On the use of noncausal LTI operators in Iterative Learning Control

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Abstract

This paper demonstrates the use of noncausal operators in Iterative Learning Control. First, it is shown that for a particular class of plants (having unstable zeros), perfect tracking can only be achieved by using noncausal operators. Then it is shown that with any converging algorithm (both causal and noncausal) we can associate a particular feedback controller. For causal algorithms this controller can be shown to be internally stabilizing. In the noncausal case however, the associated controller is found to be generally destabilizing which implies that the existing notion of an *equivalent* controller for causal ILC does not extend to *noncausal* ILC.

1 Introduction

The key observation that started the field of Iterative Learning Control (ILC) [9, 10] was the fact that, under certain conditions, a plant operating on the same task time and again continually reproduces the same error. This observation stimulated the intuitive idea that it should somehow be possible to exploit experience from the past in order to improve the future performance. The number of publications that have been published since by itself provides evidence that this idea was correct and fruitful. The flip side is that many were led to believe that as a method of control, ILC was fundamentally different from any other method.

In a recent paper however [5], it was shown that at least causal ILC can do no better than conventional feedback control and that both methods are in fact equivalent. That is, the asymptotic performance that is obtained through ILC can always be obtained directly through feedback. To put things in a right perspective, it should be noted that when applying (high-gain) feedback the performance will become sensitive to noise that might be injected through sensor data. In ILC, chances of performance degradation are expected to be less, because the method is essentially open loop and the effects of noise are likely to be averaged out. The result never-

theless implies that the information feedback in causal ILC is essentially the same as in conventional feedback control and hence subject to the same limitations. It also shows the importance of relating different methods to evaluate relative merits.

A characteristic feature that may still distinguish ILC from feedback control is the possibility to exploit the information preview, i.e. to use information from the 'future' of previous trials to determine the control action in the current trial. In this paper we mean to investigate ways to exploit the information preview and to determine its limitations in terms of performance. We will focus on noncausal LTI operators; the analysis is done in an infinite time setting.

This paper is organized as follows. Section 2 starts off with some notation and provides the necessary background material on noncausal operators. This is followed by a formal problem definition and a general condition for convergence in Section 3. Section 4 discusses an example that demonstrates the potential of using noncausal operators. Section 5 shows that an equivalent stabilizing feedback controller generally does not exist. Section 6 concludes this paper.

2 Notation & Preliminaries

Let $\mathcal{U}$ denote the space of inputs on which a recurrence relation is to be defined. Throughout this paper we will assume that $\mathcal{U}$ equals $\mathcal{L}_{2+}$ or the isomorphic space $\mathcal{H}_2$. The space $\mathcal{L}_{2+}$ is interpreted as a subspace of $\mathcal{L}_2(-\infty, \infty)$ with functions that are zero for negative time. This allows us to treat all signals as if they were defined on the whole of the real axis. Signals are denoted lower case (u, e, y) and will generally depend on the iteration number ' k ' and the time ' t ' or the Laplace variable ' s ' (e.g. $u = u_k(t), y = y_k(s)$). The upper case is used to denote operators. Let T be an operator, then T^* and $\bar{\sigma}(T)$ denote respectively the adjoint operator of T (or the complex conjugate transpose of T) and the largest singular value of T .

Recall that every LTI operator H , either causal or non-causal has a convolution representation [7]

$$\begin{aligned} (Hw)(t) &= (h * w)(t) \\ &= \int_{-\infty}^{\infty} h(t - \tau)w(\tau)d\tau \end{aligned} \quad (1)$$

in which $h(t)$ denotes the impulse response. H is *causal* if its impulse response is zero for negative time and it is *noncausal* if it is not causal. Furthermore H is (BIBO) *stable* if every bounded input generates a bounded output. A necessary and sufficient condition for an LTI system H to map $\mathcal{L}_2$ into $\mathcal{L}_2$ is that $h \in \mathcal{L}_1$ [3].

