On Strict Passivity and its Application to Interpolation and $\mathcal{H}_\infty$ Control

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Abstract. We introduce the $\mathcal{L}_2$ -system and derive necessary and sufficient conditions for these systems to be strictly passive. Strictly passive $\mathcal{L}_2$ -systems are characterized as having a representation in terms of a co- J -lossless matrix. A state space proof is developed and provides a Riccati equation characterization of a strictly passive $\mathcal{L}_2$ -system, as well as a formula for the co- J -lossless matrix representation. Applications to Nevanlinna Pick interpolation and an $\mathcal{H}_\infty$ filtering problem are considered.

1 Introduction

The theory of positive (or negative) subspaces of an indefinite inner product space provides powerful results in interpolation theory and related areas such as $\mathcal{H}_\infty$ control (see [2], [3], [6]). Yet these ideas have remained the preserve of a handful of researchers, perhaps due to the abstractness of the problem formulation. Such abstractness, although necessary if the most general theorems are to be derived, is unnecessary in the more concrete world of rational systems.

In this paper, we consider rational systems described by ordinary linear differential equations. We use the Willems behavioral systems approach (see [17]) and represent systems by the AR description

$$R\left(\frac{d}{dt}\right)w(t) = 0$$

in which R is a polynomial matrix and the vector w is the external signal. In other words, the external signal w is 'orthogonal' to the differential operator $R(d/dt)$.

In the applications we have in mind, closed loops are required to be stable. This leads us to focus our attention on the *unstable* 'behavior' of the system. For example, right half plane zeros determine the possible closed loops in the optimal sensitivity problem; right half plane poles do the same in optimal robust stabilization problems; and both can be formulated as optimal interpolation problems with the interpolation points in the right half plane ([4], [10], [12], [18]). An $\mathcal{L}_2$ -system is the idea we introduce to focus attention on this unstable 'behavior'. Roughly speaking, the behavior of an $\mathcal{L}_2$ -system is an unstable orthogonal complement of a differential operator $R(d/dt)$. We will argue that this provides a very convenient language for all sorts of $\mathcal{H}_\infty$ optimization problems. (See Section 4).

To illustrate our approach, consider the following constant matrix optimization problem. Given A and B , find Q such that $H = A + QB$ and $\|H\| < 1$, where $\|\cdot\|$ is the Euclidean induced norm. This problem boils down to showing that certain matrices have a specific inertia. If they have not, then no solution exists; if they have, then all solutions can be generated. This is most easily seen by using orthogonal complements.

If B is square and nonsingular, the problem always has a solution (set $Q = -AB^{-1}$), because there is no part of the space that cannot be affected by Q . When there are parts of the space Q cannot affect, these subspaces provide necessary conditions for the existence of a solution. Let $B_\perp$ be an orthogonal complement of B , so $BB_\perp = 0$. Then, for any Q , $HB_\perp = AB_\perp$. If $\|H\| < 1$, then

$$0 < B_\perp^*(I - H^*H)B_\perp = B_\perp^*(I - A^*A)B_\perp.$$

The right hand side is independent of Q , so the inequality $B_\perp^*(I - A^*A)B_\perp > 0$ is a necessary condition for the problem to have a solution. An equivalent statement is that

$$P^*JP > 0; \quad P = \begin{pmatrix} I & \\ & -A \end{pmatrix} B_\perp; \quad J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

To see that $P^*JP > 0$ is also sufficient for a solution to exist, note that

$$GP = 0; \quad G = \begin{pmatrix} B & 0 \\ A & I \end{pmatrix}.$$

Using inertia arguments, it may be shown that $P^*JP > 0$ and $GP = 0$ imply that $GJG^* = W\tilde{J}W^*$ for some nonsingular matrix W and a signature matrix $\tilde{J}$ which has the same number of negative eigenvalues as J . Define $(H_1 \ Q_2) = (U \ I)W^{-1}G$ in which $\|U\| < 1$ (for example, $U = 0$). Then $H_1H_1^* - Q_2Q_2^* = (U \ I)\tilde{J}(U \ I)^* = UU^* - I < 0$. Therefore Q_2 is nonsingular and $\|Q_2^{-1}H_1\| < 1$. Define Q by $(Q \ I) = Q_2^{-1}(U \ I)W^{-1}$. Then $H = A + QB = Q_2^{-1}H_1$, so $\|H\| < 1$. (It is easily seen that all solutions are generated as U varies over the space if contractive matrices.)

This approach has a positive subspace interpretation: consider the subspace

$$\mathcal{B} = PC^n$$

or, equivalently,

$$\mathcal{B} = \{w \mid Gw = 0\}.$$

Then $\mathcal{B}$ is a positive subspace, where the indefinite inner product is $[x, y] = y^*Jx$, since there is an $\epsilon > 0$ such that $[w, w] = w^*Jw \geq \epsilon w^*w$ for all $w \in \mathcal{B}$. Note that $\mathcal{B}$ has representation in terms of a matrix M satisfying $MJM^* = \tilde{J}$:

$$MB = 0.$$

(Take $M = W^{-1}G$.)

Consider now the situation where G and Q are stable, rational matrices. The problem is now more difficult, since inverting stable matrices may introduce unstable poles. An other way of saying the same thing is that the essential properties of G can not always be recovered from an orthogonal complement P of G . Kimura *et al.* [11] have been particularly active in developing approaches to the solution of $\mathcal{H}_\infty$ control problems based on what they call J -lossless conjugation. Their J -lossless conjugation approach and our approach share the same fundamental idea and goal (using the simple orthogonal complement ideas which work for constant matrices in a more general setting), but differ in implementation. The crucial difference is the introduction of the $\mathcal{L}_2$ -system.

