Compliant Manipulators on Graphs

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Abstract—This paper proposes a modeling method for generic serial-chain compliant robotic manipulators. It is based on graph theory and port-Hamiltonian systems, which allows a modular approach to the interconnection of rigid bodies with compliant actuators by means of kinematic pairs. This modularity allows a very simple and straightforward change in a manipulator’s actuator morphology. An example of a two degree of freedom planar manipulator shows that this modeling method is more suitable for modeling changes in actuator placement than traditional Euler-Lagrange models.

I. INTRODUCTION

Recent developments in the field of compliant robotics and the ever increasing desire for human-robot interaction [1] have caused a paradigm shift from robots that are designed as stiff as possible, to intrinsically compliant robots. Intrinsic compliance increases the level of safety during human-robot interaction and may provide energy efficiency for repetitive or high velocity (explosive) tasks. Compliance may be introduced by, for instance, flexible links [2] or compliant joints like series elastic actuators (SEA) and variable stiffness actuators (VSA) [3], [4]. Current models of compliant manipulators are based on traditional modeling techniques [2], [5], [6]. These models, which have their roots at the Euler-Lagrange equations, are widely used because of their suitability in control design. However, these models are not flexible with respect to system changes, since a change in, e.g., the kinematic structure or actuator placement basically requires reanalyzing the complete system.

This paper proposes a modeling method for generic serial-chain compliant robotic manipulators. The model is based on graph theory which allows to describe low as well as high(er) dimensional (and more complex) systems in an identical way, since a graph captures the topology of a system and embodies physical interconnection laws. This makes the model inherently modular. Due to the modularity and explicit modeling of the topology of a system, the articulation type of an actuator can be easily defined on the graph model, meaning that the actuator placement can be easily synthesized. A graph theoretic description of a VSA is given in [7], where the application of the laws imposed by graph theory leads to a one-dimensional Euler-Lagrange model. A one-dimensional model can be properly described using that approach, as was also shown in [8], but when the model is extended to higher dimensions, the complexity increases since rotations and translations are to be taken into account by separate graphs [9], [10], [11], [12], [13], [14]. On the contrary, when a unified approach is taken by using the port-Hamiltonian modeling framework, the use of only one graph that captures generalized rigid body movements suffices. A rigid body’s movement is then described by a single screw motion, i.e., a twist, represented by a six-dimensional configuration variable [15]. Moreover, the port-based modeling naturally allows the interconnection of several systems through power conjugated effort and flow variables associated to the power ports. Previous work [16], [17] has treated rigid manipulators and systems with only passive compliant elements. In this paper, our previous work in [7] is extended to the port-Hamiltonian modeling framework. The one-dimensional model is extended to a six-dimensional model that captures general rigid body rotations and translations. It focuses on the inclusion of a generic compliant actuator in a serial chain manipulator. This generic compliant actuator is not fixed a priori to a certain joint or configuration, but is a modular entity that can be connected as desired by changing the topology of the system. This topology is given by a mapping that the graph defines.

Section II first elaborates on the basics of port-Hamiltonian systems and graph theory. The model is explained in Section III and its strength is made explicit by comparing it to a traditional Euler-Lagrange model in Section IV. The model and its results are discussed in Section V and the paper concludes with Section VI.

II. PORT-HAMILTONIAN SYSTEMS AND GRAPH THEORY

This section summarizes the necessary and existing concepts from port-Hamiltonian systems and graph theory.

A. Port-Hamiltonian Systems

A Hamiltonian system [15], [18] is characterized by an energy function $H(q,p)$, called the Hamiltonian, which is a function of the energy variables, i.e., $i$ generalized configuration coordinates $q = [q_1, \ldots, q_i]^T$ and $j$ generalized momentum coordinates $p = [p_1, \ldots, p_j]^T$. The time derivatives of the energy variables $(\dot{q}, \dot{p})$ are called flows, and the partial derivatives of the energy function with respect to the energy variables $(\frac{\partial H(q,p)}{\partial q}, \frac{\partial H(q,p)}{\partial p})$ are called efforts. This can be given by an input-state-output port-Hamiltonian system:

