NUMERICAL MODELING OF THERMOACOUSTIC SYSTEMS

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Numerical modeling of thermoacoustic systems

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SAMENVATTING

Het onderwerp van dit proefschrift is een relatief nieuwe klasse warmtemotoren en warmtepompen, die gezamenlijk thermo-akoestische (TA) systemen genoemd worden. TA systemen zijn commercieel interessant door hun lage aantal bewegende delen en het potentieel hoge rendement. Het hart van een TA systeem maakt gebruik van de interactie tussen warmte en sterke geluidsgolven. In het geval van de TA motor wordt warmte omgezet in geluid. Met behulp van bijvoorbeeld een lineaire alternator (een soort omgekeerde luidspreker) kan dit geluid vervolgens worden omgezet in elektriciteit. In het geval van een TA warmtepomp worden sterke geluidsgolven gebruikt om warmte te pompen van een koude naar een warme buffer. Een erg interessante toepassing is een TA motor die een TA warmtepomp van vermogen voorziet. In dit geval kan een systeem gemaakt worden zonder bewegende - en dus mogelijk slijtende - onderdelen.

Zoals gebruikelijk bij thermodynamische machines, en dus ook bij TA systemen, worden de prestaties gemeten in termen van rendement, geleverd vermogen, of een hoeveelheid opgepompte warmte per tijdseenheid. Om deze grootheden tijdens de ontwerpfase te voorspellen kan gebruik worden gemaakt van rekenmodellen. Met behulp van driedimensionale numerieke stromingsleermodellen (CFD) kunnen de natuurkundige effecten in het vloeistofdomein van het TA systeem tot in detail voorspeld worden. Echter, het doorrekenen van een volledig TA systeem met CFD is erg tijdsintensief. Daarom dient het rekindomein beperkt worden tot enkele onderdelen van het systeem. Hiervoor zijn randvoorwaarden nodig die de interactie beschrijven tussen de delen van het systeem die wel en niet gemodelleerd worden. Deze randvoorwaarden zijn echter moeilijk te bepalen.

Een andere strategie is het vereenvoudigen van het rekenmodel om de rekentijd te reduceren. Dit heeft reeds geresulteerd in het bekende thermo-akoestische model van Swift en Rott, dat geïmplementeerd is in de computercode DeltaEC. Met dit model kunnen snelle voorspellingen van een systeem gedaan worden voor lage oscillatie-amplitudes. Echter, om een hoge vermogensdichtheid te bereiken, worden in praktische thermo-akoestische systemen juist relatief hoge amplitudes gebruikt. Om voor dit geval ook goede voorspellingen te geven van de prestaties, moeten in het model ook de - over het algemeen - rendementverlagende niet-lineaire effecten meegenomen worden.

Dit proefschrift beschrijft twee bijdragen op het gebied van het modelleren van thermo-akoestische systemen. Ten eerste is er een eendimensionaal TA warmteoverdrachtsmodel ontwikkeld. Dit model kan gebruikt worden om de prestaties in te schatten van de, in TA systemen veel gebruikte, parallelle-plaatwarmtewisselaars. Deze warmtewisselaars zitten dichtbij de stack, of regenerator, en realiseren de benodigde warmte in- en uitvoer. Het model is tweede orde accuraat, wat betekent dat het geldend is voor lage oscillatie-amplitudes. Het model is geschikt om te implementeren in een eendimensionale frequentiedomein TA code, zoals het eerder genoemde DeltaEC.

Ten tweede is er een niet-lineaire frequentiedomein-methode toegepast die het mogelijk maakt om bij een simulatie de opstartfase van een TA systeem over te slaan: de methode kan gebruikt worden om direct de periodieke oplossing te vinden van een TA systeem. Dit zorgt voor een aanzienlijke verlaging van de rekentijd, omdat de opstartfase vaak enkele honderden cycli bedraagt, en deze hoeft dus niet te worden gesimuleerd. De methode is toegepast op een eendimensionaal, niet-lineair TA model. Een belangrijk resultaat is dat de toepassing van de niet-lineaire frequentiedomein-methode het model consistent maakt met het lineaire TA model. Dit was niet mogelijk met de tijddomein variant van dit model. Het model is gebruikt om een experimentele staande golf thermo-akoestische motor uit de literatuur te simuleren. De verkregen resultaten zijn in overeenstemming met de resultaten uit deze literatuur.

Voor het uiteindelijk modelleren van thermo-akoestische Stirlingmotoren, zijn verschillende toevoegingen nodig in het model die verder onderzoek vereisen. Zo zal er bijvoorbeeld, door de relatief hoge snelheidsamplitudes een turbulentiemodel benodigd zijn. Ook zal er nog een bewegende wand-randvoorwaarde geïmplementeerd moeten worden om mechanische vermogensproductie te kunnen modelleren.

Het niet-lineaire TA model met de niet-lineaire frequentiedomein-methode is een snelle combinatie voor het simuleren van eendimensionale niet-lineaire akoestische systemen. De methode maakt het doen van optimalisatieberekeningen mogelijk met een niet-lineair thermo-akoestisch model. Bovendien is de methode niet beperkt tot thermo-akoestische systemen. De niet-lineaire akoestische velden in vrije-zuiger-Stirlingmotoren, pulsbuskoe- lers en ook sommige blaasinstrumenten kunnen hiermee op een efficiënte manier gemodel- leerd worden.

SUMMARY

The subject of this thesis is a relatively new class of heat engines and refrigerators, called thermoacoustic (TA) systems. TA systems have gained commercial interest due to their low number of moving parts and potentially high efficiency. The heart of a TA system utilizes the physical interaction between heat and strong sound waves. In the case of a TA engine, heat is converted to acoustic power. This power can subsequently be converted to electricity using a “reversed” loudspeaker, called a linear alternator. In a TA refrigerator, a speaker or linear alternator is used to generate a strong acoustic wave, which is used to pump heat. Alternative applications are a TA engine, which drives a TA refrigerator. The latter uses no moving parts at all.

Like all other thermodynamic machines, the performance of thermoacoustic systems is expressed in terms of power output, heat lifted and/or efficiency. To predict this performance beforehand, the design of thermoacoustic systems needs to be assisted with accurate physical models. To model all physics in a thermoacoustic engine, a three-dimensional computational fluid dynamics (CFD) simulation is required of the full system. However, three-dimensional CFD simulations of a full thermoacoustic system are too time-consuming to be used as a design tool, hence the computational domain need to be restricted to some parts of the system. This requires assumptions on the interaction between the simulated domain and the part of the system which is not modeled.

Another strategy is to make simplifications on the physical model to reduce the computational cost. This has resulted in the famous second order thermoacoustic model of Swift and Rott, as implemented in the computer code DeltaEC. Using this model, fast predictions of engine performance can be made for low amplitude oscillations. However, to achieve competitive power density, thermoacoustic systems are generally run at such high amplitudes, that performance deteriorating nonlinear effects can no longer be neglected. To predict performance in this regime more accurately, a nonlinear model is required.

This thesis describes two contributions to the field of thermoacoustic system modeling. Firstly, a one-dimensional heat transfer model has been developed. This model can be used to estimate the performance of often used parallel-plate heat exchangers for thermoacoustic systems. These heat exchangers are located close to the stack or regenerator of a TA system and are responsible for the heat in/output required to let the system execute its thermodynamic cycle. The model is second order accurate, which means that it is valid for low amplitude oscillations. It is suitable to be implemented in frequency domain thermoacoustic codes, such as the previously mentioned DeltaEC. The results show a good match with a different heat transfer model from the literature, and the model provides guidelines for future heat exchanger design.

Secondly, a nonlinear frequency domain method is developed with which the initial transient start-up process can be skipped in the simulations. The method can be used to

directly simulate a TA system in its periodic steady-state. This significantly reduces computational cost, since the initial transient regime often involves several hundred oscillation cycles. The method is applied to a one-dimensional nonlinear model of TA systems. An important finding is the fact that by applying this method to the model, consistency with linear thermoacoustic theory can be realized. This was not possible with the original nonlinear time domain form of the model. The model is used to simulate an experimental standing wave thermoacoustic engine from the literature. The obtained results are in agreement with literature results.

For modeling of thermoacoustic Stirling engines, several additions need to be made to the model, which require further research. For example, the high velocity amplitudes in these machines require a turbulent flow model. Furthermore, moving wall boundary conditions need to be included to provide the means for power extraction from the TA system.

The nonlinear TA model including the nonlinear frequency domain method is a fast combination to solve one-dimensional nonlinear acoustic systems. The method opens the possibility of optimization parameter studies with a nonlinear thermoacoustic model. Moreover, it is not restricted to thermoacoustic systems. Nonlinear acoustic fields such as present in free-piston Stirling engines, pulse tube cryocoolers and some musical wind instruments can also be modeled in a computationally efficient way.

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AN INTRODUCTION TO THERMOACOUSTIC SYSTEMS

This chapter comprises two parts. First, a general introduction to the research topic of thermoacoustic systems is given, with a focus on the thermodynamical aspects. The second part starts at Sec. (1.4) and describes the research project, motivation and objective of this thesis. The chapter concludes with an overview of the upcoming chapters.

During the age of the industrial revolution, man started the large-scale transition from the use of human and animal mechanical power to machine power. Nowadays, machine power is all around us, literally available with a flick of the switch. The machines producing this power are mainly placed in electrical power plants, under the bonnet of our cars and trucks, and under the wings of our airplanes.