Throughout this paper we use the following notation:

$\mathcal{C}_-, \mathcal{C}_+, \mathcal{C}_0$	Open left half complex plane, open right half complex plane, imaginary axis.
A^*	Complex conjugate transpose of A .
$[z]$	The number of components of a vector signal z .
$\ z\ $	Euclidean norm of a vector z ($\ z\ ^2 = z^*z$).

$D^{-L}, D^{-R}, D_{\perp}$	Left inverse, right inverse and orthogonal complement of a constant matrix D : $D^{-L}D = I$, $DD^{-R} = I$, $D_{\perp}$ is a maximal full rank matrix such that $DD_{\perp} = 0$ or $D_{\perp}D = 0$, depending on whether D has more columns than rows or vice versa.
$H^{\sim}, H^*$ $\ H\ _{\infty}$	$H^{\sim}(s) = H(-\bar{s})^*$ and $H^*(s) = (H(s))^*$. Supremum of the largest singular value of $H(s)$ over all $s \in \mathbb{C}_0$.
$\mathcal{L}_2(a, b), \mathcal{L}_{2-}, \mathcal{L}_{2+}$	$\{w : (a, b) \rightarrow \mathbb{C}^m \mid \int_a^b w^*(t)w(t)dt < \infty\}$, $\mathcal{L}_{2-} = \mathcal{L}_2(-\infty, 0)$, $\mathcal{L}_{2+} = \mathcal{L}_2(0, \infty)$.
$\mathcal{H}_2, \mathcal{H}_2^{\perp}$	The set of functions f analytic in $\mathbb{C}_+$ such that $\sup_{\sigma>0} \int_{-\infty}^{\infty} \ f(\sigma + j\omega)\ _2^2 d\omega < \infty$, the set of functions f such that $f^{\sim} \in \mathcal{H}_2$.
$\mathcal{L}_2$ $\mathcal{H}_{\infty}^{m \times p}, \mathcal{GH}_{\infty}^m$	$\mathcal{L}_{2-} \oplus \mathcal{L}_{2+}$, or $\mathcal{H}_2^{\perp} \oplus \mathcal{H}_2$. The set of $m \times p$ matrix valued functions that are analytic and bounded in $\mathbb{C}_+$, the set of invertible elements in $\mathcal{H}_{\infty}^{m \times m}$ (we frequently omit explicit mention of dimensions).
π_-, π_+	Orthogonal projection from $\mathcal{L}_2$ to $\mathcal{H}_2^{\perp}$ (or $\mathcal{L}_{2-}$), $\pi_+ = 1 - \pi_-$.
$\hat{w}$	Left, right or two-sided Laplace transform of a time signal w .

A rational matrix G is *stable* if $G \in \mathcal{H}_{\infty}$; G is *antistable* if $G^{\sim} \in \mathcal{H}_{\infty}$. A rational matrix H is *contractive* if $\|H\|_{\infty} < 1$. If A is the constant 'A-matrix' in a state space description, then A *stable* (*antistable*) means that all eigenvalues of A lie in $\mathbb{C}_-$ ($\mathbb{C}_+$). $J_{p,q}$ is a diagonal signature matrix of the form

$$J_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}. \quad (1)$$

An $(r+q) \times (p+q)$ rational matrix M is *co- $J_{p,q}$ -lossless* if $MJ_{p,q}M^{\sim} = J_{r,q}$, and $MJ_{p,q}M^* \leq J_{r,q}$ in $\mathbb{C}_+$. Note that $J_{p,q}$ and $J_{r,q}$ are not the same, but they do have the same number of negative eigenvalues.

The notation $\|\cdot\|_2$ denotes the well known 2-norm. For the most part we will be dealing with the signal space $\mathcal{L}_{2-}$, so we will have $\|w\|_2^2 := \int_{-\infty}^0 w(t)^*w(t) dt$.

2 $\mathcal{L}_{2-}$ -Systems

A dynamical system $\Sigma = (\mathbf{T}, W, \mathcal{B})$ is a triple, where $\mathbf{T}$ is the time axis, W the signal space, and $\mathcal{B} \subset W^{\mathbf{T}}$ the behavior, that is, the set of all possible trajectories that may occur. The systems we will look at are systems $\Sigma_- = (\mathbf{R}_-, \mathbf{R}^p, \mathcal{B}_-)$, with

$$\mathcal{B}_- = \{w \mid R(d/dt)w(t) = 0\} \cap \mathcal{L}_{2-}, \quad (2)$$

in which R is a polynomial matrix. The frequency domain counterpart is a system $\hat{\Sigma}_- = (\mathbb{C}_-, \mathbb{C}^p, \hat{\mathcal{B}}_-)$ with

$$\hat{\mathcal{B}}_- = \{\hat{w} \mid G\hat{w} \in \mathcal{H}_2\} \cap \mathcal{H}_2^{\perp}, \quad (3)$$

in which G is a stable rational matrix. It can be shown that for every polynomial matrix R there exists a stable G , and vice versa, such $\mathcal{B}_-$ and $\hat{\mathcal{B}}_-$ are isomorphic under the Laplace transform. The polynomial matrix R in (2) is said to define an *AR representation* of the $\mathcal{L}_{2-}$ -system. We will drop the 'hats' as long as no confusion can arise, and we switch from time domain to frequency domain whenever it is convenient. We say that stable G in (3) defines an AR representation of the $\mathcal{L}_{2-}$ -system.