$$\dot{x} = [J(x) - R(x)] \frac{\partial H(x)}{\partial x} + g(x)u + h(x)d$$

$$y = g^T(x) \frac{\partial H(x)}{\partial x}$$

$$z = h^T(x) \frac{\partial H(x)}{\partial x}$$

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where \( x = [x_1, \ldots, x_k]^T = [q_1, \ldots, q_i, p_1, \ldots, p_j] \) is the \( k \)-dimensional vector of elements of the state manifold \( \mathcal{X} \), \( J(x) \) is the \( k \times k \) skew-symmetric structure matrix, i.e., \( J(x) = -J^T(x) \), defining the interconnection between efforts and flows, \( R(x) \geq 0 \) is the \( k \times k \) symmetric dissipative structure matrix, \( q(x) \) is the \( k \times r \) input matrix, \( u \in \mathbb{R}^r \) is the vector of the generalized control inputs, \( h(x) \) is the \( k \times w \) interaction matrix, \( d \in \mathbb{R}^w \) is the vector of generalized interaction inputs, \( y \in \mathbb{R}^r \) is the generalized control output vector collocated with control input \( u \) and \( z \in \mathbb{R}^w \) is the generalized interaction output vector collocated with interaction input \( d \). Note that \( y^T u \) is the power flow through the control port and \( z^T d \) is the power flow through the interaction port.

### B. Graph Theory

A directed graph \( \mathcal{G} \) is defined by a set of \( n \) vertices \( V_{\mathcal{G}} = \{v_1, v_2, \ldots, v_n\} \) and a set of \( m \) directed edges \( E_{\mathcal{G}} = \{e_1, e_2, \ldots, e_m\} \subseteq V_{\mathcal{G}} \times V_{\mathcal{G}} \), i.e., a set of ordered pairs of vertices defined by the Cartesian product \( \times \), which defines the connections between the vertices. An undirected graph consists of edges that are unordered pairs of vertices.

1) Incidence matrix: The incidence matrix \( B_{\mathcal{G}} \in \mathbb{R}^{n \times m} \) defines whether edges are incident to vertices. For a directed graph, element \( (j,l) \) is given by:

\[
(B_{\mathcal{G}})_{jl} := \begin{cases} 
1 & \text{if edge } l \text{ enters vertex } v_j, \\
-1 & \text{if edge } l \text{ leaves vertex } v_j, \\
0 & \text{otherwise.}
\end{cases}
\]

For an undirected graph, an arbitrary direction can be assigned to find the incidence matrix. Note that the edge weights do not appear in the incidence matrix, since it only captures the direction of the edges.

2) Boundary and internal vertices: It is possible to make a distinction between boundary and internal vertices [14]. Define the set \( N_{\mathcal{G}}(v_i) \) of all neighboring vertices of vertex \( v_i \), i.e., vertices that are coupled to \( v_i \) by an edge. The distance \( d(v_1, v_2) \) between two vertices \( v_1, v_2 \in V_{\mathcal{G}} \) of graph \( \mathcal{G} \) is the length of a shortest path between vertex \( v_1 \) and \( v_2 \). A vertex \( v_1 \) is a boundary vertex of vertex \( v_2 \) if no neighboring vertex \( v_3 \in N_{\mathcal{G}}(v_1) \) of \( v_1 \) is further from \( v_2 \) than \( v_1 \). A vertex \( v_1 \) is a boundary vertex of graph \( \mathcal{G} \), if \( v_1 \) is a boundary vertex of some vertex \( v \in V_{\mathcal{G}} \). Internal vertices are the vertices that are not boundary vertices.

The boundary vertices are useful to describe systems that are open to interconnection to other systems.

### III. Compliant Manipulators on Graphs

Compliant manipulators are defined and characterized here by an arbitrary serial connection of coupled rigid bodies with power supplying compliant actuators.

#### A. Port-Hamiltonian Variable Stiffness Actuator on Graphs

A VSA is a generic compliant actuator capable of delivering power to an attached load while allowing potential energy storage through an adjustable storage element [19]. A schematic representation of a generic VSA is shown in Figure 1, where a rotor, a stator, an adjustable elastic element, and an output can be distinguished. Interaction with a VSA should not be possible through interaction with only the output like in [19], but also through interaction with the stator. When considering serial chain manipulators, the VSA interacts with bodies using both its stator (connected to a preceding body) and its output (connected to a subsequent body). Thus, interaction is associated to a two-dimensional port with conjugated effort and flow variables \( (e_{int}, f_{int}) \).