For every $g \in L_1(-\infty, \infty)$ we define the 2-sided Laplace transform $G(s)$ as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}dt \quad (2)$$

The two-sided Laplace transform, applied to the impulse response of a stable finite dimensional LTI operator induces two spaces of interest. The first space, corresponding to the class of causal operators is $\mathcal{RH}_{\infty}$ [13]. It consists of all proper and real rational transfer matrices with no poles in the closed right half plane. The second space consists of all proper and real rational transfer matrices with no poles on the imaginary axis. This space is generally denoted as $\mathcal{RL}_{\infty}$ and corresponds to the class of all (causal and noncausal) stable finite dimensional LTI operators. We want to stress that transfer matrices with poles in the open right-half plane are also stable systems, but then not causal. For example $G(s) = 1/(1-s)$ is the two-sided Laplace transform of

$$g(t) = \begin{cases} e^t & \text{if } t < 0 \\ 0 & \text{elsewhere} \end{cases}$$

It defines a stable system $y = g * u$ but one that is non-causal. In general $G(s) \in \mathcal{RL}_{\infty}$ may have poles both in the open left-half and open right-half plane. The poles in the open right-half plane correspond to the anticausal part of the impulse response that typically consists of exponentials that decrease with negative time. Moreover, every element $G(s)$ in $\mathcal{RL}_{\infty}$ can be decomposed into a stable part $G_s(s)$ and an antistable (but not unstable) part $G_{as}(s)$ such that $G(s) = G_s(s) + G_{as}(s)$ with $G_s(s), G_{as}(-s) \in \mathcal{RH}_{\infty}$. The antistable part $G_{as}(-s)$ can be implemented as a causal operation on a time-reversed signal. The location of the poles hence only determines the causality of the operation.

For every element $G(s)$ of either $\mathcal{RH}_{\infty}$ or $\mathcal{RL}_{\infty}$, the $\mathcal{L}_2(-\infty, \infty)$ -induced operator norm equals

$$\|G\|_{\infty} = \sup_{\omega \in \mathbb{R}} \bar{\sigma}(G(j\omega)) \quad (3)$$

To summarize the important points of this section: we have introduced the notions of causality and stability

for a class of LTI operators in terms of its impulse response. It turned out that every stable LTI operator can be uniquely identified with an element of the space $\mathcal{RL}_{\infty}$. We have thus established a mathematical framework for the analysis of noncausal Iterative Learning Control.

3 Iterative Learning Control

The problem of Iterative Learning Control can be formulated as follows. Given a plant $H: \mathcal{U} \rightarrow \mathcal{Y}$, $y = Hu$, along with a desired output or behaviour y_d . Find a recurrence relation (or iteration) on $\mathcal{U}$ that defines a convergent sequence $\{u_k\}_{k \in \mathbb{N}}$ such that the corresponding output sequence $\{y_k\}$ converges to an output that is close to y_d in some sense.

More precisely posed, we will consider a family of iterations $T(Q, L): \mathcal{H}_2 \rightarrow \mathcal{H}_2$,

$$\begin{aligned} u_{k+1} &= Tu_k \\ &= Qu_k + L(y_d - Hu_k) \end{aligned} \quad (4)$$

with Q and L both LTI but not necessarily causal. We assume $H \in \mathcal{RH}_{\infty}$ and look for those pairs (Q, L) that define a convergent sequence of inputs on $\mathcal{U}$. Those pairs will be called admissible.

Definition 1 (Admissible pair) Consider the family of iterations $T(Q, L)$ defined by (4). We say that the pair $(Q, L) \in \mathcal{RL}_{\infty} \times \mathcal{RL}_{\infty}$ is admissible if for every $y_d \in \mathcal{H}_2$, $\exists \bar{u}(y_d) \in \mathcal{H}_2$ s.t. $\lim_{k \rightarrow \infty} u_k = \bar{u}(y_d)$ for every $u_0 \in \mathcal{H}_2$.

The set of all admissible pairs will be denoted by $\mathcal{A}$. In the following theorem we will restate some well-known convergence results [9, 11] that will tell us whether or not a given pair (Q, L) belongs to $\mathcal{A}$.

Theorem 2 Given $H \in \mathcal{RH}_{\infty}$ and $Q, L \in \mathcal{RL}_{\infty}$. The following propositions hold true.

1. $\|Q - LH\|_{\infty} < 1 \Rightarrow (Q, L) \in \mathcal{A}$
2. $(Q, L) \in \mathcal{A} \Rightarrow \|Q - LH\|_{\infty} \leq 1$
3. $\|y_d - H\bar{u}\| = 0 \forall y_d \in \mathcal{H}_2 \Leftrightarrow \begin{cases} Q = I \\ (I, L) \in \mathcal{A} \end{cases}$

Proof: (Part 1.) Take any $u, v \in \mathcal{U}$. Then

$$\begin{aligned} \|Tu - Tv\| &= \|(Q - LH)(u - v)\| \\ &\leq \|Q - LH\|_{\infty} \|u - v\| \end{aligned}$$

By assumption $\|Q - LH\|_{\infty} < 1$. Now the fixed point theorem [6] says

1. The equation $u = Tu$ has a unique solution $\bar{u} \in \mathcal{U}$ and moreover
2. The sequence $\{u_k\}_{k \in \mathbb{N}}$ defined by

$$u_{k+1} = Tu_k$$

converges to $\bar{u}$ for every initial $u_0 \in \mathcal{U}$.