Definition 2.1 A dynamical system $\Sigma_- = (\mathbf{T}, W, \mathcal{B}_-)$ is an $\mathcal{L}_{2-}$ -system if $\mathbf{T} = \mathbf{R}_-$, $W = \mathbf{R}^p$, and $\mathcal{B}_- = \mathcal{B} \cap \mathcal{L}_{2-}$, for some $\mathcal{B} \subset W^{\mathbf{T}}$. Behaviors (2) are abbreviated to $\mathcal{B}_R$.

Dynamical systems $\hat{\Sigma}_- = (\mathbb{C}_-, \mathbb{C}^p, \hat{\mathcal{B}}_- \cap \mathcal{H}_2^{\perp})$ are also referred to as $\mathcal{L}_{2-}$ -systems. Behaviors (3) are abbreviated to $\mathcal{B}_G$. ■

Note: the definition allows for more general $\mathcal{L}_{2-}$ -systems than can be described by (2). In this paper, however, only $\mathcal{L}_{2-}$ -systems described by (2) are considered. Signals w in a behavior are sometimes referred to as *external* signals, so as to distinguish them from other signals we may like to introduce.

Example 2.2

1. Suppose $y = Hu$ for some stable transfer matrix H . Let $w := \begin{pmatrix} y \\ u \end{pmatrix}$ be the external signal. Then $G = (-H \ I)$ defines an AR representation of an $\mathcal{L}_{2-}$ -system, with

$$\mathcal{B}_G = \begin{pmatrix} I \\ \pi_- H \end{pmatrix} \mathcal{H}_2^{\perp},$$

where π_- is the orthogonal projection onto $\mathcal{H}_2^{\perp}$.

2. Suppose $G(s) = C(sI - (I - C))^{-1} - I$ is stable. Then $G\hat{w} \in \mathcal{H}_2$ for $\hat{w} \in \mathcal{H}_2^{\perp}$, iff $w(t) = Ce^{tI}x_0$, $x_0 \in \mathbb{C}^n$. So $\mathcal{B}_G = C(sI - I)^{-1}\mathbb{C}^n \subset \mathcal{H}_2^{\perp}$ has finite dimension. ■

The advantages of using AR representations instead of the usual transfer matrices are: (1) AR representations are defined without an input/output partitioning of the external signal; and (2) AR representations can represent systems that can not be represented by a transfer matrix from inputs to outputs. The second example in Example 2.2 can not be written as $y = Hu$, with external signal $w = \begin{pmatrix} y \\ u \end{pmatrix}$. Transfer matrices mapping inputs to outputs give rise to infinite dimensional behaviors (the input u can be chosen from an infinite dimensional space, usually). With AR representations behaviors in general consist of an infinite dimensional part and a finite dimensional part.

Lemma 2.3 ([14]) Suppose G and $\tilde{G}$ are two stable matrices that have full row rank on $\mathbb{C}_0 \cup \infty$. Then $\mathcal{B}_G = \mathcal{B}_{\tilde{G}}$ if and only if $G = W\tilde{G}$ for some $W \in \mathcal{GH}_{\infty}$. Furthermore, there exists a stable E such that $\mathcal{B}_G = \mathcal{B}_E$ with E having all its zeros in $\mathbb{C}_+$. ■

Definition 2.4 An $\mathcal{L}_{2-}$ -system $\Sigma_- = (\mathbf{R}_-, \mathbf{R}^p, \mathcal{B}_-)$ is *strictly passive (SP)* with respect to partitioning $w = \begin{pmatrix} y \\ u \end{pmatrix}$ if there exists an $\epsilon > 0$ such that every w in $\mathcal{B}_-$ satisfies

$$\int_{-\infty}^T w^*(t)J_{[u],[y]}w(t) dt \geq \epsilon \int_{-\infty}^T w^*(t)w(t) dt, \quad (4)$$

for all time $T \in \mathbf{R}_-$. Inequality (4) is referred to as the strict passivity inequality (SP inequality). ■

The systems we consider are time invariant, so we may restrict our attention to the case $T = 0$.

Example 2.5 Suppose $y = Hu$, $H \in \mathcal{H}_{\infty}$. Let $G = (H \ -I)$ and consider the $\mathcal{L}_{2-}$ -system Σ_- with behavior $\mathcal{B}_- = \mathcal{B}_G$. Then Σ_- is SP with respect to the partitioning $w = \begin{pmatrix} y \\ u \end{pmatrix}$ iff $\|H\|_{\infty} < 1$. ■

The main result is formulated next.

Theorem 2.6 Let G be a stable matrix that has full row rank on $\mathbb{C}_0 \cup \infty$. The $\mathcal{L}_{2-}$ -system $\Sigma_- = (\mathbf{R}_-, \mathbf{R}^p, \mathcal{B}_G)$ is strictly passive with respect to the partitioning $w = \begin{pmatrix} y \\ u \end{pmatrix}$ if and only if $\mathcal{B}_G = \mathcal{B}_M$ for some $M \in \mathcal{H}_{\infty}$ that is co- $J_{[u],[y]}$ -lossless. ■

Proof 2.7 That SP implies the existence of a co- J -lossless matrix that defines the system is proven in the next section. Sufficiency is easy, see [14]. ■

A co- J -lossless M can be constructed from a given G , if it exists:

Corollary 2.8 ([14]) Suppose G is stable and has full row rank m on $\mathbb{C}_0 \cup \infty$. Then $\Sigma_- = (\mathbf{R}_-, \mathbf{R}^p, \mathcal{B}_G)$ is SP with respect to the partitioning $w = \begin{pmatrix} y \\ u \end{pmatrix}$ if and only if $GJ_{[u],[y]}G^{\sim} = WJ_{m-[u],[y]}W^{\sim}$ has a solution $W \in \mathcal{GH}_{\infty}$ and $W^{-1}G$ is co- $J_{[u],[y]}$ -lossless. ■

3 State Space Analysis of Strictly Passive $\mathcal{L}_2$ -Systems

In this section we determine necessary and sufficient conditions under which an $\mathcal{L}_2$ -system is strictly passive, expressed in terms of state space descriptions of the system. To begin, we consider the state space descriptions with which we work.