The internal rotor with mass \( m \) stores kinetic energy, which is a function of the rotor’s momentum \( p \). Moreover, it is assumed that the coefficient \( k \) of the elastic element can be directly adjusted by control \( u_k \), which means that the change of stored potential energy in the elastic element is a function of both its elongation \( s \) and its elastic coefficient \( k \) (the VSA’s output stiffness). This is more generic than [19], in which the VSA’s output stiffness is adjusted through specific internal degrees of freedom.

The state of a generic VSA is given by the individual storage states \( x := [s \ k \ p]^T \), and the quadratic Hamiltonian of a VSA becomes \( H(x) := H(s,k,p) = \frac{1}{2} k s^2 + \frac{1}{2} \frac{p^2}{m} \). Hence, the storage port is represented by a three-dimensional port \( \left( \frac{\partial H(x)}{\partial x}, x \right) \). The control port through which the VSA can be controlled, i.e., the force between the stator and rotor and the adjustment of the elastic coefficient, is represented by a two-dimensional port \( (y, u) \).

In line with our work in [7], the proposed undirected graph model of the VSA is shown in Figure 2a, where three vertices and two edges can be distinguished. The middle vertex \( v_1 \) is the internal (rotor mass) vertex of the graph, shown with a solid circle, and the other two vertices \( v_2 \) and \( v_3 \) are the massless boundary vertices, shown with open circles, representing the stator and output nodes to interact with the model. Figure 2b and Figure 2c show the directed input subgraph \( \mathcal{G}_u \) and stiffness subgraph \( \mathcal{G}_k \), respectively.

The incidence matrices of these subgraphs can be found and, following the reasoning in Section II, they can be split in an internal and boundary part. For subgraph \( \mathcal{G}_u \), the incidence matrix is given by:

\[
B_{\mathcal{G}_u} := \begin{bmatrix} B_{\mathcal{G}_u^{wi}} \\ B_{\mathcal{G}_u^{ub}} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},
\]

and the incidence matrix for the subgraph \( \mathcal{G}_k \) is given by:

\[
B_{\mathcal{G}_k} := \begin{bmatrix} B_{\mathcal{G}_k^{ki}} \\ B_{\mathcal{G}_k^{kb}} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.
\]

Fig. 1: Schematic representation of a VSA, showing a rotor (1), a stator (2), an adjustable elastic element and an output.
The input-state-output port-Hamiltonian model of a VSA can now be written as:

\[
\begin{bmatrix}
\dot{\mathbf{s}} \\
\dot{\mathbf{k}} \\
\dot{\mathbf{p}}_{\text{int,}2} \\
\dot{\mathbf{p}}_{\text{int,}3}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & \left[-B\Phi_k^T\right] \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
B\Phi_k & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\partial H(x) \\
\partial H(x) \\
\partial \mathbf{p} \\
\partial e_{\text{int,}2} \\
e_{\text{int,}3}
\end{bmatrix},
\]

\[
\begin{bmatrix}
\mathbf{y}_p \\
\mathbf{y}_k
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & \left[B\Phi_u^T\right] \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\partial H(x) \\
\partial \mathbf{p} \\
e_{\text{int,}2} \\
e_{\text{int,}3}
\end{bmatrix}.
\]

The two-dimensional interaction port with variables \((e_{\text{int,}2}, f_{\text{int,}2})\) and \((e_{\text{int,}3}, f_{\text{int,}3})\), corresponding to vertices \(V_2\) and \(V_3\), respectively, can be used to interact with the environment, where the flows \(f_{\text{int}}\) represent the force on the boundary vertices and the efforts \(e_{\text{int}}\) represent the velocities of the boundary vertices. Control port variables \((y_p, u_p)\) and \((y_k, u_k)\) are the conjugated power variables to the internal mass (with momentum \(p\)) and to the elastic element with elastic coefficient \(k\), respectively.

