Nonlinear systems which have finite-dimensional
 $\mathcal{H}_\infty$ suboptimal central controllers

A.J. van der Schaft

Department of Applied Mathematics, University of Twente
P.O. Box 217, 7500 AE Enschede, The Netherlands
and CWI, Amsterdam, The Netherlands

Abstract

Following up the work of Başar and Bernhard [2], we have recently derived in [12] the nonlinear central controller solving the nonlinear (standard) $\mathcal{H}_\infty$ suboptimal control problem. This nonlinear central controller is an *infinite-dimensional* system, and resembles very much the solution in nonlinear stochastic filtering or nonlinear deterministic filtering. After showing that in the *linear* case the nonlinear central controller reduces to the finite-dimensional central controller as obtained in [4], we consider in the present note the question if there are truly nonlinear systems having finite-dimensional central controllers. Guided by similar considerations in nonlinear stochastic and deterministic filtering, see especially [5], we characterize a specific class of nonlinear systems having finite-dimensional central controllers. This class can be regarded as the deterministic $\mathcal{H}_\infty$ analogue of the class of nonlinear systems admitting finite dimensional filters as identified by Benes [3].

We consider nonlinear control systems of the form

$$\begin{aligned}\dot{x} &= a(x) + b(x)u + g(x)d_1 \\ y &= c(x) + d_2 \\ z &= \begin{bmatrix} h(x) \\ u \end{bmatrix}\end{aligned}\quad (1)$$

where $x = (x_1, \dots, x_n)$ are local coordinates for a smooth state space manifold M . Furthermore, $u \in \mathbb{R}^m$ denote the *control inputs*, $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in \mathbb{R}^r$ the *exogenous inputs* (disturbances and/or references), $y \in \mathbb{R}^p$ the *measured outputs*, and $z \in \mathbb{R}^s$ the *to-be-controlled outputs* (tracking errors, cost variables). The maps $a(x), b(x), g(x), c(x), h(x)$ are all assumed to be C^k , with $k \geq 2$. Throughout we assume the existence of a fixed equilibrium x_0 , i.e. $a(x_0) = 0$, and without loss of generality we set $x_0 = 0$, and also we let $c(0) = h(0) = 0$. Now let γ be a fixed positive constant. The $\mathcal{H}_\infty$ suboptimal control problem (for disturbance attenuation level γ) is to find a compensator

$$\begin{aligned}\dot{\xi} &= k(\xi, y) \\ u &= m(\xi, y)\end{aligned}\quad (2)$$

where $\xi = (\xi_1, \dots, \xi_\nu)$ are local coordinates for a manifold M_c (the state space of the compensator), with $k(0, 0) = 0$ and $m(0, 0) = 0$, such that the closed-loop system (1), (2) has L_2 -gain *less than or equal to* γ , in the sense that there exists a nonnegative constant K depending on $x(0), \xi(0)$ and zero for $x(0) = 0, \xi(0) = 0$, such that

$$\int_0^T \|z(t)\|^2 dt \leq \gamma^2 \int_0^T \|d(t)\|^2 + K \quad (3)$$

for all $d(\cdot)$ and $T \geq 0$, with $z(\cdot)$ denoting the closed-loop response for initial condition $x(0), \xi(0)$. If some *observability* conditions are satisfied (with regard to the outputs z !), then property (3) will also imply internal stability of the closed-loop system. For further motivation and details we refer e.g. to [1], [7], [9], [10], [12].

The state of the art for this problem is, very roughly, as follows. The state feedback problem (i.e. $y = x$) is reasonably well-understood [9], [10], [7]. For the full (dynamic output feedback) problem appealing necessary conditions have been found generalizing the famous necessary and sufficient conditions for linear systems obtained in [4], see [1], [11], [12]. Much effort and ingenuity has been put in obtaining various sufficient conditions, see e.g. [1], [6], [7], but the problem is still largely open.