All these devices share the property of producing power from heat and are therefore commonly called heat engines, or *prime movers*. These engines can be classified by the thermodynamic cycle on which they operate. The electrical power plants often use water as a working fluid and the thermodynamic cycle is essentially the same as the cycle of steam locomotives, called the Rankine cycle. Car engines use the Otto cycle or Diesel cycle and modern airplane engines use the Brayton cycle. Brayton cycles are also found in gas-fired electricity plants.

Figure 1.1 – Left: photograph of the inventor of the Stirling engine: Rev. Robert Stirling (1790- 1878). Right: mechanical configuration of his Stirling engine. The numbers are 1: flue gas exit, 2: piston, 3: transmission rods, 4: crankshaft, 5: flywheel, 6: connecting rods, 7: expansion space, 8: compression space, 9: gas displacer (described in Sec. (1.1)), 10: fire. Pictures copied and adapted from [47].

What all cycles mentioned so far have in common is that none of these are able to achieve the maximum amount of power from the available input heat. More technically, the thermal efficiency of these cycles is lower than the highest physically possible thermal efficiency between two heat reservoirs at different temperatures. This efficiency is called the *Carnot efficiency* [9, p. 253].

A thermodynamic power cycle that is able to reach the Carnot efficiency is the Stirling cycle, named after the Scotsman Rev. Robert Stirling, who patented his “hot air engine” in 1816 [47, p. V]. Figure (1.1) shows a photograph of Rev. Robert Stirling alongside of his engine. The advantages of this engine, such as low noise, potentially high efficiency and fuel independency, have fascinated researchers and engineers for nearly two centuries. However, its low specific power¹ compared to the internal combustion engine made the engine lose the “dollars-per-horsepower” race in the first half of the twenty-first century [13]. An important step forward in Stirling engine research started in the 1960s after the invention of the free-piston Stirling engine by William Beale in 1964 [5]. This engine eliminates wear and sealing problems with non-contact operation of the power piston and mechanical displacer.

1.1 The Stirling cycle

Figure (1.2) shows the physical arrangement for a thermal core of a Stirling cycle machine. The cold and hot space are assumed to be in thermal equilibrium with the cold and hot wall, respectively. Two pistons are shown, which are able to push the gas through the regenerator. For the analysis it is assumed that the amount of gas in the regenerator can be neglected. Hence, if both pistons are in the leftmost position, all gas is in the cold space, and if both pistons are in the rightmost position, all gas is in the hot space. The regenerator functions as a “thermal sponge”, absorbing heat when the hot gas moves through the regenerator from the hot space to the cold space, and releasing heat when gas moves in the opposite direction. As part of the idealized analysis, the motion of the pistons is assumed to be frictionless and there exists no leakage through the pistons.

The Stirling cycle consists of four discrete steps, as shown in Figure (1.3). Each step converts the gas to a new thermodynamic state. The states are numbered 1 through 4. On the right side of the figure, T - S diagrams and p - V diagrams are shown. These diagrams help with the description of the steps in the cycle, as they indicate what stays constant and what changes in a step. From the T - S diagram, the heat input can be evaluated as

$$Q_{\text{in}} = \oint T dS, \quad (1.1)$$

where Q_{in} is the net heat input, T is the temperature and S is the entropy. From the p - V diagram, the work output can be evaluated as

$$W_{\text{out}} = \oint p dV, \quad (1.2)$$

¹In Watts of mechanical power per cubic meter of engine, or Watts per kilogram of engine.

Figure 1.2 – Physical arrangement for performing a Stirling cycle. The cold and hot wall are assumed to be perfect heat reservoirs at different temperatures. Gas is moved through the regenerator from the cold side to the hot side of the volume by the pistons.

where W_{out} is the net work output, p the pressure and V the volume. Note that the first law of thermodynamics states that $Q_{\text{in}} = W_{\text{out}}$ and from the second law it can be deduced that

$$\oint \frac{dQ_{\text{in}}}{T} \leq 0, \quad (1.3)$$

which is known as the Clausius inequality [23, p. 39].

In step $1 \rightarrow 2$, the gas is isothermally compressed. For a thermally perfect gas, the internal energy of the gas stays constant during this step. Pressure increases and the gas volume decreases. In Step $2 \rightarrow 3$, the gas is pushed through the regenerator, absorbing heat from the regenerator, while the gas volume stays constant. Due to the increase in temperature at constant volume, the gas pressure increases further. The work step is $3 \rightarrow 4$, in which the gas expands isothermally at the high temperature. Finally, in step $4 \rightarrow 1$ the expanded gas is pushed back through the regenerator to the cold space, rejecting heat to the regenerator in the meantime. Reaching stage 1 again, the cycle is complete and a new cycle begins.

For the analysis of the thermal efficiency of this cycle, the system boundaries can be taken around the regenerator. Then, the only places where heat is exchanged with the surroundings is at the hot and cold wall. Hence $Q_{\text{in}} = Q_{3 \rightarrow 4} + Q_{1 \rightarrow 2}$. Then the efficiency is

$$\eta = \frac{W_{\text{out}}}{Q_{3 \rightarrow 4}} = \frac{Q_{3 \rightarrow 4} + Q_{1 \rightarrow 2}}{Q_{3 \rightarrow 4}}. \quad (1.4)$$

Note that all processes are reversible and therefore the inequality of Eq. (1.3) becomes an equality. Moreover, the heating and cooling in steps $3 \rightarrow 4$ and $1 \rightarrow 2$, respectively, are isothermal. As a result, Eq. (1.3) can be simplified to

$$\frac{Q_{3 \rightarrow 4}}{T_{3 \rightarrow 4}} + \frac{Q_{1 \rightarrow 2}}{T_{1 \rightarrow 2}} = 0. \quad (1.5)$$

Figure 1.3 – Diagram showing the four steps of the Stirling cycle. A dashed piston denotes the position of that piston at the start of the stage. On the right, the corresponding paths in the T - S diagram and p - V diagram are shown. 1 $\rightarrow$ 2: Isothermal compression, 2 $\rightarrow$ 3: isochoric heating, 3 $\rightarrow$ 4: isothermal expansion, and 4 $\rightarrow$ 1: isochoric cooling. The arrows in the figure denote the direction in which heat flows during the stage. 1 $\rightarrow$ 2: heat rejected to cold buffer. 2 $\rightarrow$ 3: heat added from regenerator. 3 $\rightarrow$ 4: heat added from hot buffer. 4 $\rightarrow$ 1: heat rejected to regenerator. For an ideal regenerator, the heat flow from the regenerator in stage 2 $\rightarrow$ 3 is the same as the heat flow to the regenerator in stage 4 $\rightarrow$ 1.

Combining Eq. (1.4) and Eq. (1.5) results in

$$\eta = 1 - \frac{T_{1 \rightarrow 2}}{T_{3 \rightarrow 4}}, \quad (1.6)$$

which equals the Carnot efficiency for these temperatures.

The Stirling cycle is a highly idealized situation and it should be noted that Stirling engines do *not* perform a Stirling cycle. In fact, each gas particle in the engine follows a different path and hence performs a different cycle, and some of them do not contribute to the power output at all. Moreover, in reality no regenerators without an internal void exist, because one way or another the hot and cold side of the engine need to transport the gas through the regenerator. Hence, some gas particles initially on the cold side never reach the hot side. The same applies to particles initially on the hot side. Additionally, as the engine runs at a finite speed, particles in the hot and cold space do not have enough time to exchange heat with the hot and cold wall, respectively. Therefore, the expansion and

Figure 1.4 – Physical arrangement of the piston-displacer free-piston Stirling engine [47, p. 261].

1

compression in the hot and cold spaces is really more close to adiabatic than to isothermal, resulting in a lower thermal efficiency. Many other practical problems arise when one starts designing a real Stirling engine, and the result will be a design in which a trade-off has been made among several loss mechanisms, such as flow friction loss in the regenerator and heat exchangers, friction loss at the piston walls, finite temperature difference conduction losses and many others.

The physical arrangement of the Stirling cycle of Figure (1.2) requires two piston drives. A more common configuration in modern Stirling systems is a concentric one in which the piston and displacer are in the same cylinder [47, p. 261]. Figure (1.4) shows a schematic of this configuration. In this arrangement, the gas is pushed through the regenerator by the mechanical *displacer*. When the displacer moves to the right, gas flows from the hot space to the cold space and vice versa. In this arrangement, only one piston is responsible for compression and expansion of the gas.

The free-piston Stirling engine differs from the conventional Stirling engine in another aspect. Where the movement of the piston and displacer in conventional Stirling engines is kinematically controlled with crankshafts and rods, the movement in the free-piston Stirling engine is dynamically controlled and is based on proper choices of the masses and spring constants, such that the Stirling cycle phasing is obtained. Kinetic energy of the piston and displacer is cyclically stored and released with – gas – springs. Moreover, the gas behind the piston functions as an extra “spring” by cyclically compressing and expanding. A means to extract power is shown by the coil and magnets. The movement of the power piston generates electromagnetic induction which results in an alternating voltage in the coil.

Since the working gas itself can both be “massy” and “springy”, the question arises if it is possible to replace the displacer with an equivalent system of working gas only. The answer is: yes. These engines are called *thermoacoustic Stirling heat engines* [2] (TASHEs), and they are the main topic of this research. To understand the operation of thermoacoustic Stirling engines, first an introduction to thermoacoustics is necessary.

Figure 1.5 – $T - s$ and $p - v$ diagram of an acoustic wave for three situations. Left: adiabatic acoustic wave. Middle: acoustic power generating cycle. Right: heat pumping cycle.