3.1 ONR's and DVR's

This subsection contains some basic results on two different state space representations of a system. The external signal will be denoted by w . Other signals, like the state, are there to enable a parameterization of all w 's. In this subsection it is *not* assumed that w is in $\mathcal{L}_2$; we will introduce this additional constraint in the next subsection.

Definition 3.1

1. An *output nulling state space system* is a system whose external signal w satisfies

$$\begin{aligned}\dot{x} &= Ax + Bw \\ 0 &= Cx + Dw.\end{aligned}\quad (5)$$

No restrictions are imposed on the 'initial' state.

The quadruple $\{A, B, C, D\}$ defines an *output nulling representation* (ONR) of the system.

2. A *driving variable state space system* is a system whose external signal w satisfies

$$\begin{aligned}\dot{x} &= \bar{A}x + \bar{B}v \\ w &= \bar{C}x + \bar{D}v.\end{aligned}\quad (6)$$

No assumptions are imposed on the signal v or on the 'initial' state.

The quadruple $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ defines a *driving variable representation* (DVR) of the system.

3. The *zeros* of a quadruple $\{A, B, C, D\}$ defining an ONR or DVR of a system is the set of values s for which

$$\begin{pmatrix} A - sI & B \\ C & D \end{pmatrix}\quad (7)$$

drops below normal rank. ■

Example 3.2 Consider the system described by

$$\begin{aligned}\dot{x} &= \hat{A}x + \hat{B}u \\ y &= \hat{C}x + \hat{D}u.\end{aligned}$$

This is a DVR if the external signal is y and the 'initial' state is arbitrary. If the external signal is $w = \begin{pmatrix} y \\ u \end{pmatrix}$, we have DVR

$$\begin{aligned}\dot{x} &= \hat{A}x + \hat{B}v \\ w &= \begin{pmatrix} 0 \\ \hat{C} \end{pmatrix} x + \begin{pmatrix} I \\ \hat{D} \end{pmatrix} v\end{aligned}$$

and ONR

$$\begin{aligned}\dot{x} &= \hat{A}x + \begin{pmatrix} \hat{B} & 0 \end{pmatrix} w \\ 0 &= \hat{C}x + \begin{pmatrix} \hat{D} & -I \end{pmatrix} w.\end{aligned}\quad \blacksquare$$

The 'output' $Cx + Dw$ in the ONR (5) is zero, so applying output injection amounts to doing nothing; DVR's are unaffected by state feedback, since this is just a redefinition of the driving variable.

An ONR or DVR is considered minimal if the associated matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

has minimal dimension (amongst all possible such representations). The following lemma describes the minimality conditions.

Lemma 3.3

1. Suppose $\{A, B, C, D\}$ defines an ONR of a system Σ .
 - (a) The quadruple $\{A + HC, B + HD, TC, TD\}$ defines an ONR of the same system Σ , for any H and any nonsingular T . (We call this transformation a *regular output injection*.)
 - (b) There exists a quadruple $\{\hat{A}, \hat{B}, \hat{C}, \hat{D}\}$ defining an ONR of the system Σ such that $(\hat{C}, \hat{A})$ is observable and $\hat{D}$ has full row rank. (We call this a *minimal ONR*.)
2. Suppose $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ defines a DVR of a system Σ .
 - (a) The quadruple $\{\bar{A} + \bar{B}F, \bar{B}R, \bar{C} + \bar{D}F, \bar{D}R\}$ defines a DVR of the same system Σ , for any F and any nonsingular R . (We call this transformation a *regular state feedback*.)
 - (b) There exists a quadruple $\{\hat{A}, \hat{B}, \hat{C}, \hat{D}\}$ defining a DVR of the system Σ which is *strongly observable* (i.e., $(\hat{C} + \hat{D}F, \hat{A} + \hat{B}F)$ observable for every F) and such that $\hat{D}$ has full column rank. (We call this a *minimal DVR*.) ■

Example 3.4 Controllability plays no role in the minimality of an ONR or DVR. Consider for example the system whose signals are of the form

$$w(t) = Ce^{At}x_0; \quad x_0 \in \mathbb{R}^n.$$

This system has ONR quadruple $\{A, 0, -C, I\}$ and DVR quadruple $\{A, I, C, \}$ (the 'driving variable' has null dimension). If (C, A) is observable, both the ONR and the DVR are minimal. ■

The following lemma describes the relationship between minimal ONR's and DVR's of a system Σ :

Lemma 3.5

1. Let $\{A, B, C, D\}$ define a minimal ONR of a system Σ . Then $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ given by

$$\begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -D^{-1}C & D^{-1} \end{pmatrix}\quad (8)$$

defines a minimal DVR of Σ . Furthermore, λ is a zero of $\{A, B, C, D\}$ if and only if it is an uncontrollable mode of $(\bar{A}, \bar{B})$.