Following up the work of Başar and Bernhard [2], and using older work on nonlinear deterministic filtering [8], [5], we have recently taken another approach to the full $\mathcal{H}_\infty$ suboptimal control problem. In fact, in [12] we have shown that under suitable technical conditions the $\mathcal{H}_\infty$ suboptimal control problem is solved by the controller

$$\begin{aligned}\dot{\hat{x}} &= \left[a(\hat{x}) - b(\hat{x})b^T(\hat{x})P_x^T(\hat{x}) + \frac{1}{\gamma^2}g(\hat{x})g^T(\hat{x})P_x^T(\hat{x}) \right] \\ &\quad + \gamma^2 [S_{xx}(\hat{x}, t)]^{-1} \frac{\partial c^T}{\partial x}(\hat{x}) [y(t) - c(\hat{x})] \\ u &= -b^T(\hat{x})P_x^T(\hat{x})\end{aligned}\quad (4)$$

with $P \geq 0$ being the minimal solution to the Hamilton-Jacobi equation

$$\begin{aligned}P_x(x)a(x) + \frac{1}{2}P_x(x) \left[\frac{1}{\gamma^2}g(x)g^T(x) - b(x)b^T(x) \right] P_x^T(x) \\ + \frac{1}{2}h^T(x)h(x) = 0, \quad P(0) = 0,\end{aligned}\quad (5)$$

(implying the solvability of the *state feedback* $\mathcal{H}_\infty$ suboptimal control problem, see [9], [10]), and with R being a solution to the non-stationary Hamilton-Jacobi equation

$$\begin{aligned}R_t(x, t) + R_x(x, t)a(x) + \frac{1}{2}\frac{1}{\gamma^2}R_x(x, t)g(x)g^T(x)R_x^T(x, t) \\ + \frac{1}{2}h^T(x)h(x) - \frac{1}{2}\gamma^2 c^T(x)c(x) + \gamma^2 c^T(x)y(t) \\ + R_x(x, t)b(x)u(t) - \frac{1}{2}\gamma^2 \|y(t)\|^2 + \frac{1}{2}\|u(t)\|^2 = 0\end{aligned}\quad (6)$$

(with $u(t)$ given as the output of (4)), such that $S(x, t) := R(x, t) - P(x)$ has a unique minimum $\hat{x}(t)$ for every t with invertible Hessian matrix $S_{xx}(\hat{x}(t), t)$.

This controller is obtained as the solution of a certain min-max optimization problem with imperfect state measurements, see [2]; and thus is called the nonlinear *central controller*, since in the linear case this optimization problem is known [2] to yield the central controller of [4]. In fact, in the sequel we will *directly* demonstrate how in the linear case the nonlinear central controller reduces to the central controller of [4].

Although the first part of the nonlinear central controller, i.e. (4), has an appealing "worst-case disturbance" observer structure (see [2], [1], [12]), in general the gain matrix $[S_{xx}(\hat{x}, t)]^{-1}$ cannot be computed off-line, since the partial differential equation (6) for $R(x, t)$ is directly driven by the measured outputs $y(t)$, as well as by $u(t)$. Thus the nonlinear central controller is *infinite-dimensional*.

Now suppose (1) is linear, i.e.

$$\begin{aligned}\dot{x} &= Ax + Bu + Gd_1 \\ y &= Cx + d_2 \\ z &= \begin{bmatrix} Hx \\ u \end{bmatrix}\end{aligned}\quad (7)$$

It is readily checked (see [5] for the same argument in the context of deterministic filtering) that the solution $R(x, t)$ of (6) in this case takes the form $R(x, t) = c(t) + \ell^T(t)x + \frac{1}{2}x^T Q(t)x$ with $R_{xx}(x, t) = Q(t) \geq 0$ satisfying

$$\dot{Q}(t) + A^T Q(t) + Q(t)A + \frac{1}{\gamma^2}Q(t)GG^T Q(t) + H^T H - \gamma^2 C^T C = 0 \quad (8)$$

It follows that $R_{xx}(x, t) = Q(t)$ can be computed off-line, and in fact can be taken to be the maximal constant matrix $Q > 0$ solving the algebraic Riccati equation

$$A^T Q + Q A + \frac{1}{\gamma^2} G G^T Q + H^T H - \gamma^2 C^T C = 0, \quad (9)$$

(Notice that the dual Riccati equation (FARE) of [4] is obtained from (9) by dividing by γ^2 and pre- and post-multiplication by $\gamma^2 Q^{-1}$). Furthermore the stationary Hamilton-Jacobi equation (5) reduces to an algebraic Riccati equation, and it is immediately seen that the resulting linear controller (4) is precisely the central controller of [4].