1.2 Thermoacoustics

Thermoacoustics (TA) is the field of acoustics in which heat conduction and entropy variations play a role [35]. An introduction to thermoacoustics starts with an introduction to acoustics, which is the field of research of sound, including its production, transmission and effects [31]. Sound is a type of mechanical energy in the form of the movement of fluid particles. The movement of fluid particles results in pressure variations which propagate through the fluid with a fixed speed; the speed of sound. Sound can be created by forcing the fluid particles into motion, for example by a moving (vibrating) structure. The mechanical energy of the moving structure is transformed into acoustic energy, or sound. This also works in the opposite direction. In that case, the acoustic pressure variations of particles push on the surface of a solid structure, such as a microphone membrane, or on our ear drums, which is the way we experience sound in daily life.

When the pressure in a gas increases adiabatically, the temperature of the gas also increases. Hence, the pressure oscillations due to sound waves, like those of speech, are coupled with a – generally small – temperature oscillation. For moderate amplitudes of the wave in the audible frequency range, pressure oscillations can be regarded as isentropic. However, close to a rigid wall, these acoustic temperature changes cause a temperature difference with the adjacent wall, causing heat transfer to occur with the wall. This layer of gas in contact with the wall is called the thermal boundary layer, with thickness (δ_κ) typically of order $\sim 10^{-4}$ m [41, p. 7]. On the other hand, if the temperature of the wall increases, this causes heating of the thermal boundary layer. This last effect can be used in the following way. Suppose an acoustic pressure oscillation causes compression and expansion of the fluid. Then, also the particles close to the adjacent wall are subject to this pressure change. If one then would be able to prescribe the wall temperature, such that the temperature is high at the moment of highest pressure, heat would flow from the wall to the acoustic boundary layer, causing the fluid to expand. Then if one would make the wall cold at the moment the pressure is lowest, a little less heat would flow from the acoustic medium back into the wall and the fluid would contract. Hence, the fluid *performs a thermodynamic cycle*. Figure (1.5) shows schematically what happens during such an “acoustic cycle” for three different situations. On the left the $p-v$ diagram of normal adiabatic compression and expansion is shown. The middle two diagrams show the engine cycle as was just described. The two rightmost diagrams show what happens if the cycle is reversed. In this case heat is absorbed at low temperature and released at high temperature, so heat is pumped.

Figure 1.6 – Sondhauss’s tube. When the transition between the bulb and the narrow tube is heated, the glass starts to “sing”: a single frequency tone is emitted from the open end of the narrow tube.

When the work done by the thermodynamic cycle overcomes the losses present in a system, a spontaneous oscillation of the gas begins. This effect was encountered by glass blowers, when a bulb is blown on the end of a narrow tube. The situation is shown in Figure (1.6). When the bulb is connected to the tube and both are cooling down, a single frequency tone is emitted from the open end of the narrow tube [14]. Spontaneous gas oscillations of this form were studied by Sondhauss in 1850 [38]. A similar type of spontaneous oscillation was discovered by Taconis [43], when the open end of the tube is cooled, instead of the closed end being heated. Merkli and Thomann demonstrated the reverse thermoacoustic effect: cooling of the resonator cylinder wall using a standing wave [25].

Other forms of thermoacoustic instabilities are flame-induced, such as “singing flames” [32], and the Rijke tube [33], in which a heated wire mesh drives acoustic oscillations using natural convection [15]. Flame-induced thermoacoustic instabilities are often unwanted and can cause severe problems in combustion systems such as gas turbine burners, industrial furnaces and domestic boilers. The oscillation amplitude can become so large that the combustion system can be damaged, or even destroyed [21].

Lord Rayleigh was the first to give a qualitative description applicable to all these phenomena. He made the well-quoted statement [24, p. 226]

“If heat be given to the air at the moment of greatest condensation or taken from it at the moment of greatest rarefaction, the vibration is encouraged.”

This is nowadays known as the Rayleigh criterion. Note that it only states that the vibration is encouraged. An instability occurs only when the encouraged vibration overcomes the losses [49].

Standing wave thermoacoustic systems

Instead of changing the local wall temperature, in thermoacoustic engines such as the Sondhauss tube, heat addition and subtraction to a fluid particle is achieved by displacing the particle to places where the wall temperature is higher or lower, respectively. Suppose a fluid particle is adjacent to a wall with a temperature gradient. Then, if such a particle would move in the direction of increasing wall temperature, heat would flow from the wall to the fluid particle. Similarly, if the particle would move in the direction of decreasing

Figure 1.7 – Schematic picture of the thermodynamic cycle of a standing wave thermoacoustic engine, and of the Sondhauss tube. The dashed lines denote the particle in the “old configuration.” The colored bar denotes a piece of wall with a temperature gradient, where red is hot and blue is cold.

temperature, the wall is colder than the particle and heat would flow from the particle to the wall. This is schematically shown in Figure (1.7). Again, if compression takes place before heat addition, and expansion before cooling, a thermodynamic cycle is performed by the fluid particles. Because the compression and movement need to be in phase, a standing acoustic wave is required for this cycle. An essential feature of the cycle is the time delay between compression/movement and heat communication with the wall, which is established by a “weak” thermal contact between the fluid particles and the wall. This will be further discussed in Sec. (1.3).

In the Sondhauss tube, the only gas that performs the thermodynamic cycle is the gas in the thermal boundary layer in the vicinity of the wall. The amount of gas participating in the cycle can be increased by introducing a stack of plates, which is simply called a *stack*. This increases the effective surface area between solid and fluid, and as such increases the amount of gas that performs a thermodynamic cycle. Figure (1.8) shows a schematic diagram of a standing wave thermoacoustic system, which can both be an engine and a refrigerator/heat pump. In the case of a refrigerator/heat pump, the driver generates a standing acoustic wave in the system. This standing wave moves the particles in the stack, which

Figure 1.8 – Standing wave thermoacoustic system.

transport heat from left to right, such that a temperature difference is generated between the cold heat exchanger (HX) and the hot HX. If the system runs as an engine, heat is added to the hot HX, while the cold HX is kept at a constant temperature. This creates a temperature gradient across the stack. This temperature gradient drives the thermodynamic cycle that generates acoustic power, which is absorbed by the alternator and used to create, for example, electrical power (see also Figure (1.4)).

Applications of these thermoacoustic systems are thermoacoustic engines, and refrigerators/heat pumps. If a thermoacoustic refrigerator is coupled with a thermoacoustic engine, all moving parts can be eliminated. An example is the solar-powered thermoacoustically driven thermoacoustic refrigerator (TADTAR) [1]. Several successful research efforts have been undertaken to build useful devices with this technology. An overview of previous research and design efforts can be found in the thesis of Tijani [44]. The most important advantages are no or a minimal number of moving parts, resulting in a high reliability and low maintenance costs. Other advantages are the use of environmentally friendly working gases (helium, argon), and inexpensive parts due to the absence of close tolerances in the design.

Standing wave thermoacoustic systems do have a disadvantage, which is the low efficiency due to the irreversible Brayton thermodynamic cycle. This cycle can only reach Carnot efficiency in the limit of no power output [50]. Furthermore, the phase delay in the transfer of heat requires a finite temperature difference, which introduces irreversibilities in the heat transfer between the wall and the fluid.

1.3 The thermoacoustic Stirling heat engine

A major step forward in the development of thermoacoustic systems was the combination of the thermodynamically advantageous Stirling cycle with the previously described practical advantages of standing wave thermoacoustic systems. The major research insight came from Ceperley [10, 11]. He discovered that a traveling acoustic wave flowing through a Stirling engine regenerator subject to an imposed temperature gradient actually performs a regenerative thermodynamic cycle similar to the Stirling cycle.

In a traveling acoustic – plane – wave, the acoustic pressure oscillation and the velocity are in phase. As discussed in Sec. (1.1), in an engine regenerator perfect thermal contact is desirable. This can be obtained when the size of the pores is much smaller than the thermal boundary layer thickness. This is contrary to a standing wave engine, whose operating principle relies on a time delay in the heat transfer induced by a poor thermal contact.

When the thermal contact in a regenerator is perfect, the temperature of the fluid particles is the same as the local regenerator material temperature. As a temperature gradient is imposed across the regenerator, movement of a particle means instantaneous heating and cooling of the subjected particle. Figure (1.9) shows the particle pressure, velocity, temperature and position of a particle in a regenerator subject to a sinusoidal traveling acoustic wave. As the pressure and velocity are in phase, the position of the particle lags the pressure with a quarter of a period. As shown in the articulated steps in the figure, this results in a thermodynamic cycle. A closer inspection reveals that the articulated cycle is not a Stirling

Figure 1.9 – Schematic diagram of a fluid particle oscillating in a regenerator subject to a temperature gradient, with the higher temperature rightwards. The bottom schematic shows the different phases “articulated”, such that the underlying thermodynamic cycle is exposed.

cycle, but a closely related cycle called the *Ericsson cycle* [47], since the two isochoric heating steps are replaced with isobaric heating steps. Nevertheless, if the two isobaric steps cycle all heat in the regenerator, the cycle is thermodynamically reversible and the efficiency equals the Carnot efficiency.