This can be visualized as depicted in Figure 3, which shows the control port, interaction port and storage port being connected by a Dirac structure \(\mathcal{D}\). This Dirac structure defines the power distribution between the connected ports, and cannot generate or dissipate power itself.

**B. Kinematic Pair**

A one-port kinematic pair defines a relative direction in which two bodies can move, while constraining the other directions to zero velocity. A two-port kinematic pair includes an additional power port through which power can be supplied by, for instance, motors. The motor force is then mapped to a wrench that is applied to both connected bodies. As elaborated in Section III-A, a compliant actuator has a two-dimensional interaction port. This means that it is necessary to split such a two-port kinematic pair in two distinct sides, i.e., sides \(a\) (associated to the stator) and \(b\) (associated to the output), as shown in Figure 4. Two body fixed coordinate frames \(\Psi_a\) and \(\Psi_b\) are shown, that connect to the kinematic pair to either the stator (side \(a\)) or the output (side \(b\)) of the VSA. In this kinematic pair definition, a Cartesian space is assumed, meaning that an intrinsic complement is defined (through orthogonality of the coordinate frame), allowing for the direct definition of the complement of the free or constraint twists or wrenches.

In this specific elaborative example, a pure moment around the \(x\)-axis of the local frame \(\Psi_a\) is assumed. In this way, the free wrenches of both sides are given by:

\[
\begin{bmatrix}
1W_{a,1}^{a,1} \\
1W_{a,2}^{b,2}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}^T.
\]

where \(nW_{a,j}^{b,j}\) indicates the \(n\)-th free wrench on body fixed frame \(\Psi_j\) expressed with coordinates in frame \(\Psi_i\). These free wrenches determine the direction in which an actuator can deliver power through the forces \(f_b \in \mathbb{R}^3\), i.e., the forces of the boundary vertices. Since a Cartesian space is assumed, the complement, i.e., the constraint wrenches \(nW_{C}^{n}\), can be directly defined as:

\[
\begin{bmatrix}
1W_{c,1}^{c,1} \\
\ldots \\
1W_{c,6}^{c,6}
\end{bmatrix} := W_{c,1}^{c,1} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\(= W_{c,b}^{b,b}.
\)
These wrenches are the directions in which the bodies should be prevented to move, which is taken care of by the joint constraint forces $\lambda \in \mathbb{R}^{5-f}$, which are Lagrange multipliers that keep the velocity in the constraint directions to zero. The kinematic pair Jacobians are then:

$$J_{kp}^a := \begin{bmatrix} W_{a}^{a,a} & W_{c}^{a,a} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$J_{kp}^b := \begin{bmatrix} W_{a}^{b,b} & W_{c}^{b,b} \end{bmatrix} = J_{kp}^a T.$$  

Note that the last equalities of (2) and (3) are due to, and only valid in case of, the alignment of frames $\Psi_a$ and $\Psi_b$ in this particular case, i.e., the local $x$-axes are coincident. Although this formulation is valid, it is easier to express the wrenches in one of the two frames $\Psi_a$ or $\Psi_b$. It then holds that $J_{kp}^a := J_{kp}^b := J_{kp}^T$. Then the total wrenches at the kinematic pair expressed in $\Psi_a$ become:

$$\begin{bmatrix} W_{a}^{a,a} \\ W_{a}^{a,a} \end{bmatrix} = \begin{bmatrix} J_{kp}^T & 0 \\ 0 & J_{kp}^T \end{bmatrix} \begin{bmatrix} f_a^a \\ \lambda \\ f_b^b \\ -\lambda \end{bmatrix}.$$  

C. Rigid bodies

The rigid bodies, with body wrench $W_{RB}$, connect to the kinematic pairs such that their relative motion is constrained. The standard port-Hamiltonian equations, as can be found in [15], [20] of a rigid body were used in this model.

D. Overall Interconnection on Graphs

The combination of the components treated in the previous sections synthesizes a compliant manipulator. In particular, a serial kinematic chain of rigid bodies coupled by kinematic pairs and powered by variable stiffness actuators is considered here. A segment of a serial chain of rigid bodies connected via kinematic pairs is depicted in Figure 5a, where the following assumptions are made:

Assumption 1. Each VSA is located at the distal end of the body with respect to which it drives another distal body.