2. Let $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ define a minimal DVR of a system Σ . Then $\{A, B, C, D\}$ given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & -\bar{B}\bar{D}^{-1}\bar{C} \\ 0 & -\bar{D} \end{pmatrix} \begin{pmatrix} \bar{A} & 0 \\ \bar{C} & -I \end{pmatrix}\quad (9)$$

defines a minimal ONR of Σ . Furthermore, λ is a zero of $\{A, B, C, D\}$ if and only if it is an uncontrollable mode of $(\bar{A}, \bar{B})$. ■

3.2 ONR's and DVR's of $\mathcal{L}_2$ -systems

Consider a DVR of a system Σ . For the external signal w to be in $\mathcal{L}_2$, the stable part of the dynamics must not appear in w ; all $\mathcal{L}_2$ -systems may be represented by DVR's which have antistable dynamics; and all $\mathcal{L}_2$ -systems may be represented by ONR's which have zeros only in $\mathbb{C}_+$:

Lemma 3.6

1. Let w be the external signal of a system Σ .

(a) $w \in \mathcal{L}_{2-}$ implies that $x \in \mathcal{L}_{2-}$ and $\lim_{t \rightarrow -\infty} x(t) = 0$, in which x is the state variable in any minimal ONR of Σ .

(b) $w \in \mathcal{L}_{2-}$ implies that $v \in \mathcal{L}_{2-}$, $x \in \mathcal{L}_{2-}$ and that $\lim_{t \rightarrow -\infty} x(t) = 0$, in which x is the state variable and v is the driving variable in any minimal DVR of Σ .

2. Let Σ_- be an $\mathcal{L}_{2-}$ -system.

(a) Σ_- can be defined by a minimal ONR $\{A, B, C, D\}$ that has all its zeros in $\mathbb{C}_+$. We call such an ONR an $\mathcal{L}_{2-}$ -minimal ONR.

(b) Σ_- can be defined by a minimal DVR $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ such that $(\bar{A}, \bar{B})$ is antistabilizable. We call such a DVR an $\mathcal{L}_{2-}$ -minimal DVR. (Note that by choice of an antistabilizing regular state feedback there exists a minimal DVR with antistable ' $\bar{A}$ -matrix'.)

Proof 3.7

1. (a) Introduce an antistabilizing output injection H . Since $\dot{x} = (A + HC)x + (B + HD)w$ in which $A + HC$ is antistable and $w \in \mathcal{L}_{2-}$, we have $x \in \mathcal{L}_{2-}$. Also, $\dot{x} = Ax + Bw$ implies $\dot{x} \in \mathcal{L}_{2-}$. Thus $\lim_{t \rightarrow -\infty} x(t) = 0$.

(b) Consider minimal DVR quadruple $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ and let E be a nonsingular matrix such that $E\bar{D} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. So $Ew = E\bar{C}x + \begin{pmatrix} 1 \\ 0 \end{pmatrix}v$. Apply regular state feedback $v = -E_1\bar{C}x + \hat{v}$, where E_1 is the upper row block of E . Now we have $E_1w = \hat{v}$, so $\hat{v}$ is in $\mathcal{L}_{2-}$. Strong observability implies that $(E_2\bar{C}, \bar{A})$ is observable (E_2 is the lower block row of E), so there exists an H such that $\bar{A} + HE_2\bar{C}$ is antistable. Rewrite the dynamics as $\dot{x} = (\bar{A} + HE_2\bar{C})x + \hat{v} - HE_2w$ and proceed as in Item (a).

2. Note that Items (a) and (b) are equivalent, by Lemma 3.5. We prove Item (b), the DVR result.

Consider any minimal DVR $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ defining Σ_- . By Item 1b, the state x and the driving variable v are both in $\mathcal{L}_{2-}$ and $x(-\infty) = 0$. The state corresponding to any uncontrollable mode $\lambda \in \mathbb{C}_- \cup \mathbb{C}_0$ of $(\bar{A}, \bar{B})$ is therefore identically zero. Removing all such modes from the DVR leaves the desired antistabilizable DVR. ■

3.3 Strictly passive $\mathcal{L}_{2-}$ -systems

Consider an $\mathcal{L}_{2-}$ -system Σ_- with external signal w partitioned as

$$w = \begin{pmatrix} u \\ y \end{pmatrix}. \quad (10)$$

Define the two signature matrices: $J = J_{[u],[y]}$, $\hat{J} = J_{m-[y],[y]}$.

Theorem 3.8 Let $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ define an $\mathcal{L}_{2-}$ -minimal DVR of an $\mathcal{L}_{2-}$ -system Σ_- .

1. Then Σ_- is SP with respect to the partitioning (10) if and only if

(a) $\bar{D}^*J\bar{D} > 0$ and

(b) There exists X such that

$$\bar{A}^*X + X\bar{A} - \bar{C}^*J\bar{C} + [X\bar{B} - \bar{C}^*J\bar{D}](\bar{D}^*J\bar{D})^{-1}[-\bar{D}^*J\bar{C} + \bar{B}^*X] = 0 \quad (11)$$

such that $\bar{A} + \bar{B}(\bar{D}^*J\bar{D})^{-1}[-\bar{D}^*J\bar{C} + \bar{B}^*X]$ is antistable and

(c) $X > 0$.

2. Furthermore, given an antistabilizing solution X to (11) and a nonsingular solution W to $W^*W = \bar{D}^*J\bar{D}$, define $\{A_2, B_2, C_2, D_2\}$ as

$$\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} \begin{pmatrix} I & 0 \\ (\bar{D}^*J\bar{D})^{-1}[-\bar{D}^*J\bar{C} + \bar{B}^*X] & W^{-1} \end{pmatrix}. \quad (12)$$

Let D_M be a constant matrix such that $D_M\bar{D} = 0$ and $D_MJ\bar{D}^* = \bar{J}$. Then $M(s) := D_M - D_MC_2X^{-1}(sI + A_2^*)^{-1}C_2^*J$ is a stable, rational matrix that defines an AR representation of Σ_- , and M is co- J -lossless iff $X > 0$. ■

Proof 3.9

(Item 1, (a)) Suppose Σ_- is SP. Consider driving variable $v(t) = \sqrt{\delta(t)}v_0$, with $x(t) = 0$ for $t < 0$. Then

$$v_0^*D^*JDv_0 = \int_{-\infty}^0 w^*(t)Jw(t) dt \geq \epsilon \int_{-\infty}^0 w^*(t)w(t) dt = \epsilon v_0^*D^*Dv_0.$$

Since D has full column rank, this implies $D^*JD > 0$. (This argument can be made precise by considering $\mathcal{L}_{2-}$ approximations to $\delta(t)$ —we omit these tedious and unenlightening details.)