At the end of the 1990s, the first successfully running thermoacoustic Stirling heat engine was designed and built by Backhaus and Swift [2, 3] at Los Alamos National Laboratory (LANL), USA. Figure (1.10) shows the topology of this TASHE. A traveling wave phasing in the regenerator is achieved using an acoustic feedback network, consisting of an acoustic compliance and inductance. Together with the compliance, the resonator dictates the fundamental oscillation frequency. A load is present in the form of a resistive valve, with which the generated acoustic power can be extracted or dissipated. The engine of Backhaus and Swift has reached a thermal-to-acoustic efficiency of 30%, which corresponds to 41% of the Carnot efficiency for the chosen operating temperature. After this successful introduction of the TASHE, many following research efforts have led to a variety of designs. An example is the 4-stage traveling wave thermoacoustic system of De Blok [6], which combines 4 regenerators in one system. This system can run at a minimum temperature difference

Figure 1.10 – Schematic topology of the thermoacoustic Stirling heat engine of Backhaus and Swift [2, 3]. The arrows denote the direction of acoustic power flow. TBT stands for thermal buffer tube, and FS stands for flow straightener.

of only 30 K and is suitable for running on industrial waste heat. Another example is the cascade thermoacoustic engine of Gardner and Swift [16], which combines a standing wave and a traveling wave heat engine. This configuration has the major advantage that Gedeon streaming (see Sec. (1.4)) is eliminated.

1.4 The research project

The goal of the research project, of which this thesis is a part, is to enable a small-scale (~ 1 kW) and low-cost thermoacoustic Stirling engine Generator (TASEG), which produces electricity from externally supplied heat [51]. The electricity generation from acoustic power is done with linear electrodynamic alternators like that schematically shown in Figure (1.4). The applications in mind are, for example, micro-combined-heat-and-power units for domestic purposes, and range extenders for electrical cars. The project is a joint effort of the University of Twente² with industrial partners Nefit B.V.³, and Aster Thermoakoestische Systemen⁴, subsidized by Agentschap NL⁵. Together with the Laboratory of Thermal Engineering, the Laboratory of Applied Mechanics of the University of Twente focuses on solving the more fundamental research questions related to the fluid domain of the TASEG.

Despite the inherent advantageous properties of the TASEG, such as its high reliability and potentially high efficiency, exploiting these properties for small-scale applications requires a deliberate design. This design can only be done with a good understanding of

²Website URL: <http://www.utwente.nl>

³Website URL: <http://www.nefit.nl>

⁴Website URL: <http://www.aster-thermoacoustics.com>

⁵Website URL: <http://www.agentschapnl.nl>

the complex thermodynamic and flow processes present in thermoacoustic systems. Hence, computer models of the TASEG are required, which accurately simulate the physics in the system. Moreover, computer models are useful for optimization of the design, because the feedback on design decisions is much faster with simulation results than with experimental results of built TASEG prototypes.

Prior to this research project, a prototype TASEG was built as proof that the technology is suitable for small-scale power generation. From this experience, it is found that there are quite a few uncertainties in the performance of the system. These uncertainties are due to the limitations of the used design tools, but also due to physical mechanisms which are not fully understood, from both a qualitative and a quantitative perspective. Focusing on the fluid domain of the TASEG, a small overview is given of topics that require special attention:

Acoustic streaming and turbulence

Acoustic streaming is a time-average fluid mass flux induced by the presence of an acoustic wave [22]. This streaming has adverse effects in thermoacoustic systems because the temperature differences in these systems allow the mass flow to transport enthalpy. This occurs, for example, in the thermal buffer tube (TBT), which is the region right below the hot HX in Figure (1.10). The function of the TBT is to insulate the hot HX, such that the heat leak in the downward direction is minimized. However, if streaming is present in the TBT, the hot and cold areas are mixed, which results in a larger heat leak from the hot heat exchanger, which decreases the efficiency of the device. Acoustic streaming in the TBT can be counteracted with a small taper angle in the TBT [27]. However, at high amplitudes it is shown that this does not work well anymore because the flow becomes (weakly) turbulent, which breaks down the theoretically expected streaming profiles.

Gedeon streaming

In thermoacoustic Stirling systems containing a closed loop, as shown in Figure (1.10), a time-averaged mass flow can circulate around the loop. That is, if no counter measures are taken to suppress it. This mass flow is called *Gedeon streaming* [18], named after the discoverer. The direction of the streaming is the same as the direction of the acoustic power flow, so it transports heat from the hot HX around the loop to the cold HX. As derived by Backhaus and Swift, the associated heat leak for helium is approximately 2.5 times the delivered acoustic power [3]. Hence, if Gedeon streaming is not suppressed an impermissible toll in performance has to be paid.

Several means can be used to suppress Gedeon streaming. For example, an elastic membrane can be used [45], which is acoustically transparent, but physically stops a time-averaged mass flow. A disadvantage of this solution might be the low lifetime of this membrane, which can fail as a result of stress fatigue. A solution which does not require any moving parts is a so-called *jet pump* [42]. This is a device which generates a flow in the opposite direction with the net result of canceling out the Gedeon streaming. The jet pump working principle is based on an asymmetry in flow losses. When the – oscillating – flow is in the negative direction, the pressure drop is different from the pressure drop when the

flow is in positive direction. This results in a net time-averaged pressure drop, which is the driving force for a time-average flow in the opposite direction of the Gedeon streaming. This way, the net mass flow around the loop is canceled. Due to the introduced resistance, both of these solutions introduce additional power loss, however, the efficiency gain due to the cancellation of Gedeon streaming makes them worth the effort.

Numerical and experimental research is being conducted at the University of Twente on the performance of these jet pumps [29].

Heat exchanger performance

Heat transfer in heat exchangers for unidirectional flow is a well-understood process, described in standard text books on heat transfer, for both laminar and turbulent flow. For oscillating flow, however, this is not the case. For example, due to the oscillating flow in heat exchangers, increasing the length of the heat exchanger does not always result in more heat transfer from the exchanger to the working fluid [17]. Poorly functioning heat exchangers reduce the efficiency of the system by introducing a high temperature difference between heat exchanger material and working gas, which decreases the effective temperature span of the TA system, thereby decreasing the efficiency. On the other hand, heat exchangers with more heat exchange area than required introduce unnecessary visco-thermal dissipation.

High amplitude motion and harmonics generation

To obtain a reasonable volumetric power density, thermoacoustic engines are often run at high drive ratios. The drive ratio is defined as the ratio of pressure oscillation amplitude to the mean pressure in the system. The power output of a system at constant mean pressure is approximately proportional to the drive ratio squared [40], and drive ratios as high as 10% are no exception. High amplitudes result in nonlinear effects, such as the generation of higher harmonics and wave steepening, possibly leading to shocks [12]. The process of harmonics generation is the result of the nonlinearities in the governing equations of fluid dynamics, as well as the nonlinear pressure-density coupling in adiabatic compression and expansion [20]. Due to the unfavorable phasing of the higher harmonics in the stack or regenerator, these higher harmonics are generally dissipative. Hence, they can be regarded as an additional dissipative load on the system.

Besides these “global” nonlinearities, local nonlinearities are also present in the system. For example, at the transition from the hot HX to the TBT, temperature harmonics can be generated due to the sudden change from a near isothermal compression/expansion in the HX to near adiabatic compression/expansion in the TBT [19]. This results in temperature discontinuities at the interface between the components [7]. Moreover, transitions from near isothermal to near adiabatic compression/expansion lead to so-called *adiabatic loss* [4, 37]. When gas particles that exit the hot HX at a pressure below the maximum pressure, these particles adiabatically compress further outside the hot HX. Then, when the flow reverses, these particles re-enter the hot HX at a higher temperature than the HX surface temperature. This results in lost work due to irreversible heat conduction. The same mechanism occurs at the cold HX.

Flow losses in compact geometries

In a small-scale TASEG, the compact geometry results in less subtle changes in the direction of the flow (besides the oscillatory change in direction of the flow). For example, in a bent tube, this leads to the generation of vortical structures [39] which contribute to the viscous dissipation of the acoustic wave [28]. Abrupt changes in the geometry lead to flow losses and periodical vortex shedding [30], which is beneficially used in a jet pump, but generally influences the performance negatively.

The design of the earlier described prototype TASEG was done using the linear thermoacoustic theory of Swift and Rott [34, 40] as implemented in a free for public use computer code developed at LANL, called DeltaEC [48]. DeltaEC is a widely used code with an extensive amount of features included to design actual thermoacoustic Stirling systems. Using the results of linear thermoacoustic theory, DeltaEC can predict performance parameters, such as acoustic/electric power output and efficiency. However, as this code uses one-dimensional and linear models, it has some inherent limitations. Therefore, the design process needs to be assisted with additional models and tools.

One such tool is computational fluid dynamics (CFD). With CFD the full three-dimensional governing equations of fluid dynamics are discretized and numerically solved. Both free for public use, as well as commercial software packages, are available to do this job. The method is able to expose the tiniest details of the fluid flow. However, the applicability of CFD is restricted due to the required computational cost. In 2014, modeling of a thermoacoustic engine from start-up to steady-state still required a super computer or computing cluster to keep the computation (wall clock) times manageable [36]. Therefore, CFD can be used for the design, however, this requires one to take into account smaller computational domains, such that only the interesting part of the domain is modeled. Yet subdomain modeling of unsteady flows is difficult due to the application of subdomain boundary conditions which are often stated in the frequency domain, while the numerical scheme of CFD is stated in time domain. Currently, research is being conducted at the University of Twente on the use of CFD for implementation of impedance boundary conditions in CFD [8].

Somewhere in between full three-dimensional CFD and the linear thermoacoustic model of DeltaEC is a one-dimensional nonlinear model. A one-dimensional model solves averaged governing equations for only one spatial direction. Closure terms for the interactions along the perpendicular directions are provided from additional models, either theoretically or experimentally obtained. When computers entered the stage for industrial use, this method (in Stirling modeling terminology called a “nodal method”) was the state of the art for modeling engines and refrigerators [47]. As such, this method can rely on about 50 years of experimental work.