Hence, (Q, L) is admissible by definition.

(Part 2.) Suppose $\|Q - LH\|_\infty > 1$. Then by continuity there exists a nonempty interval $\mathcal{I} \subset j\mathbb{R}$ on which $\bar{\sigma}(Q - LH) > 1$ and we can find $u_0 \in \mathcal{U}$ such that the sequence $\{u_k\}$ becomes unbounded. Hence, (Q, L) is not admissible.

(Part 3.) To prove sufficiency, suppose $Q = I$ and $(I, L) \in \mathcal{A}$. By definition of admissibility, the sequence $\{u_k\}$ converges to the unique solution of the equation

$$\begin{aligned} \bar{u} &= T(I, L)\bar{u} \\ &= (I - LH)\bar{u} + Ly_d \end{aligned} \quad (5)$$

from which we conclude that $L(y_d - H\bar{u}) = 0$. Since this holds for every y_d , we conclude that $y_d - H\bar{u} = 0$.

To prove necessity, we need to show that if $(Q, L) \in \mathcal{A}$ (which is clearly a necessary condition) and $\|y_d - H\bar{u}\| = 0$ for all $y_d \in \mathcal{H}_2$ then necessarily $Q = I$. The proof reads as follows. By definition of admissibility the equation

$$(I - Q)\bar{u} = L(y_d - H\bar{u}) \quad (6)$$

has a unique solution in $\mathcal{U}$ for every $y_d \in \mathcal{H}_2$. By assumption, the right hand side of (6) vanishes. But then also the left hand side has to vanish, i.e. $\|(I - Q)\bar{u}(y_d)\| = 0$ for every y_d . Hence $Q = I$. This completes our proof. ■

Remark 3 Note that the first and second proposition in theorem 2. give us a sufficient and necessary condition for admissibility respectively. It turns out to be hard to find a condition that is both sufficient and necessary. The difficulty lies in the fact that of all pairs (Q, L) that satisfy the condition $\|Q - LP\|_\infty = 1$ some are admissible and some are not. Fortunately we don't need a complete characterization of $\mathcal{A}$ to state our main result.

4 Causal vs. Noncausal

In this section we will show that for a particular class of plants, perfect tracking (zero error for all trajectories) can only be achieved using noncausal operators.

Definition 4 H is called a unit of $\mathcal{RH}_\infty(\mathcal{RL}_\infty)$ if $H, H^{-1} \in \mathcal{RH}_\infty(\mathcal{RL}_\infty)$.

By means of example consider the non-minimum phase plant $H = \frac{s-1}{s+1}$ which is a unit of $\mathcal{RL}_\infty$ but not of $\mathcal{RH}_\infty$.

The following theorem restates a well-known result that says that a if a plant has an unstable zero (non-minimum phase), perfect tracking can not be achieved when restricting to causal operators. See also [1, 12]

Theorem 5 Let H be a unit of $\mathcal{RL}_\infty$. Then perfect tracking can be achieved using just causal operators if and only if H is also a unit of $\mathcal{RH}_\infty$.

Proof: Convergence to zero error can be achieved iff $Q = I$ and $(I, L) \in \mathcal{A}$ (see theorem 2). First we prove sufficiency. Suppose $Q = I$. A sufficient condition for (I, L) to be admissible is given by $\|I - LH\|_\infty < 1$. Now choose $V \in \mathcal{RH}_\infty$ with $\|V\|_\infty < 1$ and set $L = (I - V)H^{-1}$. Clearly, $L \in \mathcal{RH}_\infty$ and by construction $\|I - LH\|_\infty < 1 \Rightarrow (I, L) \in \mathcal{A}$.

To prove necessity, suppose H is not a unit of $\mathcal{RH}_\infty$ and $(I, L) \in \mathcal{A}$. The conditions on H imply that H^{-1} has a pole in the open RHP or equivalently: (see lemmas 3.38 and 3.27 in [13]) H loses rank for some $s_0 \in \mathbb{C}_+$. By assumption $L \in \mathcal{RH}_\infty \Rightarrow s_0$ is also transmission zero of $LH \Rightarrow \bar{\sigma}(I - LH)(s_0) = 1$. Admissibility of (I, L) implies that $\|I - LH\|_\infty \leq 1$ and by the Maximum Modulus Theorem (or rather its generalization in terms of subharmonic functions, see [2]) we know that in fact $\bar{\sigma}(I - LH)(j\omega) = 1$ for all $\omega \in \mathbb{R} \cup \infty$. Now it is easy to see that we can always find u_0, y_d such that $\{u_k\}$ becomes unbounded. This contradicts the assumption of admissibility and hence we conclude that perfect tracking cannot be obtained using just causal operators. ■