To simplify the algebra for Items 2 and 3, consider the following regular state feedback. Let

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} \begin{pmatrix} I & 0 \\ -(\bar{D}^*J\bar{D})^{-1}\bar{D}^*J\bar{C} & W^{-1} \end{pmatrix}. \quad (13)$$

in which W is a nonsingular solution to $W^*W = \bar{D}^*J\bar{D}$. The quadruple $\{A_1, B_1, C_1, D_1\}$ defines a DVR of the same system as the two DVR's differ by a regular state feedback. It suffices therefore to consider the DVR

$$\begin{aligned} \dot{x} &= A_1x + B_1v_1 \\ w &= C_1x + D_1v_1. \end{aligned}$$

It is easily checked that $D_1^*JD_1 = I$ and $D_1^*JC_1 = 0$.

(Item 1, (b)) Suppose the system Σ_- is SP. Let H be the Hamiltonian matrix

$$H = \begin{pmatrix} A_1 & B_1B_1^* \\ C_1^*JC_1 & -A_1^* \end{pmatrix}. \quad (14)$$

It may be shown that H has no eigenvalues on the imaginary axis. Since $(-A_1, B_1)$ is stabilizable, $B_1B_1^* \geq 0$ and H in (14) has no imaginary axis eigenvalue, it follows from standard Hamiltonian matrix results (see, for instance, [6], Chapter 7 or [5], Lemma 2) that there exists a Y such that

$$Y(-A_1) + (-A_1^*)Y + YB_1B_1^*Y - C_1^*JC_1 = 0$$

and $-A_1 + B_1B_1^*Y$ is stable. Setting $X = -Y$, we have

$$XA_1 + A_1^*X + XB_1B_1^*X - C_1^*JC_1 = 0 \quad (15)$$

and $A_1 + B_1B_1^*X$ is antistable. A straightforward calculation shows that X also satisfies (11) and $\bar{A} + \bar{B}(\bar{D}^*J\bar{D})^{-1}[-\bar{D}^*J\bar{C} + \bar{B}^*X]$ is antistable.

(Item 1, (c)) Suppose Σ_- is SP. Let X be as in Item (b) and consider the DVR $\{A_1, B_1, C_1, D_1\}$. The signals w , x and v_1 are in $\mathcal{L}_{2-}$ and $x(-\infty) = 0$. Completing the square gives

$$\int_{-\infty}^0 w(t)^*Jw(t) dt = x(0)^*Xx(0) + \|v_1 - B_1^*Xx\|_2^2. \quad (16)$$

Now consider the driving variable $v_1 = B_1^*Xx$, which is in $\mathcal{L}_{2-}$ since X is antistabilizing. Let w be the external signal resulting from v_1 , with 'initial' state $x(0)$. Then

$$\begin{aligned} x(0)^*Xx(0) &= \int_{-\infty}^0 w(t)^*Jw(t) dt \quad \text{by (16)} \\ &\geq \epsilon \int_{-\infty}^0 w(t)^*w(t) dt = \epsilon x(0)^*Qx(0) \end{aligned}$$

where $\hat{Q} = (A_1 + B_1 B_1^* X) + (A_1 + B_1 B_1^* X)^* Q = (C_1 + D_1 B_1^* X)^* (C_1 + D_1 B_1^* X)$. Note that $(C_1 + D_1 B_1^* X, A_1 + B_1 B_1^* X)$ is observable (by strong observability), so $\hat{Q} > 0$. Hence $X \geq \epsilon \hat{Q} > 0$.

Conversely, suppose $X > 0$. Write

$$\begin{aligned} \dot{x} &= (A_1 + B_1 B_1^* X)x + B_1(v_1 - v_{\text{opt}}) \\ w &= (C_1 + D_1 B_1^* X)x + D(v_1 - v_{\text{opt}}) \end{aligned}$$

in which $v_{\text{opt}} = B_1^* X x$. Since $A_1 + B_1 B_1^* X$ is antistable, there exist constants $\beta \geq 0$ and $\gamma > 0$ such that

$$\|w\|_2^2 \leq \gamma \|v_1 - v_{\text{opt}}\|_2^2 + \beta \|x(0)\|^2; \quad \gamma > \frac{\beta}{\lambda_{\min}(X)}.$$

Hence

$$\begin{aligned} \int_{-\infty}^0 w(t)^* J w(t) dt &= x(0)^* X x(0) + \|v_1 - v_{\text{opt}}\|_2^2 \\ &\geq (\lambda_{\min}(X) - \frac{\beta}{\gamma}) \|x(0)\|^2 + \frac{1}{\gamma} \|w\|_2^2 \geq \frac{1}{\gamma} \|w\|_2^2 \end{aligned}$$

which proves the system is SP.

(Item 2) Let $A_M = -A_2^*$, $B_M = C_2^* J$ and $C_M = -D_M C_2 X^{-1}$, so $M(s) = C_M(sI - A_M)^{-1} B_M + D_M$. The DVR $\{A_2, B_2, C_2, D_2\}$ is normalized in the sense that $A_2^* X + X A_2 - C_2^* J C_2 = 0$, $B_2^* X - D_2^* J C_2 = 0$, $D_2^* J D_2 = I$. This can be used to show that $A_M X + X A_M^* + B_M J B_M^* = 0$, $B_M J D_M^* + X C_M^* = 0$, $D_M J D_M^* = J$. It follows from [8], Theorem 5.3 that M is co- J -lossless iff $X > 0$.