In thermoacoustic systems, full three-dimensional effects often have only local influences, such that for the majority of the system components a one-dimensional model will suffice. Local three-dimensional effects can be taken into account by using additional models. An example of a one-dimensional computer code is Sage⁶.

⁶Website URL: <http://www.sageofathens.com>

1.5 Objective and scope of this thesis

The main objective of the research described in this thesis is the development of mathematical and numerical modeling techniques for thermoacoustic systems, both to better understand the behavior, and to predict and optimize the performance when designing new TA systems. Within this main objective, two contributions are made:

Development of a heat transfer model for parallel-plate thermoacoustic heat exchangers. In the initial stage of the design of TA systems, fast computational models are required and one-dimensional codes such as DeltaEC are used to create an initial design of the acoustic topology. In this initial design, estimations on the heat exchanger performance must be made. However, the parallel-plate heat exchanger model of DeltaEC is regarded by some as the weakest aspect of the software [46]. In Chapter (3), a new heat exchanger model is derived for parallel-plate TA heat exchangers. This model can be used in the one-dimensional and second order thermoacoustic theory framework (such as implemented in DeltaEC). The model is valid for low-amplitude oscillations i.e. when the particle displacement amplitude is still small compared to the heat exchanger length. It should be noted that this is often still the case in TA systems running at full load. The model is implemented in an in-house linear thermoacoustic code [52], which uses an extension of the transfer matrix method (acoustic two-ports) [26, p. 119] for both the acoustic- and the second order heat equation [41, p. 194]. This code is developed as part of this PhD work and has similar capabilities as DeltaEC.

Development of a one-dimensional nonlinear TA model with low computational cost. To compute the periodic steady-state of a TA system, a time domain code needs to integrate the equations from the initial disturbance to the final saturation amplitude. Even for a one-dimensional model, this takes a considerable amount of computation time, as the number of cycles to reach periodic steady-state is on the order of 100. To omit the computation of this initial transient, a method needs to be developed to directly compute the periodic steady-state of a dynamic system. The development of such a method is explained in Chapter (4), where it is applied to solve the classical problem [20, p. 66] of nonlinear plane progressive acoustic waves. In Chapter (5) the method of Chapter (4) is applied to a one-dimensional thermoacoustic model. A demonstration is shown, where the periodic steady-state of a standing wave thermoacoustic engine is simulated.

The foundation of the used mathematical models of Chapters (3) and (5) are laid out in Chapter (2). Finally, in Chapter (6), the conclusions from the individual chapters are summarized and recommendations for further research are given. As Chapters (3),(4) and (5) were originally written as journal contributions, the author apologizes for a small overlap in the content.

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ONE-DIMENSIONAL THERMOACOUSTIC THEORY

In this chapter, the mathematical modeling of thermoacoustic systems using one-dimensional theory is explained. This chapter is supposed to serve as a common basis and reference for the work described in Chapters (3) and (5).

2.1 Outline

The next section starts by presenting the governing equations, including the constitutive equations for the working gas and the solid in a TA system. The governing equations will be made dimensionless in Sec. (2.3). The dimensionless form of the governing equations is beneficial for the subsequent scaling analysis. However, for modeling of real systems, a dimensional form is more easy. Therefore, starting in Sec. (2.5), the equations are transformed back into their dimensional equivalent form.

An important simplification of the governing equations is made in Sec. (2.4), which is based on the preferred direction of motion due to the geometry of the segments in thermoacoustic systems. The required assumptions for this simplified model are all put under the common denominator *long pore approximation*, where a pore is a fluid channel with a small aspect ratio, which is the cross-sectional length scale divided by the pore length.

The governing equations can be linearized for small oscillation amplitudes. This is done in Sec. (2.5). In itself, linearizing the governing equations is not useful, because in thermoacoustic systems the interest lies in the nonlinear effects of finite amplitude waves, such as power and efficiency. However, the linear model is also the first step of a perturbation expansion of the governing equations in the acoustic Mach number. Taking this expansion to second order, the time-averaged effects of a finite amplitude wave can be estimated. Expanding the model to second order is the subject of Chapter (3). The linear thermoacoustic model can also be used to determine the stability limit of a thermoacoustic engine, which is important as it will tell whether a TA engine will run or not.

For higher oscillation amplitudes, nonlinear effects become important and have to be taken into account. In Sec. (2.6), a nonlinear one-dimensional thermoacoustic model will be derived. This model will be used in Chapter (5) to simulate the periodic steady-state of a standing wave engine.

2.2 Governing equations

The motion and state of a fluid are described by conservation of mass, momentum and energy, collectively called the governing equations for fluid dynamics. The governing equations are given in conservative vector form notation, assuming a Newtonian and thermally perfect gas. For a derivation of these equations, the reader is referred to the book by Bird,

Stewart and Lightfoot [4], or Landau and Lifshitz [13], or any other standard work on fluid mechanics. Body forces, such as the gravitational force, bulk viscosity and thermal radiation are all neglected. All dependent variables are a function of time (t) and position ($\mathbf{x}$) in a (yet undefined) reference coordinate system. For brevity, these dependencies are not stated in the equations, unless explicit attention is drawn to a dependency. The continuity equation is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (2.1)$$

where ρ is the density of the medium and $\mathbf{u}$ is the local (Eulerian) velocity vector. ∇ is the gradient operator and $\nabla \cdot$ takes the divergence of its operand. The momentum equation is

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \nabla \cdot \underline{\tau}, \quad (2.2)$$

where p is the pressure and $\underline{\tau}$ is the viscous stress tensor. The operator $\otimes$ denotes the dyadic product. The dyadic product of two vectors results in a second order tensor. The term $(\rho \mathbf{u} \otimes \mathbf{u})$ is referred to as the momentum flux tensor. For Newtonian fluids, the viscous stress tensor is

$$\underline{\tau} = \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{3}(\nabla \cdot \mathbf{u})\mathbf{I}), \quad (2.3)$$

where μ is the dynamic viscosity and $\mathbf{I}$ is the unit tensor. The superscript T denotes the transpose operator. μ is a function of the local temperature $\mu = \mu(T)$, where T is the temperature. For gases, the dynamic viscosity increases monotonically with temperature. The temperature dependence of the dynamic viscosity is assumed to be a power law with constant exponent b_μ [23, p. 45]:

$$\mu = \mu_0 \left(\frac{T}{T_0} \right)^{b_\mu}, \quad (2.4)$$

where subscript $_0$ denotes the reference state, for which the viscosity and temperature are given. More accurate models can be used with cryogenic applications in mind, for example Sutherland's law [14]. The fact that μ is temperature dependent results in a streaming source term in places with strong temperature gradients, such as the thermal buffer tube of a TA engine or the pulse tube in pulse tube refrigerators [16].

The energy equation is

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot (\rho \mathbf{u} H) = -\nabla \cdot \mathbf{q} + \nabla \cdot (\underline{\tau} \cdot \mathbf{u}), \quad (2.5)$$

where E is the total energy per unit mass

$$E = e(T) + \frac{1}{2} \|\mathbf{u}\|^2, \quad (2.6)$$

e the temperature-dependent internal energy

$$e(T) = \int_{T_0}^T c_v(T) dT, \quad (2.7)$$

where c_v is the specific heat at constant density and T_0 the reference temperature. H is the total enthalpy per unit mass

$$H = h(T) + \frac{1}{2}\|\mathbf{u}\|^2, \quad (2.8)$$

where h is the static enthalpy per unit mass

$$h = \int_{T_0}^T c_p(T) dT, \quad (2.9)$$

where c_p is the specific heat at constant pressure. In Eq. (2.5), $\mathbf{q}$ is the heat flux. Heat conduction can be modeled with Fourier's law of heat conduction, which states

$$\mathbf{q} = -\kappa \nabla T, \quad (2.10)$$

where $\kappa = \kappa(T)$ is the temperature-dependent thermal conductivity. This temperature dependence is modeled with a similar power law as Eq. (2.4):

$$\kappa = \kappa_0 \left(\frac{T}{T_0} \right)^{b_\kappa}. \quad (2.11)$$

The system of equations describing the gas dynamics is closed with the equation of state for a perfect gas:

$$p = \rho R_s T, \quad (2.12)$$

where R_s is the specific gas constant. For a perfect gas, the specific heats are related as $c_p = c_v + R_s$. Helium has specific heats which are nearly independent of temperature [17] ($c_p = 5195 \text{ J} \cdot \text{kg}^{-1} \cdot \text{K}^{-1}$). For air however, the specific heat at constant pressure is about 10% higher at 900 K than at 300 K [5]. In Chapter (4) the working gas is air at ambient conditions and in Chapter (5) helium is the working gas. As for both these situations the specific heat does vary significantly, the gas may safely be regarded as calorically perfect. For the second order heat exchanger model in Chapter (3), this assumption is not necessary and has therefore not been done.