Assumption 2. The rotors are uniform massless inertias with their center of mass on the rotation axis, i.e., the rotors only contribute to the rotational kinetic energy, but not to the translational kinetic energy. In other words: the mass of the rotors is negligible with respect to the link to which the VSA is attached according to Assumption 1.

Assumption 3. There can be an inertial coupling, i.e., the rotation of the rotors is not only due to their self rotation, but also due to the rotation of the links.

The kinematic pairs graph (or primary graph) is shown in Figure 5b. It defines the interconnection of the rigid bodies (vertices) by the kinematic pairs (edges). VSAs can interact with the kinematic pairs by introducing boundary vertices inside the kinematic pair edges, as shown in Figure 5c. These vertices are interfaces for the connection of VSAs and have no mass associated to them, i.e., they are massless. As elaborated in Section III-B, the wrenches generated in the kinematic pairs are mapped (by the kinematic pairs) to wrenches on the rigid bodies (internal vertices). Therefore, the interconnection of rigid bodies with kinematic pairs, through the internal vertex incidence matrix, is given by:

$$W_{RB} = B_{\theta_{kp}} W_{kp},$$  

where $W_{RB}$ are the wrenches on the rigid bodies and $W_{kp} = \begin{bmatrix} W_{a}^{a,a} & W_{c}^{b,b} \end{bmatrix}^T$ are the wrenches generated in the kinematic pairs on the a-side and b-side. Because the boundary vertices are massless, a map from the kinematic pair wrenches $W_{kp}$ to a force in the boundary vertices $f_b$ is not possible. However, the dual holds, i.e.:

$$e_b = B_{\theta_{kp}} T_{kp},$$  

where $e_b$ is the boundary vertex effort and $T_{kp}$ the kinematic pair’s twist. A serial kinematic structure, where the connection is defined by the graph, is now obtained, allowing the connection of compliant actuators as shown in Section III-A.

The interacting kinematic pair allows the connection of power supplying elements, which are in general actuators. The connection of actuators to the system is traditionally done in a uniaxial way, where every actuator delivers power to only one joint. This situation is shown in Figure 6a, together with the actuators connection graph defining indeed this connection, shown in Figure 6b. When considering the graph of Figure 6, the VSA edges map to the boundary vertices of the kinematic pair and the internal vertex using

$$W_{tc} = B_{\theta_{kp}} T_{kp},$$  

where $W_{tc}$ are the torque generated at the internal vertices.

Note that the first and last body of a serial chain are formally not boundary vertices of a graph. However, to make a consistent distinction between vertices representing rigid bodies and vertices representing interconnection points of kinematic pairs, they are referred to as internal vertices.

Note that Equations (4) and (5) are purely of topological nature, since $W_{tc}$ do not actually represent six-dimensional wrenches, but represent wrench entities. Therefore, it is also allowed to map $T_{kp}$ to $e_b$ in (5).
the incidence mappings. These forces are mapped through a boundary vertex incidence matrix to the kinematic pair’s boundary vertex flows, i.e.:

\[ f_b = B_{\Theta} f_{act}, \]

where \( f_b \equiv [f_b^a, f_b^b]^T \) are the kinematic pair’s boundary vertex flows and \( f_{act} \) are the actuators force flows. Likewise, the actuator forces map to their internal vertex (the rotor):

\[ f_p = B_{\Theta} f_{act_i} \]

where \( f_p \) is the flow of the internal (rotor mass) vertex.

The total incidence matrix \( B_{\Theta} \) of a graph is now:

\[ B_{\Theta} = \begin{bmatrix} B_{\Theta kp} & 0 \\ \frac{B_{\Theta kp}}{B_{\Theta act}} & \frac{B_{\Theta act}}{B_{\Theta act}} \end{bmatrix}. \] (6)

The total connection of \( n \) rigid body (excluding a reference body), \( n \) kinematic pair and \( m \) actuator flows becomes:

\[ \begin{bmatrix} W_{RB}(0) \\ \vdots \\ W_{RB}(n) \\ f_b(1) \\ \vdots \\ f_b(2n) \\ f_p(1) \\ \vdots \\ f_p(m) \end{bmatrix} = \begin{bmatrix} B_{\Theta kp} & 0 \\ \frac{B_{\Theta kp}}{B_{\Theta act}} & \frac{B_{\Theta act}}{B_{\Theta act}} \end{bmatrix} \begin{bmatrix} W_{kp}(1) \\ \vdots \\ W_{kp}(2n) \\ f_{act}(1) \\ \vdots \\ f_{act}(2m) \end{bmatrix}. \] (7)

