ and $d\mathbf{e} = (\mathbf{H} - \frac{1}{3}\mathbf{II}) : d\boldsymbol{\epsilon}$ the consistent stiffness matrix is now defined as:

$$\mathbf{D} = \frac{1}{1 + \frac{3G}{\sigma_{yield}}\Delta\kappa} \left[2G\mathbf{H} - \left(1 - \frac{\Delta\kappa}{\sigma_{yield}}h \right) \frac{3G\mathbf{s}\mathbf{s}}{(1 + \frac{h}{3G})\sigma_{yield}^2} \right] + (K - \frac{2}{3}G \frac{1}{1 + \frac{3G}{\sigma_{yield}}\Delta\kappa})\mathbf{II} \quad (31)$$

This relation differs from the stiffness matrix based on the continuum equations. Only for $\Delta\kappa = 0$ both matrices are equal.

4 THE MIXED ELASTOPLASTIC / RIGID-PLASTIC MODEL

For the elastoplastic model the iterative process may fail due to large strain increments. For the rigid-plastic model two situations may lead to numerical failure for small strain increments. If the lower bound value δ in (17) is set to a small value e.g. $\delta = 10^{-10}$ then the stress is forced to the yield surface, even if in reality the situation would be elastic. This will lead to very bad convergence and locally wrong results. This may be alleviated by setting δ to a value that represents the strain increment from zero stress to the yield surface. However in this case the rigid-plastic model degenerates to a viscous model for small strain increments. Consequently the strains will be considerably overestimated in regions with (almost) rigid body motions. Convergence is usually very good in this case, but the results may be (very) wrong in large parts of the model. Basically the rigid-plastic model fails because of the inability to model elastic strains.

It is the purpose of the newly introduced mixed model to combine the accuracy of the elastoplastic model and the stability of the rigid-plastic model over a large range of strain increments. The starting point of the mixed model is the elastoplastic equation (25). The value of α however is not fixed beforehand, but is chosen depending on the value of $\Delta\kappa$. Note that this approach is still a full elastoplastic approach if the value of α is bounded between 0 and 1.

It is argued that for large strain increments, the influence of the initial stress $\mathbf{s}_0$ must vanish. This leads to the condition:

$$\lim_{\Delta\kappa \rightarrow \infty} (1 - \alpha) \frac{3G}{\sigma_{yield}} \Delta\kappa = 1 \quad (32)$$

We define a reference strain increment $\Delta\kappa^{ref}$ (corresponding to an elastic stress increment from zero to the yield stress):

$$\frac{\sigma_{yield}}{3G} = \Delta\kappa^{ref} \quad \Rightarrow \quad \frac{3G}{\sigma_{yield}} \Delta\kappa = \frac{\Delta\kappa}{\Delta\kappa^{ref}} \quad (33)$$

Substituting this in (32) yields

$$\lim_{\Delta\kappa \rightarrow \infty} \alpha = 1 - \frac{\Delta\kappa^{ref}}{\Delta\kappa} \quad (34)$$

For small plastic strain increments α may still have any value between 0 and 1, based on a monotonous increase from $\mathbf{s}_0$ to $\mathbf{s}_1$.

After substitution of (33) in (25) we get:

$$\mathbf{s}_1 = \mathbf{s}_0 \left[1 - \frac{\Delta\kappa}{\Delta\kappa^{ref}}(1 - \alpha) \right] + 2G\Delta\mathbf{e} - \frac{\Delta\kappa}{\Delta\kappa^{ref}}\alpha\mathbf{s}_1 \quad (35)$$

The weighing factor α is now defined as a function of $\Delta\kappa$. For large $\Delta\kappa$ we still require that the final stress is independent of the initial stress. Therefore $\alpha \rightarrow 1 - \frac{\Delta\kappa^{ref}}{\Delta\kappa}$, not only for $\Delta\kappa \rightarrow \infty$ but for $\Delta\kappa > \gamma$ where γ is some limit value. Bringing all terms with $\mathbf{s}_1$ to the left gives

$$\mathbf{s}_1 \left(1 + \frac{\Delta\kappa}{\Delta\kappa^{ref}}\alpha \right) = \mathbf{s}_0 \left[1 - \frac{\Delta\kappa}{\Delta\kappa^{ref}}(1 - \alpha) \right] + 2G\Delta\mathbf{e} \quad (36)$$

so that

$$\mathbf{s}_1 = \frac{1 - \frac{\Delta\kappa}{\Delta\kappa^{ref}}(1 - \alpha)}{1 + \frac{\Delta\kappa}{\Delta\kappa^{ref}}\alpha} \mathbf{s}_0 + \frac{2G}{1 + \frac{\Delta\kappa}{\Delta\kappa^{ref}}\alpha} \Delta\mathbf{e} \quad (37)$$