The surrounding solid is assumed to be a non-moving rigid body, so a no-slip boundary condition is assumed at the interface between solid and fluid, which is called the (fluid) wall:

$$\mathbf{u}|_{\text{wall}} = \mathbf{0}, \quad (2.13)$$

The temperature in the solid T_s is governed by the heat equation

$$\rho_s c_s \frac{\partial T_s}{\partial t} = \kappa_s \nabla^2 T_s, \quad (2.14)$$

where $\nabla^2 \equiv \nabla \cdot \nabla$ is the Laplacian, c_s is the specific heat of the solid, ρ_s the solid's density and κ_s its thermal conductivity, which is assumed to be constant. Heat can freely flow between solid and fluid. At the solid-fluid interface, the temperature is assumed to be continuous $T_s|_{\text{wall}} = T|_{\text{wall}}$. The normal heat flux is also continuous at the interface:

$$-\kappa \nabla T|_{\text{wall}} \cdot \mathbf{n} = -\kappa_s \nabla T_s|_{\text{wall}} \cdot \mathbf{n}, \quad (2.15)$$

Figure 2.1 – Schematic drawing of the considered geometry. The transverse direction vector is $\mathbf{r} = (0, r_1, r_2)^T$ with associated length scale $r_{h,0}$. The axial length scale is x , with associated length scale L . $S_f(x)$ is the local cross-sectional area filled with fluid and $S_s(x)$ the cross-sectional area part occupied with solid.

where $\mathbf{n}$ is the unit normal vector pointing *from* the fluid *into* the solid. In case the length scale of the solid is much larger than the thermal boundary layer thickness in the solid, and the ratio of “thermal inertia” (square of ratio of thermal effusivity), is

$$\frac{\rho c_p \kappa}{\rho_s c_s \kappa_s} \ll 1, \quad (2.16)$$

then temperature changes in the solid on the time scale of the acoustic oscillations may be neglected [22]. In that case the solid can be assumed to be a perfect (temporary) buffer for heat. In Chapters (3) and (5) the temperature at the interface (wall) is called T_w .

2.3 Dimensionless governing equations

In this section, the governing equations are made dimensionless by scaling the dependent variables. Moreover, the coordinates are split into the wave propagation direction and the transverse direction. This allows for introducing the specific approximations done in Sec. (2.4).

Figure (2.1) shows a schematic of a typical domain of interest. The domain includes both solids and fluid. The position vector $\mathbf{x}$ is split into its components in the transverse directions and the axial (wave propagation) direction as

$$\mathbf{x} = x\mathbf{e}_x + \mathbf{r}, \quad (2.17)$$

where x is the coordinate in the wave propagation direction, $\mathbf{e}_x$ is the unit direction vector in this direction, and $\mathbf{r}$ the position in the transverse direction. As a result, the component of $\mathbf{r}$ in the direction $\mathbf{e}_x$ is zero. Applying this decomposition, the local velocity of the gas can be written as

$$\mathbf{u}(\mathbf{x}, t) = u(\mathbf{x}, t)\mathbf{e}_x + \mathbf{v}(\mathbf{x}, t), \quad (2.18)$$

Figure 2.2 – Drawing of three cross-sectional shapes with the corresponding definition of the hydraulic radius (r_h). The gray area in the tubes represents solid. For the free tube, the hydraulic radius changes as a function of axial position (x). For the left and middle case, the hydraulic radius and porosity is constant along x , although the total cross-sectional area can be a function of x .

where u is the velocity (scalar) in the wave propagation direction and $\mathbf{v}$ the velocity (vector) in the transverse directions.

The cross section of the domain has a total cross-sectional area $S(x)$, of which the part occupied by the fluid is $S_f(x) \equiv \phi(x)S(x)$, where ϕ is the position-dependent porosity, which is assumed to be $0 < \phi \leq 1$. Hence, the cross-sectional area part occupied by the solid is $S_s(x) \equiv (1 - \phi(x))S(x)$. The contact area between fluid and solid at the wall per unit of length is denoted by Π , which is also called the *wetted perimeter* (dimensions in m).

Figure (2.2) shows examples for the shape of the cross section. In the case of the left and middle configuration, fluid flows through separate channels, which are called *pores*. The hydraulic radius is a measure for the fluid's cross-sectional length scale. It is defined as

$$r_h = S_f / \Pi. \quad (2.19)$$

The fluid flow in all pores is assumed to be the same. Therefore, a single reference pore can be used as a representative for the total cross section. For example, the cross-sectional average of a quantity can be computed as:

$$\frac{1}{S_f} \int_{S_f} \dots dS = \frac{1}{S_{f, \text{single pore}}} \int_{S_{f, \text{single pore}}} \dots dS. \quad (2.20)$$

From now on, for area-averaging over the fluid cross section the abbreviated notation $\langle \dots \rangle$ will be used.

The geometry of Figure (2.1) introduces two length scales: L and $r_{h,0}$. L is used as the length scale for the wave propagation direction (x) and the reference hydraulic radius $r_{h,0}$

is used for the transverse direction ($\mathbf{r}$). The physical quantities are scaled according to

$$\begin{aligned}\hat{t} &= \omega_0 t, & \hat{x} &= x/L, & \hat{\mathbf{r}} &= \mathbf{r}/r_{h,0}, \\ \hat{T} &= T/T_0, & \hat{p} &= p/p_0, & \hat{\rho} &= \rho/\rho_0, \\ \hat{u} &= u/u_0, & \hat{\mathbf{v}} &= \mathbf{v}/u_0, & \hat{\kappa} &= \kappa/\kappa_0, \\ \hat{\mu} &= \mu/\mu_0,\end{aligned}\quad (2.21)$$

where the subscript $_0$ denotes the physical quantity at a chosen reference condition. As such, u_0 is a measure for the velocity in the system. The time is normalized with ω_0 , which is the reference oscillation frequency (in $\text{rad}\cdot\text{s}^{-1}$).

Using the scaling definition of Equations (2.21), the gradient operator is written in terms of the dimensionless gradient as

$$\nabla = r_{h,0}^{-1} \hat{\nabla} \equiv r_{h,0}^{-1} \left(\hat{\nabla}_T + \varepsilon \mathbf{e}_x \frac{\partial}{\partial \hat{x}} \right), \quad (2.22)$$

where ε is the aspect ratio

$$\varepsilon = \frac{r_{h,0}}{L}, \quad (2.23)$$

and $\hat{\nabla}_T$ is the dimensionless gradient operator in transverse direction only. Applying the scaling to the continuity equation yields

$$\frac{k_0}{M} \frac{\partial \hat{\rho}}{\partial \hat{t}} + \varepsilon \frac{\partial \hat{\rho} \hat{u}}{\partial \hat{x}} + \hat{\nabla}_T \cdot (\hat{\rho} \hat{\mathbf{v}}) = 0, \quad (2.24)$$

where $M = u_0/c_0$ is the reference acoustic Mach number, in which c_0 is the reference speed of sound. k_0 is called the reference *reduced frequency* [24]:

$$k_0 = \frac{\omega_0 r_{h,0}}{c_0}. \quad (2.25)$$

The momentum equation is split in the different directions. The dimensionless momentum equation in $\hat{x}$ -direction is

$$k_0 M \frac{\partial \hat{\rho} \hat{u}}{\partial \hat{t}} + M^2 \hat{\nabla} \cdot (\hat{\rho} \hat{u} \hat{\mathbf{u}}) + \frac{\varepsilon}{\gamma} \frac{\partial \hat{p}}{\partial \hat{x}} = \frac{M k_0}{s_0^2} \left(\hat{\nabla} \cdot \hat{\underline{\underline{\tau}}} \right)_x, \quad (2.26)$$

where

$$\hat{\underline{\underline{\tau}}} = \hat{T}^{b\mu} \left(\hat{\nabla} \hat{\mathbf{u}} + \left(\hat{\nabla} \hat{\mathbf{u}} \right)^T - \frac{2}{3} \left(\hat{\nabla} \cdot \hat{\mathbf{u}} \right) \mathbf{I} \right), \quad (2.27)$$

in which

$$\hat{\mathbf{u}} = u \mathbf{e}_x + \hat{\mathbf{v}}, \quad (2.28)$$

which is the full dimensionless velocity vector. The subscript $_x$ in Eq. (2.26) denotes the component in the x -direction. In Eq. (2.27), the relation for the dynamic viscosity (Eq. (2.4)) is used. s_0 is the reference *shear wave number*

$$s_0 = r_{h,0} \sqrt{\frac{\rho_0 \omega_0}{\mu_0}}, \quad (2.29)$$

and $\gamma = c_p/c_v$, the ratio of specific heats. Scaling of the dimensionless momentum equation in the transverse direction yields

$$Mk_0 \frac{\partial \hat{\rho} \hat{\mathbf{v}}}{\partial \hat{t}} + M^2 \left[\varepsilon \frac{\partial}{\partial \hat{x}} (\hat{\rho} \hat{\mathbf{u}} \hat{\mathbf{v}}) + \hat{\nabla}_T \cdot (\hat{\rho} \hat{\mathbf{v}} \otimes \hat{\mathbf{v}}) \right] + \frac{1}{\gamma} \hat{\nabla}_T \hat{p} = \frac{Mk_0}{s_0^2} (\hat{\nabla} \cdot \hat{\underline{\underline{\mathbf{t}}}})_T, \quad (2.30)$$

where the subscript T denotes the components in the T -direction. The dimensionless equation of state is

$$\hat{p} = \hat{\rho} \hat{T}, \quad (2.31)$$

and finally, the dimensionless energy equation becomes

$$\begin{aligned} & \frac{\partial}{\partial \hat{t}} \left[\hat{\rho} \left(\frac{\hat{T}}{\gamma - 1} + \gamma M^2 \frac{1}{2} \|\hat{\mathbf{u}}\|^2 \right) \right] + \frac{M}{k_0} \hat{\nabla} \cdot \left[\hat{\rho} \hat{\mathbf{u}} \left(\frac{\gamma}{\gamma - 1} \hat{T} + \gamma M^2 \frac{1}{2} \|\hat{\mathbf{u}}\|^2 \right) \right] \\ &= \frac{\gamma}{\gamma - 1} \frac{1}{\text{Pr}_0 s_0^2} \left(\hat{\nabla}_T \cdot (\hat{T}^{b_\kappa} \hat{\nabla}_T \hat{T}) + \varepsilon^2 \frac{\partial}{\partial \hat{x}} \left(\hat{T}^{b_\kappa} \frac{\partial \hat{T}}{\partial \hat{x}} \right) \right) + \frac{\gamma M^2}{s_0^2} \hat{\nabla} \cdot (\hat{\underline{\underline{\mathbf{t}}}} \cdot \hat{\mathbf{u}}), \end{aligned} \quad (2.32)$$

where Pr_0 is the reference Prandtl number.