Matrix \( B_{\Theta kp} \in \mathbb{R}^{(n+1) \times 2n} \) is the internal incidence matrix of the connection between the kinematic pairs’ wrenches and the body. Matrix \( B_{\Theta act} \in \mathbb{R}^{(2n+m) \times 2m} \) is the total incidence matrix (when combining the boundary and internal mappings) of the connection between the actuators’ output forces and the kinematic pairs’ boundary vertex flows and the internal mass vertices. Note that a specific enumerating order within grouped vertices and edges, determines the specific form of the incidence matrices. Moreover, note that the block matrix \( B_{\Theta kp} \) at element \((2,1)\) vanishes, because it defines an effort map as given in (5). Hence, the mapping in (7) is strictly no longer a mapping through an incidence matrix.

This modeling framework based on graphs allows the straight-forward inclusion of biarticulation or even multi-articulation actuators in the current kinematic structure, as shown in Figure 7a where two actuators are uniarticulation, and one is biarticulation, delivering power to two consecutive joints. This is done by simply adjusting the mapping \( B_{\Theta act} \) according to the actuators connection graph shown in Figure 7b. This is possible without any further adjustments of the kinematic pairs graph defining the kinematic structure.

### IV. Comparison to Traditional Modeling

This section shows the benefits of using a modular model based on port-Hamiltonian systems and graphs, as opposed to the more traditional Euler-Lagrange model.

The model of a two degree of freedom planar manipulator with compliant actuators is derived, as shown in Figure 8. Firstly, a model is considered in the situation where both joints are actuated by a uniarticulation VSA (seen in Figure 8a), and, secondly, a model is considered where the first joint is uniarticulated and both joints are biarticulated (seen in Figure 8b). In both models, a frictionless system under the absence of gravity is considered, with linear elastic behavior (the elastic force is \( \tau_{E1} = k_1 l(q) \) and where \( k_1 = k_2 = 10 \text{ Nm/rad} \)). A constant torque of 1 Nm is applied by the second VSA (the uniarticulation on the second joint in Figure 8a and the biarticulation in Figure 8b). No external disturbance forces are present.
A. Uniartiliated manipulator

1) Euler-Lagrange model: A generic Euler-Lagrange model of a compliant manipulator can be given by [2]:

\[
\begin{bmatrix}
  M_m & S^T(q) \\
  S(q) & M(q)
\end{bmatrix}
\begin{bmatrix}
  \dot{\theta} \\
  \dot{\hat{q}}
\end{bmatrix}
+ \begin{bmatrix}
  0 \\
  C_q(q, \dot{q})
\end{bmatrix}
\begin{bmatrix}
  C(q, \dot{q})
\end{bmatrix}
\begin{bmatrix}
  \hat{\theta} \\
  \dot{\hat{q}}
\end{bmatrix}
+ \begin{bmatrix}
  D_m \\
  0
\end{bmatrix}
\begin{bmatrix}
  \dot{\theta} \\
  \dot{\hat{q}}
\end{bmatrix}
+ \begin{bmatrix}
  0 \\
  g(q)
\end{bmatrix}
+ \begin{bmatrix}
  \tau_{E1} \\
  \tau_{E2}
\end{bmatrix}
= \begin{bmatrix}
  \tau_u \\
  \tau_{ext}
\end{bmatrix},
\]

where \( \theta \) are the generalized rotor coordinates, \( \hat{q} \) are the generalized link coordinates, and where \( q = [ \theta \ \hat{q} ]^T \).

The motor equations and the link equations are in general dynamically coupled through the elastic torques \( \tau_{Ei} \) as well as through the inertial coupling term \( S(q) \). These equations can be found by mapping the inertia tensor and stiffness (using the potential elastic energy) in world coordinates to generalized coordinates, through the Jacobians of the forward kinematics maps, and calculating the Christoffel’s symbols to find the Coriolis and centrifugal term. It is assumed that the stiffness is directly changed by a control input \( \tau_{ext} \), in accordance with the port-Hamiltonian model of the VSA in Section III-A, i.e., \( \dot{k} = \tau_{ext} \).