Remains to show that M defines an AR representation of Σ_- . We first show that $\{A_M, B_M, C_M, D_M\}$ defines an ONR of Σ_- . It follows from Lemma 3.5, Item 2 that

$$\begin{pmatrix} A_2 - B_2 D_2^{-L} C_2 & B_2 D_2^{-L} \\ -D_{21} C_2 & D_{21} \end{pmatrix} = \begin{pmatrix} A_2 - B_2 D_2^* J C_2 & B_2 D_2^* J \\ -D_M C_2 & D_M \end{pmatrix}$$

defines an ONR of Σ_- . Here we used that $D_{21} := D_M$ is an orthogonal complement of D_2 and that $D_2^{-L} := D_2^* J$ is a left inverse of D_2 . It is easily checked that this ONR transforms under output injection $H = C_2^* D_M^* J$ into

$$\begin{pmatrix} A_2 - X^{-1} C_2^* J C_2 & X^{-1} C_2^* J \\ -D_M C_2 & D_M \end{pmatrix} = \begin{pmatrix} X^{-1} A_M X & X^{-1} B_M \\ C_M X & D_M \end{pmatrix}.$$

A state transformation gives the desired ONR.

In order to see the connection with AR representations, write $C_M x + D_M w$ as

$$\begin{aligned} y(t) &:= C_M x(t) + D_M w(t) \\ &= C_M e^{A_M(t+T)} x(-T) + \int_{-T}^t C_M e^{A_M(t-\tau)} B_M w(\tau) d\tau + D_M w(t). \end{aligned} \quad (17)$$

Letting T go to infinity and taking Laplace transforms shows that $\hat{y}(s) = M(s)\hat{w}(s)$. Due to the Paley-Wiener Theorem we have $y(t) = 0$ for $t < 0$ iff $\hat{y} = M\hat{w}$ is in $\mathcal{H}_2$. So $w \in \mathcal{L}_2$ satisfies the ONR equations iff $\hat{w} \in B_M$. ■

Theorem 3.8 can be easily translated into a corresponding ONR result using Lemma 3.5. It is frequently the case that we would like to determine whether or not a given ONR defines an SP $\mathcal{L}_2$ -system without having to find an $\mathcal{L}_2$ -minimal representation. This is quite easily done:

Corollary 3.10 Let $\{A, B, C, D\}$ define an ONR of an $\mathcal{L}_2$ -system Σ_- . Suppose D has full row rank, (C, A) is detectable and that

$$\begin{pmatrix} A - sI & B \\ C & D \end{pmatrix}$$

has full row rank for all $s \in \mathbb{C}_0 \cup \infty$.

1. Then Σ_- is SP if and only if

- (a) There is a nonsingular matrix W such that $DJD^* = WJW^*$ and
- (b) There exists Q such that

$$AQ + QA^* + BJB^* - [QC^* + BJD^*](DJD^*)^{-1}[CQ + DJB^*] = 0 \quad (18)$$

with $A - [QC^* + BJD^*](DJD^*)^{-1}C$ stable and

(c) $Q \geq 0$.

2. Furthermore, given a stabilizing solution Q to (18) and a nonsingular solution W to $WJW^* = DJD^*$, define $\{A_M, B_M, C_M, D_M\}$ as

$$\begin{pmatrix} A_M & B_M \\ C_M & D_M \end{pmatrix} = \begin{pmatrix} I & -[BJD^* + QC^*](DJD^*)^{-1} \\ 0 & W^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then $M(s) := C_M(sI - A_M)^{-1} B_M + D_M$ is a stable rational matrix that defines an AR representation of Σ_- , and M is co- J -lossless iff $Q \geq 0$. ■

4 Examples

Example 4.1 (Nevanlinna Pick Interpolation.) The aim of this example is to show a connection between SP $\mathcal{L}_2$ -systems and Nevanlinna Pick Interpolation problems (NPIP). We show that a certain NPIP has a solution iff a corresponding $\mathcal{L}_2$ -system is SP.

Consider the problem of finding $H \in \mathcal{H}_\infty$ such that $\|H\|_\infty < 1$ and such that a set of interpolation conditions is satisfied: $H(\zeta_i)a_i = b_i$ for $i \in \{1, \dots, n\}$. We assume that all ζ_i lie in $\mathbb{C}_+$ and that $\zeta_i \neq \zeta_j$ if $i \neq j$. The claim is that this NPIP has a solution iff the $\mathcal{L}_2$ -system $\Sigma_- = \{C_-, C^*, B_-\}$ with the n dimensional behavior

$$B_- = \begin{pmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{pmatrix} \left(sI - \begin{pmatrix} \zeta_1 & & \\ & \ddots & \\ & & \zeta_n \end{pmatrix} \right)^{-1} C^n \quad (19)$$

is SP with respect to partitioning $w = \begin{pmatrix} u \\ v \end{pmatrix}$, with $|u| = |a_1|$, and $|v| = |b_1|$.

Suppose NPIP has a stable contractive solution H . The interpolation conditions on H imply that $(H - I)w \in \mathcal{H}_2$, for every $w \in B_-$. In other words, B_- is a subset of the behavior of a system $\tilde{\Sigma}_-$ with behavior $B_{(H-I)}$. As H is stable and contractive, the system $\tilde{\Sigma}_-$ is SP. As a result the SP inequality also holds on the subset B_- . This proves that solvability of the NPIP implies SP of Σ_- .