It should be noted that Eqs. (2.24)-(2.32) are still the exact dimensionless equivalents of Eqs. (2.1), (2.2), (2.5) and (2.12).

Summarizing, the governing equations for the fluid and the constitutive equations have been made dimensionless and the momentum equation has been split into two parts, one for the wave propagation direction (x) and one for the transverse direction (T). Accordingly, the gradient is separated into the derivative in propagation direction and a transverse gradient. The dimensionless numbers are γ , Pr_0 , b_μ , b_κ , ε , k_0 , s_0 , and M . This is the same quantity of dimensionless numbers as used in the governing equations for the fluid in the similitude analysis carried out by Olson [15].

2.4 Long pore approximation

The cross-sectional dimensions of the segments in a thermoacoustic system are typically small compared to the wavelength, which makes the wave propagation in the system essentially one-dimensional. This can be used to simplify the governing equations. These assumptions are listed below:

- The cross-sectional area is assumed to vary slowly: $\frac{L}{S} \frac{dS}{dx} \ll 1$.
- The acoustic wavelength is small compared to the cross-sectional length scale: $k_0 \ll 1$.
- The viscous penetration depth is small compared to the acoustic wavelength: $k_0/s_0 \ll 1$.
- The velocity in transverse direction is small compared to the velocity in propagation direction: $|\hat{\mathbf{v}}|/|\hat{u}| \ll 1$.
- Long pores: $\varepsilon \ll 1$, so entrance effects may also be neglected.

- The wavelength is comparable to the segment length, or longer: $k_0/\varepsilon \leq 1$.
- The acoustic Mach number (M) is smaller than 10^{-1} .
- Laminar flow.
- The reference Prandtl number and the specific heat ratio are order unity: $\text{Pr}_0, \gamma \sim 1$.

Comparing the order of magnitude of the terms in the dimensionless governing equations and taking into account the list of assumptions, the continuity and momentum equations can be reduced to:

$$k_0 \frac{\partial \hat{\rho}}{\partial \hat{t}} + M \left(\varepsilon \frac{\partial \hat{\rho} \hat{u}}{\partial \hat{x}} + \hat{\nabla}_T \cdot (\hat{\rho} \hat{\mathbf{v}}) \right) = 0, \quad (2.33a)$$

$$k_0 M \frac{\partial \hat{\rho} \hat{u}}{\partial \hat{t}} + M^2 \hat{\nabla} \cdot (\hat{\rho} \hat{u} \hat{\mathbf{u}}) + \frac{\varepsilon}{\gamma} \frac{\partial \hat{p}}{\partial \hat{x}} = \frac{k_0 M}{s_0^2} \hat{\nabla}_T \cdot (\hat{T}^{b_\mu} \hat{\nabla}_T \hat{u}), \quad (2.33b)$$

$$\hat{\nabla}_T \hat{p} = 0. \quad (2.33c)$$

Hence, the pressure variation in the transverse direction can be neglected. The energy equation reduces to

$$\begin{aligned} & \frac{1}{\gamma - 1} \frac{\partial}{\partial \hat{t}} \left[\hat{\rho} \left(\hat{T} + \gamma M^2 \frac{1}{2} \hat{u}^2 \right) \right] + \frac{M}{k_0} \hat{\nabla} \cdot \left[\hat{\rho} \hat{\mathbf{u}} \left(\frac{\gamma}{\gamma - 1} \hat{T} + \gamma M^2 \frac{1}{2} \hat{u}^2 \right) \right] \\ &= \frac{\gamma}{\gamma - 1} \frac{1}{\text{Pr}_0 s_0^2} \left[\hat{\nabla}_T \cdot (\hat{T}^{b_\kappa} \hat{\nabla}_T \hat{T}) + \varepsilon^2 \frac{\partial}{\partial \hat{x}} \left(\hat{T}^{b_\kappa} \frac{\partial \hat{T}}{\partial \hat{x}} \right) \right] \\ & \quad + \frac{\gamma M^2}{s_0^2} \hat{\nabla}_T \cdot (\hat{u} \hat{T}^{b_\mu} \hat{\nabla}_T \hat{u}). \end{aligned} \quad (2.34)$$

The last term on the right-hand side of Eq. (2.34) is the only term left from the divergence of the viscous work flux. Due to Gauss's divergence theorem, the average over the cross section of this term equals zero, which means that area-averaged, the viscous work flow cancels out. It should be stressed that this does not mean that *viscous dissipation* is neglected. Viscous dissipation is the process of transforming kinetic energy to thermal energy, as this process does not change the *total energy*, the area-average of Eq. (2.34) does not include a source term for this process.

Eqs. (2.31), (2.33) and (2.34) are the basis for a variety of models. For example, for small sinusoidal oscillations, the classical one-dimensional linear thermoacoustic theory can be derived from this collection of equations. This will be done in Sec. (2.5) and the resulting equations are required in Chapter (3). By using a perturbation expansion in the acoustic Mach number, models for Rayleigh streaming [10] can be derived and relations can be obtained for the evolution of second order time-averaged quantities, such as acoustic power and thermoacoustic streaming [20].

By area-averaging Eqs. (2.33)-(2.34), a one-dimensional nonlinear model can be derived. This will be done in Sec. (2.6) and the resulting model is used in Chapter (5).

2.5 Linear thermoacoustics

For small disturbances around an equilibrium state, the governing equations may be linearized. This is the main assumption of the linear thermoacoustic model developed by Swift and Rott [19, 22], which will be derived in this section. For this, the acoustic Mach number is treated as a small quantity. A quiescent mechanical equilibrium state is assumed, but no thermal equilibrium. Hence the mean pressure is constant, but the mean temperature can vary in the domain. The disturbance is assumed to be harmonic in time. Based on these considerations, the following perturbation expansion is constructed:

$$\hat{\rho}(\hat{\mathbf{x}}, \hat{t}) = \rho_m(\hat{\mathbf{x}}) + M \Re \left[\hat{\rho}_1(\hat{\mathbf{x}}) e^{i\hat{t}} \right] + \mathcal{O}(M^2), \quad (2.35a)$$

$$\hat{u}(\hat{\mathbf{x}}, \hat{t}) = \Re \left[\hat{u}_1(\hat{\mathbf{x}}) e^{i\hat{t}} \right] + \mathcal{O}(M), \quad (2.35b)$$

$$\hat{\mathbf{v}}(\hat{\mathbf{x}}, \hat{t}) = \Re \left[\hat{\mathbf{v}}_1(\hat{\mathbf{x}}) e^{i\hat{t}} \right] + \mathcal{O}(M), \quad (2.35c)$$

$$\hat{p}(\hat{\mathbf{x}}, \hat{t}) = \hat{p}_m(\hat{x}) + M \Re \left[\hat{p}_1(\hat{x}) e^{i\hat{t}} \right] + \mathcal{O}(M^2), \quad (2.35d)$$

$$\hat{T}(\hat{\mathbf{x}}, \hat{t}) = \hat{T}_m(\hat{\mathbf{x}}) + M \Re \left[\hat{T}_1(\hat{\mathbf{x}}) e^{i\hat{t}} \right] + \mathcal{O}(M^2), \quad (2.35e)$$

for the dependent variables. $i^2 = -1$ and $\Re[\dots]$ denotes the real part of its argument. The notation $\Re[\dots e^{i\hat{t}}]$ is the commonly used phasor notation [23, p. 54]. The phasor notation can be converted to real sines and cosines using

$$\Re[\dots e^{i\hat{t}}] = |\dots| \cos(\hat{t} + \phi) \quad , \text{ with } \phi = \text{phase}(\dots). \quad (2.36)$$

To compute the phase angle ϕ , the inverse tangent of the imaginary part divided by the real part needs to be computed.¹

The mean equilibrium field is the zeroth order solution of the expanded governing equations, denoted with subscript m . The quiescent mean state is reflected in the absence of a zeroth order velocity term. Substituting Eqs. (2.35) in the dimensionless governing equations, and collecting terms up to zeroth order yields

$$\frac{\partial \hat{p}_m}{\partial \hat{x}} = 0, \quad (2.37a)$$

$$\hat{\nabla} \cdot \left(\hat{T}_m^{b_\kappa} \hat{\nabla} \hat{T}_m \right) = 0, \quad (2.37b)$$

$$\hat{p}_m = \hat{\rho}_m \hat{T}_m. \quad (2.37c)$$

The zeroth order continuity equation yields a $0 = 0$ identity. Eq. (2.37a) combined with Eq. (2.33c) state that the mean pressure is constant in the domain, as is expected from the

¹In many programming languages, a function is predefined which computes $\phi \in [0, 2\pi)$ for all four quadrants of the complex plane, avoiding the singularities of $\arctan(\dots)$. For example in **Matlab**, **Python** and **C++**, this function is called **atan2**.

mechanical equilibrium assumption. The solution of Eqs. (2.37) with their boundary conditions is called the mean state of the system.