Remark 6 As suggested by the proof of theorem 5, perfect tracking can always be obtained using noncausal operators in case H is a unit of $\mathcal{RL}_\infty$. To see this, let Q be the identity and choose $V \in \mathcal{RL}_\infty$ such that $\|V\|_\infty < 1$. Take L according to $L = (I - V)H^{-1}$. Clearly, $L \in \mathcal{RL}_\infty$ and moreover $(I, L) \in \mathcal{A}$. Then perfect tracking is guaranteed by theorem 2.

With theorem 5 and remark 6 we have established a result that says that at least for a particular class of plants it proves to be useful to process the data obtained from previous past trials in a noncausal way. In closing this section, we state a theorem (without proof, see [9]) that gives a bound on the achievable performance when using noncausal operators. The formulation is such that it coincides with its causal counterpart [8],

but note that the optimization problem as it stands is generally ill-posed if we do not put a bound on L .

Theorem 7 Consider the iteration defined by (4) and let $Q, L \in \mathcal{RL}_\infty$. Then the achievable performance (in terms of the associated error) is bounded by

$$\|e\| \geq \|(I - L_{\text{opt}}H)y_d\|$$

where L_{opt} is a solution of

$$L_{\text{opt}} = \arg \min_{L \in \mathcal{RL}_\infty} \|(I - HL)y_d\|$$

5 Equivalent Feedback

In a recent paper [4], the concept of an equivalent controller for causal ILC was introduced. In this paper we extend this idea to include noncausal operators. We introduce an equivalence relation on $\mathcal{A}$ and show that the induced equivalence classes can be associated with a feedback controller. When restricting to causal operators, this controller has the property of being internally stabilizing. Without causality constraints, this controller is shown to be generally destabilizing.

We will need a notion of equivalence on the set of admissible pairs.

Definition 8 (Equivalence on $\mathcal{A}$) Two elements $(Q_1, L_1), (Q_2, L_2) \in \mathcal{A}$ are said to be equivalent if

$$(I - Q_1)^{-1}L_1 = (I - Q_2)^{-1}L_2 \quad (7)$$

It is easy to verify that (7) defines an equivalence relation in the sense that it is reflexive, symmetric and transitive. However, one may wonder whether this relation is well-defined since for some $(Q, \cdot) \in \mathcal{A}$, $(I - Q)^{-1}$ might not exist. The following lemma shows that the equivalence relation is indeed well-defined for all elements of $\mathcal{A}$, provided H is strictly proper.

Lemma 9 Suppose H is strictly proper. Then, for all $(Q, L) \in \mathcal{A}$, $(I - Q)^{-1}L$ defines a proper transfer function.

Proof: By definition of admissability, the equation $(I - Q + LH)\bar{u} = Ly_d$ has a unique (and bounded) solution for every $y_d \in \mathcal{H}_2$. This implies that $(I - Q + LH)^{-1}L \in \mathcal{RL}_\infty$. Now suppose H is strictly proper, i.e. all its elements are strictly proper. Properness of L then implies $L(\infty)H(\infty) = 0$. Hence $(I - Q)^{-1}L$ is proper if and only if $(I - Q + LH)^{-1}L$ is proper. The latter is proper by assumption of admissability as we showed before. Hence $(I - Q)^{-1}L$ defines a proper transfer function. ■

Remark 10 Note that $(I - Q)^{-1}L$ need not belong to $\mathcal{RL}_\infty$. It can have poles on the imaginary axis.

Now that we have established a well-defined equivalence relation on $\mathcal{A}$ let us try to interpret the result. What does it mean to say that two admissible pairs are equivalent? The answer is simple. Equivalence in the sense of definition 8 implies that two (different) sequences converge to the same fixed point. To see this, note that for all admissible pairs the iteration $u_{k+1} = T(Q, L)u_k$ defined by (4) converges to a fixed point given by

$$\begin{aligned} \bar{u} &= (I - Q + LH)^{-1}Ly_d \\ &= \left(I + (I - Q)^{-1}LH\right)^{-1}(I - Q)^{-1}Ly_d \end{aligned}$$

where it is clear that the fixed point is completely determined by $(I - Q)^{-1}L$.