Now suppose that Σ_- is SP. Then $B_- = B_M$ for some stable co- J -lossless M . Define $(H_1 - H_2) := (U - I)M$ where H_2 is square and H_1 and U have the same size as the solution H we are trying to find. Take U stable and contractive. Then $H_1 H_1^* - H_2 H_2^* \leq UU^* - I < 0$ in the closed right-half plane, by co- J -losslessness of M . It follows that H_2 is in $\mathcal{GH}_\infty$, and that $H = H_2^{-1} H_1$ is stable and contractive. For $w \in B_-$, $Mw \in \mathcal{H}_2$, so $(H - I)w = H_2^{-1}(U - I)(Mw) \in \mathcal{H}_2$ if $w \in B_-$ as H_2^{-1} and U are stable. This holds in particular for

$$w = \begin{pmatrix} a_j \\ b_j \end{pmatrix} \frac{1}{s - \zeta_j} \in B_-.$$

Thus $(H(s)a_j - b_j)/(s - \zeta_j)$ is in $\mathcal{H}_2$, which implies $H(s)a_j - b_j$ is zero at $s = \zeta_j$. In other words, H is stable contractive and satisfies the interpolation conditions as well. That is, H is a solution to the NPIP.

Note that there is a freedom in the construction of H . Every stable contractive U will give rise to a solution H to the NPIP. In fact, it may be shown that all solutions H to the NPIP are generated this way (see e.g. [1], [12]).

Using the state space results of the previous section, it follows (see (11)) that this NPIP is solvable iff the solution X to the Lyapunov equation

$$\begin{pmatrix} \zeta_1^* & & \\ & \ddots & \\ & & \zeta_n^* \end{pmatrix} X + X \begin{pmatrix} \zeta_1 & & \\ & \ddots & \\ & & \zeta_n \end{pmatrix} = \begin{pmatrix} a_1^* & b_1^* \\ \vdots & \vdots \\ a_n^* & b_n^* \end{pmatrix} J \begin{pmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{pmatrix}$$

is positive definite. This is a standard result and X is known as the Pick matrix. The co- J -lossless matrix M so that $B_- = B_M$ may be chosen to be

$$M(s) = \begin{pmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{pmatrix} X^{-1} \left(sI + \begin{pmatrix} \zeta_1^* & & \\ & \ddots & \\ & & \zeta_n^* \end{pmatrix} \right)^{-1} \begin{pmatrix} a_1^* & -b_1^* \\ \vdots & \vdots \\ a_n^* & -b_n^* \end{pmatrix} - I.$$

The general NPIP where interpolation with multiplicities are allowed may be handled using behaviors $C(sI - A)^{-1}C^n$, with A a matrix in Jordan form. ■

Figure 1: The $\mathcal{H}_\infty$ filtering configuration.

Example 4.2 ($\mathcal{H}_\infty$ filtering. Necessity results.) The $\mathcal{H}_\infty$ filtering problem is to find filters F such that the closed-loop system map is stable and contractive. (We estimate z with error $\|e\|_2 < \|d\|_2$.) We allow causal, homogeneous filters F . By homogeneous we mean that F maps the zero signal into the zero signal. The given system Σ is assumed to be of the usual type:

$$\Sigma: \begin{cases} \dot{x} = Ax + Bd \\ z = C_1x \\ y = C_2x + Dd. \end{cases} \quad (20)$$

We assume without loss of generality that $D \begin{pmatrix} B^* & D^* \end{pmatrix} = \begin{pmatrix} 0 & I \end{pmatrix}$ (see [5], [13]).

The idea is to show that there exists a subset of the closed-loop behavior that does not depend on the filter. If the $\mathcal{H}_\infty$ filtering problem has a solution, i.e., if there is a filter such that the closed-loop system is SP, then certainly the SP inequality must hold on this filter independent subset. This provides a necessary condition for the $\mathcal{H}_\infty$ filtering problem to have a solution.

The $\mathcal{L}_2$ -behavior that is important here is the set of external signals (d, e) in $\mathcal{L}_2$ that do not 'activate' the output $y(t)$ for $t < 0$, so that there is nothing to filter for $t < 0$. Then $u(t) = 0$, $t < 0$ and $e = z$, by causality and homogeneity of F . Consider therefore the behavior

$$\mathcal{B}_- = \{ (d, e) \mid \begin{array}{l} \dot{x} = Ax + Bd \\ 0 = C_1x \\ 0 = C_2x + Dd \end{array} - Ie, (d, e) \in \mathcal{L}_2 \}.$$

The behavior is of the ONR type. In order to be able to apply Corollary 3.10, we have to assume

1. $\left(\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, A \right)$ is detectable;
2. $\begin{pmatrix} A - sI & B & 0 \\ C_1 & 0 & -I \\ C_2 & D & 0 \end{pmatrix}$ has full row rank for all $s \in \mathbb{C}_0 \cup \infty$.

We assumed in the first place that $D(B^* \ D^*) = \begin{pmatrix} 0 & I \end{pmatrix}$ and this implies that the second assumption is equivalent to (A, B) not having uncontrollable modes on the imaginary axis.

It follows from Corollary 3.10 that under these assumptions the system with behavior $\mathcal{B}_-$ is SP iff

$$AQ + QA^* + Q(C_1^*C_1 - C_2^*C_2)Q + BB^* = 0 \quad (21)$$

has a stabilizing nonnegative definite solution Q .

Conversely, given a stabilizing nonnegative definite solution Q to (21) a filter that solves the problem can be constructed (see [5], [13]):

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + QC_2^*(y - C_2\hat{x}) \\ u &= C_1\hat{x}. \end{aligned}$$

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