Due to the assumptions, it holds that \( D_m = D = 0 \), \( g(q) = 0 \), \( \tau_u = [ 0 \ 1 \ 1 ]^T \), \( \tau_{ext} = [ 0 \ 0 ]^T \), and \( \tau_{ext} = [ 0 \ 1 ]^T \). Note that Assumption 3 explicitly states that a complete model is used, where \( S(q) = S \neq 0 \) holds in this planar situation. Since \( S \) is constant, \( C_0(q) = C_q(q) = 0 \).

2) Graph model: The model of Section III is used, where the system’s topology is given by the graph in Figure 9.

The corresponding incidence matrix is given by:

\[
B_{\theta} = \begin{pmatrix}
  1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
  -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
  0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
  0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}.
\]

This completely defines the dynamics of the system.

Both identical simulations can be seen in Figure 11.

B. Biarticated manipulator

1) Euler-Lagrange model: The equations in (8) are still valid, but since the second uniarticulation actuator has changed to a biarticulation actuator, the map from actuator world coordinates to generalized coordinates has changed. This means that the inertia tensor in generalized coordinates changes, the Christoffel symbols change as a result of that, and the stiffness in generalized coordinates changes as well. Hence, the equations of motion are different and have to be recalculated due to the actuator relocation.

2) Graph model: The graph in Figure 10 defines the topology of the biarticulated manipulator; the rest of the model remains identical.

![Graph model](image)

Therefore, this biarticulation model only requires a change in the incidence matrix, obtained from the graph. The incidence matrix now becomes:

\[
B_{\theta} = \begin{pmatrix}
  1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
  -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
  0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
  0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
  0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}.
\]

Note that only a change of two numbers (in bold) is required to define the biarticulation model.

Both identical simulations, albeit obviously different from the uniarticulation model, can be seen in Figure 12.

The simulations shown here validate the proposed model, since this graph model behavior and the well established Euler-Lagrange model behavior are identical in both situations. It has been made explicit that the changes required in the graph model are very basic and much less severe than the changes that have to be made to the Euler-Lagrange model. Although the example treats two degrees of freedom, the model shows potential for larger and more complex systems.

V. DISCUSSION

Any serial chain manipulator with an arbitrary amount of compliant actuators connected to arbitrary joints can be easily modeled by adjusting the incidence matrix in (7). This is because the kinematic structure of the system is described independently from the actuators connection topology, facilitated by the usage of the port-Hamiltonian framework on graphs. With this, various actuator connection topologies are possible on the same kinematic structure.

It has been shown that the proposed model can be easily adapted to different actuator connection topologies by a minor change in the incidence matrix, which is a result of the port-based modeling and graph theoretic topology description which decouples a system’s morphology from its dynamics. This is in contrast with the more fundamental adaptation needed in the Euler-Lagrange model. Surely, automation and software tools reduce the effort of changing a model to suit a specific situation, but nonetheless does this model provide new possibilities for designing manipulators and their actuator placement and configuration.

This ease of adaptation also holds for changing a kinematic pair, i.e., changing the free relative motion of the connection of two bodies is simply done by adapting (1).

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Although the model proposed here allows an easier adaptation, the control techniques developed with the Euler-Lagrange model are very well established and widely used. However, the more recent control methods like Intrinsically Passive Control [16] can be directly applied to this proposed model, but is outside the scope of this work.

VI. CONCLUSION

This paper proposed a model based on graph theory and the port-Hamiltonian framework for generic compliant serial chain manipulators. Previous work, which showed the modeling of a variable stiffness actuator on a graph, has been extended and a kinematic pair that allows the connection of compliant actuators was introduced. It was shown that a manipulator’s topology is decoupled from its dynamics, which allows a straightforward change of actuators’ connection by changing elements in an incidence matrix. An example, where a uni- and biarticulated two degree of freedom compliant manipulator were modeled using the proposed model and the traditional Euler-Lagrange model, made this ease of adaptation explicit.

REFERENCES