A logical question is now: are there any other systems, apart from the linear ones, for which $R_{xx}(x, t)$ and thus $S_{xx}(x, t)$ can be computed off-line, and therefore the nonlinear central controller reduces to the finite-dimensional controller (4)? In order to study this problem we take the same approach as used in [5] for the deterministic nonlinear filtering problem, i.e. we consider the Hamiltonian function $H(x, p)$ corresponding to the Hamilton-Jacobi equation (6):

$$\begin{aligned} H(x, p) = & p^T a(x) + \frac{1}{2\gamma^2} p^T g(x) g^T(x) p + \frac{1}{2} h^T(x) h(x) \\ & - \frac{1}{2} \gamma^2 c^T(x) c(x) + \gamma^2 c^T(x) y(t) \\ & + p^T b(x) u(t) - \frac{1}{2} \gamma^2 \|y(t)\|^2 + \frac{1}{2} \|u(t)\|^2, \end{aligned} \quad (10)$$

with $p \in \mathbb{R}^n$ denoting the co-state. (Notice that (6) can be rewritten as $\bar{R}_t + H(x, \bar{R}_x^T) = 0$.) It is natural (see [5]) to consider the following type of canonical transformations $(x, p) \mapsto (x, \bar{p})$, where

$$p = \bar{p} + V_x^T(x) \quad (11)$$

for some C^k function $V(x)$. Then the Hamiltonian $H(x, p)$ transforms into $\bar{H}(x, \bar{p}) := H(x, \bar{p} + V_x^T(x))$, leading to the transformed Hamilton-Jacobi equation $\bar{R}_t + \bar{H}(x, \bar{R}_x^T) = 0$ given as

$$\begin{aligned} \bar{R}_t(x, t) + \bar{R}_x(x, t) \left[a(x) + \frac{1}{\gamma^2} g(x) g^T(x) V_x^T(x) \right] \\ + \frac{1}{2\gamma^2} \bar{R}_x(x, t) g(x) g^T(x) \bar{R}_x^T(x, t) + \frac{1}{2} h^T(x) h(x) \\ - \frac{1}{2} \gamma^2 c^T(x) c(x) + \frac{1}{2\gamma^2} V_x(x) g(x) g^T(x) V_x^T(x) + V_x(x) a(x) \\ + \gamma^2 c^T(x) y(t) + \bar{R}_x(x, t) b(x) u(t) + V_x(x) b(x) y(t) \\ - \frac{1}{2} \gamma^2 \|y(t)\|^2 + \frac{1}{2} \|u(t)\|^2 = 0 \end{aligned} \quad (12)$$

It immediately follows that $\bar{R}$ is a solution of (12) if and only if $\bar{R} + V$ is a solution of (6). Now assume that

$$\begin{aligned} b(x) \text{ and } g(x) \text{ are constant,} \\ c(x) \text{ is linear in } x, \end{aligned} \quad (13)$$

and suppose that the nonlinear system (1) has the property that $V(x)$ can be found such that

$$\begin{aligned} a(x) + \frac{1}{\gamma^2} g(x) g^T(x) V_x^T(x) \text{ is at most linear in } x, \\ \frac{1}{2} h^T(x) h(x) - \frac{1}{2} \gamma^2 c^T(x) c(x) + \frac{1}{2\gamma^2} V_x(x) g(x) g^T(x) V_x^T(x) \\ + V_x(x) a(x) \text{ is at most quadratic in } x, \\ V_x(x) b(x) \text{ is at most linear in } x. \end{aligned} \quad (14)$$

Then, as in the linear case, it follows that the solution $\bar{R}(x, t)$ of (12) can be written as $\bar{R}(x, t) = \bar{c}(t) + \bar{\ell}(t)x + \frac{1}{2} x^T \bar{Q}(t)x$, where $\bar{Q}(t) = \bar{R}_{xx}(x, t)$ is the solution of a differential Riccati equation of the same type as (8), and thus can be computed off-line (without knowing $y(t)$ and $u(t)$). Therefore also the Hessian $R_{xx}(x, t)$ of the solution $R(x, t) = \bar{R}(x, t) + V(x)$ of (6) can be computed off-line, and thus in this case the nonlinear central controller reduces to the finite-dimensional controller (4)! Summarizing:

Theorem Consider the nonlinear system (1). Suppose the $\mathcal{H}_\infty$ sub-optimal control problem is solvable by the nonlinear central controller (4), (6). Assume that $b(x)$, $g(x)$ and $c(x)$ are as in (13), while $V(x)$

can be found such that (14) holds. Then there exists a solution $R(x, t)$ of (6) for which $R_{xx}(x, t)$ can be computed off-line, and the nonlinear central controller reduces to the finite-dimensional controller (4).

Example Consider the almost linear system

$$\begin{aligned} \dot{x}_1 &= x_2 + u + d_1, & y &= x_2 + d_2 \\ \dot{x}_2 &= -x_2 & z &= \begin{bmatrix} x_2^3 \\ u \end{bmatrix} \end{aligned}$$

Then $V(x_1, x_2) = \frac{1}{12} x_2^6$ satisfies (12), and thus the nonlinear central controller reduces to a finite dimensional controller. $\square$

It would be nice to characterize the class of nonlinear systems covered by the above theorem in a more explicit and coordinate free way. Notice however that (as in nonlinear filtering, see [3], [5]) it is expected to be only a small subset of all nonlinear systems. On the other hand, we recall from [12] that even if the central controller is inherently infinite-dimensional, then still there may exist finite dimensional controllers also solving the (sub-)optimal $\mathcal{H}_\infty$ control problem.

Acknowledgements I thank J. de Does and J.M. Schumacher for some stimulating conversations.

References

- [1] J. Ball, J.W. Helton, M. Walker, " $\mathcal{H}_\infty$ control for nonlinear systems with output feedback", *IEEE Trans. Aut. Contr.*, vol. 38, pp. 546-559, 1993.
- [2] T. Başar and P. Bernhard, *$\mathcal{H}_\infty$ -optimal control and related minimax design problems*, Birkhäuser, Boston, 1990.
- [3] V.E. Benes, "Exact finite-dimensional filters for certain diffusions with nonlinear drift", *Stochastics*, 5, pp. 65-92, 1981.
- [4] J.C. Doyle, K. Glover, P.P. Khargonekar and B.A. Francis, "State space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems", *IEEE Trans. Automat. Contr.*, vol. 34, pp. 831-846, 1989.
- [5] O.B. Hijab, *Minimum Energy Estimation*, Ph.D. dissertation, University of California, Berkeley, 1980.
- [6] A. Isidori, " $\mathcal{H}_\infty$ control via measurement feedback for affine nonlinear systems", Preprint November 1992.
- [7] A. Isidori, A. Astolfi, "Disturbance attenuation and $\mathcal{H}_\infty$ control via measurement feedback in nonlinear systems", *IEEE Trans. Autom. Contr.*, AC-37, pp. 1283-1293, 1992.
- [8] R.E. Mortensen, "Maximum likelihood recursive nonlinear filtering", *J. Optimization Theory and Appl.*, vol. 2, pp. 386-394, 1968.
- [9] A.J. van der Schaft, "On a state space approach to nonlinear $\mathcal{H}_\infty$ control", *Syst. Contr. Lett.*, vol. 16, pp. 1-8, 1991.
- [10] A.J. van der Schaft, " L_2 -gain analysis of nonlinear systems and nonlinear state feedback $\mathcal{H}_\infty$ control", *IEEE Trans. Autom. Contr.* AC-37, pp. 770-784, 1992.
- [11] A.J. van der Schaft, "Nonlinear $\mathcal{H}_\infty$ control and Hamilton-Jacobi inequalities", Proc. IFAC NOLCOS '92 (ed. M. Fließ), Bordeaux, pp. 130-135, 1992.
- [12] A.J. van der Schaft, "Nonlinear state space $\mathcal{H}_\infty$ control theory", pp. 153-190 in *Essays on Control: Perspectives in the Theory and its Applications Control* (eds. H.L. Trentelman, J.C. Willems), PSCT 14 Birkhäuser, Boston, 1993.