Choosing $\gamma = \Delta\kappa^{ref}$ and $\alpha = 0$ for $\Delta\kappa < \gamma$ results in:

$$\text{for } \Delta\kappa \leq \Delta\kappa^{ref} : \quad \mathbf{s}_1 = \left(1 - \frac{\Delta\kappa}{\Delta\kappa^{ref}} \right) \mathbf{s}_0 + 2G\Delta\mathbf{e} \quad (38)$$

$$\text{for } \Delta\kappa > \Delta\kappa^{ref} : \quad \mathbf{s}_1 = \frac{2G}{\frac{\Delta\kappa}{\Delta\kappa^{ref}}} \Delta\mathbf{e} = \frac{2G}{\frac{3G}{\sigma_{yield}} \Delta\kappa} \Delta\mathbf{e} = \frac{2}{3} \frac{\sigma_{yield}}{\Delta\kappa} \Delta\mathbf{e} \quad (39)$$

Comparing (39) with (14) we see that for large strain increments this model equals the rigid-plastic model. For small strain increments this model equals the Euler forward explicit time integration. In figure 1 the choice for $\alpha(\Delta\kappa)$ is presented graphically, together with another possible choice, combining the mean-normal integration ($\alpha = 0.5$) with the rigid-plastic model.

The consistent stiffness matrix for large strain increments naturally equals the consistent stiffness matrix for the rigid-plastic model.

5 APPLICATIONS

In this section three examples are presented. The first example is a small tension test with nonuniform deformations, due to the boundary conditions. The convergence behavior of the different methods are presented for different sizes of strain increments. The rigid-plastic model is used with a small lower bound $\delta = 10^{-10}$ leading to a bad convergence for small strain increments. The second example is a deep drawing simulation. This example suffers from an inaccurate plastic strain calculation in the rigid-plastic analysis because of a relatively high lower bound $\delta = \sigma_{yield}/3G = \kappa^{ref}$. In the third example the influence of δ is demonstrated in a plane strain compression test.

Figure 1: Two possible α -functions.

5.1 Tension test

In this example a small patch of 9 plane strain elements is stretched. The material model is ideally plastic with a yield stress of 30 MPa and if applicable a Young's modulus of 78000 MPa and a Poisson's ratio of 0.3. The total height and width of the patch are initially 100 mm.

At the upper and lower boundary the displacements are suppressed in horizontal direction. Due to contraction in the central part of the patch a nonuniform deformation occurs, see figure 2(a). The analysis was performed with different magnitudes of displacement increments and with the radial return, rigid-plastic and mixed elastoplastic/rigid-plastic model. The Newton-Raphson iterations were stopped at a mechanical unbalance ratio of $2 \cdot 10^{-6}$. For the rigid-plastic model a lower bound $\delta = 10^{-10}$ was used. One displacement increment was taken to enter the plastic regime in all models. After that 10 displacement increments were imposed. For every case the number of iterations in the last increment is presented in figure 2(b).

It can be clearly seen that for small strain increments, the elastoplastic model converges fast and the rigid-plastic model does not converge at all. This is due to the fact that $\Delta\kappa$ in the denominator becomes very small. At the other hand for large strain increments the elastoplastic model fails and the rigid-plastic model converges fast. The mixed elastoplastic / rigid-plastic model equals the rigid-plastic model for large strain increments and has a good convergence for small strain increments.

5.2 Deep drawing of a rectangular product

Deep drawing is a deformation process in which a sheet metal, the blank, is forced to deform plastically. The specific shape of the punch and die is transferred to the sheet during the forming operation. A blank holder is used to avoid wrinkling of the blank. A principle outline of this process is given in figure 3. The deep drawing of a specific rectangular product will be used to illustrate the behavior of the mentioned material models, i.e. the

Figure 2: Nr. of iterations needed for different models.

Figure 3: Deep drawing scheme.

rigid-plastic material model, the elastoplastic material model and the mixed elastoplastic / rigid-plastic material model. In the rigid-plastic model the lower bound is related to the maximum elastic deformation: $\delta = \sigma_{yield}/3G$. Simulations are performed with the implicit finite element code DIEKA.