Another way to see this is to observe that $\bar{u}$ also satisfies

$$\begin{aligned} \bar{u} &= (I - Q)^{-1}L(y_d - H\bar{u}) \\ &= (I - Q)^{-1}L\bar{e} \end{aligned} \quad (8)$$

where $\bar{e}$ denotes the ultimate error. Now $(I - Q)^{-1}L$ has the interpretation of a feedback controller.

In [4, 5] it is shown that for every $(Q, L) \in \mathcal{A}$ with the restriction that both Q and L are causal, the feedback compensator defined by $K = (I - Q)^{-1}L$ is internally stabilizing. In the following theorem we provide an alternative proof for this statement and after that we show that the converse also holds true.

Theorem 11 Suppose $(Q, L) \in \mathcal{A}$ with $Q, L \in \mathcal{RH}_\infty$. Then the equivalent controller $K = (I - Q)^{-1}L$ is well-defined (proper) and yields the closed loop internally stable.

Proof: The controller is well-defined by lemma 9. Internal stability is guaranteed iff $K(I + HK)^{-1} \in \mathcal{RH}_\infty$ [13]. By definition of K this condition is equivalent to $(I - Q + LH)^{-1}L \in \mathcal{RH}_\infty$. By assumption of admissability, $(I - Q + LP)^{-1}Ly_d$ is bounded for every $y_d \in \mathcal{H}_2$ and with Q, L causal this can only hold true if $(I - Q + LH)^{-1}L \in \mathcal{RH}_\infty$. ■

Theorem 12 Suppose the feedback interconnection $(H - K)$ is internally stable. Then there exists an admissible pair (Q, L) satisfying $(I - Q)^{-1}L = K$.

Proof: Take $Q = K(I + HK)^{-1}H$ and $L = K(I + HK)^{-1}$ and note that $(Q, L) \in \mathcal{RH}_\infty \times \mathcal{RH}_\infty$ by assumption of internal stability. Furthermore it is easy to verify that $Q - LH = 0$, which is a sufficient condition for admissability (see theorem 2). ■

Remark 13 *Theorem 12 says that to every stabilizing controller K there corresponds at least one causal admissible pair (L_0H, L_0) , $L_0 \in \mathcal{RH}_\infty$. In actual fact this correspondence is one-to-many since by definition every (causal) pair $(Q, L) \in \mathcal{A}$, equivalent to (L_0H, L_0) , also yields the same stabilizing controller.*

Let us summarize the important points so far. We noted that of all sequences induced by admissible pairs, many will converge to the same fixed point. By introducing an equivalence relation on $\mathcal{A}$ we can group together sequences into equivalence classes which form a partition of $\mathcal{A}$. In case Q, L are both causal, we can associate an equivalence class with a stabilizing feedback controller (theorem 11) and vice versa by theorem 12 and remark 13.

In the remaining part of this section we will answer the question as to what extent these results generalize in case we consider noncausal admissible pairs. To this end, suppose that for a particular noncausal admissible pair (Q_{nc}, L_{nc}) the associated feedback controller $K = (I - Q_{nc})^{-1} L_{nc}$ turns out to be stabilizing. Then theorem 12 says that there also exists a *causal* admissible pair (in fact a whole lot by remark 13) inducing a sequence that will converge to the same fixed point as the one induced by the noncausal pair. In other words, those noncausal pairs are of no interest. If we therefore discard (or ignore) those pairs, it is clear that every element in the remaining set of noncausal admissible pairs corresponds to a feedback controller that is stabilizing. It is easy to see that this set is nonempty. In fact, we can easily parameterize it as

$$\{(Q, L) \in \mathcal{A} | (Q, L) \simeq (LH, L), L \notin \mathcal{RH}_\infty\}$$

6 Conclusions

In a CDC paper [4], Goldsmith included a closing statement saying that: "... a noncausal approach is the only viable route for ILC."

The results in this paper can be considered as a first step on this route. We showed that at least for a particular class of plants (having unstable zeros), noncausal ILC clearly outperforms causal ILC. But even more important perhaps was the fact that the 'equivalent' controller associated with a class of noncausal algorithms was found to be generally destabilizing. The implication of the latter result is that ILC can be very effective in cases where the causality constraint imposed by the closed loop is the limiting factor in the performance of conventional feedback methods, i.e. in the context of non-minimum phase systems.

Concerning implementability we remark that in any practical situation the trial length is finite. This im-

plies that the information preview is also finite and is in fact getting smaller towards the end of the trial. Hence we are facing a difficulty since we lack the necessary information to implement the algorithm. To overcome this difficulty we could simply assume that all signals have returned to rest by the time we reach the end of the trial. This is indeed a reasonable assumption provided that the trial length is long enough and includes a period of relaxation.

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