The geometry of the tools, used for the deep drawing of the rectangular product, is given in figure 4. The product depth is 100 mm. The used blank is 600 mm * 470 mm and has a thickness of 0.7 mm. The blank is meshed with 4160 three node triangular plate elements based on membrane theory with 1 integration point in its plane. Contact between

Figure 4: Tool geometry of the rectangular product.

the sheet and the tools is described with contact elements,⁹ in which a friction coefficient of 0.16 is assumed.

A first set of simulations is performed with an incremental step size of 0.3 mm in which the three material models are applied separately. The plastic thickness strain distributions in the rectangular product after 75 mm deep drawing are depicted in figure 5 for the rigid-plastic material model, in figure 6 for the elastoplastic material model and figure 7 for the mixed elastoplastic / rigid-plastic material model.

Figure 5: Plastic thickness strain distribution, using the rigid-plastic material model.

One can note that the used material model influences the plastic thickness strain distribution drastically. The thickness reduction is the highest for the rigid-plastic material model and the lowest for the elastoplastic material model. The plastic thickness strain

Figure 6: Plastic thickness strain distribution, using the elastoplastic material model.

Figure 7: Plastic thickness strain distribution, using the mixed elastoplastic / rigid-plastic material model.

distribution of the mixed material model inclines towards the plastic thickness strain distribution of the elastoplastic material model. The difference in thickness strain is most notable at the bottom of the product. As a result of the higher calculated strain in the bottom the draw-in at the right side of the flange is much lower for the rigid-plastic model than for the other two models.

The convergence behavior of the simulations differs significantly as well. The mechanical unbalance ratio is set at 2%. The simulation with the rigid-plastic material model needs 1 iteration per incremental step for convergence. The simulation with the elastoplastic material model needs 1 to 5 iterations per step for convergence. The simulation with the mixed material model needs 3 iterations per step for convergence.

The final product depth of 100 mm is successfully reached in the simulation with the rigid-plastic material model and the mixed material model. However, the simulation with the elastoplastic material model diverges after 93 mm deep drawing. The plastic thickness strain distribution after 100 mm deep drawing along line A-B, see figure 4, is depicted in figure 7 for the rigid-plastic material model and the mixed material model.

Figure 8: The plastic thickness strain distribution along line A-B.

In figure 8 it can be seen clearly that the calculated plastic thickness strain in case of the rigid-plastic material model is higher than the plastic thickness strain in case of the mixed material model, especially in the bottom of the product. This can be attributed to the relatively high lower bound value of δ . Effectively the rigid-plastic model reduces to a viscous model in regions where the strain rate is low (the bottom), leading to an overestimation of the total strain. With a lower bound of $\delta = 10^{-7}$ the simulation did not converge.

5.3 Compression test

The third example is the 3D simulation of a plane strain compression (PSC) test. It is commonly used to determine the material behavior of metals because of the simple geometry and the fact that the deformation is quite similar to that of rolling. The PSC

test is outlined in figure 9. In finite elements it is sufficient to model only one eighth

Figure 9: PSC test before and after deformation.

because of the triple symmetry. In the current example the die width w is 15 mm and the initial length l_0 and height h_0 are 60 and 10 mm respectively. 1400 linear hexahedral and 140 3D contact elements are used. The height reduction is 60% with a punch speed v of 10 mm/s. With the implicit code DIEKA simulations are performed with the elastoplastic

Figure 10: Calculated edges in top view.

model, the rigid-plastic model with a tiny lower bound $\delta = 10^{-10}$ and high (classic) lower bound $\delta = \sigma_y/3G$ respectively, and with the mixed elastoplastic / rigid-plastic model. The calculated deformations show good agreement for all methods except for the classic rigid-plastic method. The calculated mid plane product contour for this model and the mixed model are depicted in figure 10. From these figures one can see that the specimen in case of the rigid-plastic model shows deformation in the rigid region, reflected by the curvature of the sides. With the mixed model the rigid region is described completely elastic and therefore no significant deformation can be seen.

After comparison of CPU times of the three remaining methods one can conclude that the mixed model should be preferred to the elastoplastic and rigid-plastic models. Indexing

the CPU time of the rigid–plastic model 100% the elastoplastic model scores 86% and the mixed model 74%.

6 CONCLUSION

A new integration method for elastoplastic material models was introduced that degenerates to the rigid–plastic model for large strain increments. The mixed elastoplastic / rigid–plastic model presents a robust way of integrating the rate equations of plasticity for a large range of strain increments. Future research will be focused on the choice of the weighing function α and on the implementation of this model for orthotropic yield functions and rate-dependent plasticity.

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