The mean temperature and density field can be regarded as approximately independent of the transverse direction:

$$\hat{T}_m(\hat{\mathbf{x}}) \simeq \langle \hat{T}_m \rangle(\hat{x}), \quad (2.38a)$$

$$\hat{\rho}_m(\hat{\mathbf{x}}) \simeq \langle \hat{\rho}_m \rangle(\hat{x}). \quad (2.38b)$$

The proof of Eqs. (2.38) is given in Chapter (3), but they will be used in the current derivation. Moreover, the area-average of the value is taken for consistence with this chapter. $\hat{p}_m$ is a constant in both space and time, which value can be regarded as a known quantity. Here, $\hat{p}_m$ is set to unity, such that the reference pressure p_0 corresponds to the undisturbed pressure in the system. Therefore $\langle \hat{T}_m \rangle = \langle \hat{\rho}_m \rangle^{-1}$.

Next, small sinusoidal disturbances of the steady quiescent equilibrium state are considered, so the expansion is taken up to first order. Performing the substitution, eliminating the static enthalpy and internal energy, taking into account Eqs. (2.37), multiplying by i for the dimensionless time derivatives and dividing out the acoustic Mach number yields

$$ik_0\hat{\rho}_1 + \varepsilon \frac{\partial \langle \hat{\rho}_m \rangle \hat{u}_1}{\partial \hat{x}} = -\langle \hat{\rho}_m \rangle \hat{\nabla}_T \cdot \hat{\mathbf{v}}_1, \quad (2.39a)$$

$$ik_0 \langle \hat{\rho}_m \rangle \hat{u}_1 + \frac{\varepsilon}{\gamma} \frac{d\hat{p}_1}{d\hat{x}} = \frac{k_0}{s_0^2} \langle \hat{T}_m \rangle^{b_\mu} \hat{\nabla}_T^2 \hat{u}_1, \quad (2.39b)$$

$$\langle \hat{\rho}_m \rangle \left(i\hat{T}_1 + \frac{\varepsilon}{k_0} \hat{u}_1 \frac{d\langle \hat{T}_m \rangle}{d\hat{x}} \right) = i \frac{\gamma - 1}{\gamma} \hat{p}_1 + \frac{\langle \hat{T}_m \rangle^{b_\kappa}}{\text{Pr}_0 s_0^2} \hat{\nabla}_T^2 \hat{T}_1, \quad (2.39c)$$

$$\hat{p}_1 = \langle \hat{\rho}_m \rangle \hat{T}_1 + \hat{\rho}_1 \langle \hat{T}_m \rangle, \quad (2.39d)$$

where additionally, the small term

$$\frac{\varepsilon^2}{\text{Pr}_0 s_0^2} \frac{\partial}{\partial x} \left[\langle \hat{T}_m \rangle^{b_\kappa} \left(\frac{\partial \hat{T}_1}{\partial x} + b_\kappa \frac{\hat{T}_1}{\langle \hat{T}_m \rangle} \frac{\partial \langle \hat{T}_m \rangle}{\partial x} \right) \right] \quad (2.40)$$

due to axial conduction in the energy equation is neglected.

Eqs. (2.39) describe the propagation of small sinusoidal disturbances in tubes superimposed on an inhomogeneous mean temperature field, where heat conduction and viscous effects cannot be neglected. For a homogeneous mean temperature field ($\nabla \hat{T}_m = \mathbf{0}$), these equations reduce to an important set of equations describing viscothermal wave propagation in tubes with arbitrary cross-sectional shapes, called the low reduced frequency model [3]. Since the phasors are complex numbers, each of the Eqs. (2.39) is actually two equations: one for the real part and one for the imaginary part.

To find solutions of Eqs. (2.39), first the differential equation for $\hat{u}_1$ (Eq. (2.39b)) is partially solved for a given cross-sectional geometry to find $\hat{u}_1$ in terms of $\hat{\mathbf{r}}$ and $\partial \hat{p}_1 / \partial \hat{x}$. Then the differential equation (2.39c) for the acoustic temperature is partially solved and expressed in terms of $\hat{u}_1$ and $\hat{p}_1$. Finally, all equations are area-averaged over the fluid cross

section and their area-averaged expressions are substituted into the area-averaged form of Eq. (2.39a). This yields a second order differential equation for the acoustic pressure as a function of $\hat{x}$, $\langle \hat{T}_m \rangle$ and $\hat{S}_f(\hat{x})$, known as “Rott’s thermoacoustic wave equation.” [23, p. 91]

In order to implement the theory, further analysis in dimensional form is required. The dimensional form is obtained by backward substitution and re-scaling of the equations. The dimensional equivalent of Eq. (2.39b) is

$$i\omega \langle \rho_m \rangle u_1 + \frac{dp_1}{dx} = \langle \mu_m \rangle \nabla_T^2 u_1. \quad (2.41)$$

Filling in the trial function

$$u_1 = \langle u_1 \rangle \frac{1 - h_\nu(s, \mathbf{r})}{1 - f_\nu(s)}, \quad (2.42)$$

where f_ν is $\langle h_\nu \rangle$, into Eq. (2.41), and using the no-slip boundary condition $u_1|_{\text{wall}} = 0$, it can be found that

$$\langle u_1 \rangle = \frac{i(1 - f_\nu)}{\langle \rho_m \rangle \omega} \frac{dp_1}{dx}, \quad (2.43)$$

when h_ν satisfies the differential equation

$$\frac{ir_h^2}{s^2} \nabla_T^2 h_\nu + h_\nu = 0, \quad (2.44)$$

where s is the local shear wave number

$$s = r_h \sqrt{\frac{\langle \rho_m \rangle \omega}{\langle \mu_m \rangle}}. \quad (2.45)$$

From Eq. (2.42) the no-slip boundary condition results in the following boundary condition for h_ν :

$$h_\nu|_{\text{wall}} = 1. \quad (2.46)$$

The function h_ν describes the shape of the axial velocity profile as a function of the transverse position. Both the magnitude and the phase of the velocity are a function of $\mathbf{r}$. Solutions for h_ν for a variety of cross-sectional geometries are derived by Arnott [2]. For example, for parallel plates the solution for h_ν is

$$h_\nu(s, y) = \frac{\cosh(\sqrt{i} sy/r_h)}{\cosh(\sqrt{i} s)}, \quad (2.47)$$

where y is zero halfway between the plates and the corresponding hydraulic radius is half the plate spacing. The corresponding $f_\nu(s)$ is found by taking the area-average of Eq. (2.47):

$$f_\nu(s) = \frac{\tanh(\sqrt{i} s)}{\sqrt{i} s}. \quad (2.48)$$

Figure 2.3 – Velocity profiles at $\hat{t} = 0$ (—), $\hat{t} = \frac{1}{4}\pi$ (— — —), $\hat{t} = \frac{1}{2}\pi$ (- - -) and $\hat{t} = \frac{3}{4}\pi$ (- · - · -) for a parallel plate cross section at different shear wave numbers. For $\hat{t} = \pi, \frac{5}{4}\pi, \frac{3}{2}\pi$ and $\frac{7}{4}\pi$, the profiles are mirrored around the axis at $\hat{u}/\langle\hat{u}_1\rangle$ of $\hat{t} = 0, \frac{1}{4}\pi, \frac{1}{2}\pi$ and $\frac{3}{4}\pi$, respectively.

Figure (2.3) shows the cross-sectional velocity profile for this parallel-plate geometry for different shear wave numbers and phases. The velocity profiles are scaled with the area-averaged velocity $\langle\hat{u}_1\rangle$ which has a reference phase of 0, so the velocity profile is evaluated as

$$\frac{\hat{u}(\hat{t})}{\langle\hat{u}_1\rangle} = \Re \left[\frac{1 - h_\nu(s, y)}{1 - f_\nu(s)} e^{i\hat{t}} \right]. \quad (2.49)$$

In the limit of $s \rightarrow 0$, the velocity profile becomes parabolic and coincides with the quadratic profile corresponding to the steady-state Hagen-Poiseuille solution [4, p. 51]. In the limit of $s \rightarrow \infty$, the profile corresponds to a plug flow profile, with a negligible boundary layer thickness.

As Figure (2.3) shows, the size of the viscous boundary layer is smaller for higher shear wave numbers. It is worth noting that the velocity in the boundary layer “leads” the velocity in the bulk in phase. Moreover, the velocity amplitude at the outside of the boundary layer is slightly higher than in the bulk. This result seems to be counter-intuitive, because the shear force at the wall damps the fluid motion. Literature results which show the velocity profile as a function of time are rare, but Gedeon [8] shows a plot of this velocity profile, which is in agreement with this result. The correctness of the model for cylindrical tube geometries has been experimentally validated by Harris et al. [11]. In these experiments, a similar overshoot in the boundary layer is found. More recently, the fact that the velocity in the boundary layer leads the bulk in phase has been experimentally confirmed with the use of Particle Image Velocimetry by Aben [1]. Although the literature results show that

Fluid	Solid	$\delta_{\kappa,s}$	$\delta_{\kappa,0}$	$\sqrt{\frac{\rho_0 c_p \kappa_0}{\rho_s c_s \kappa_s}}$
Helium (40 bar)	Stainless steel	$1 \cdot 10^{-4}$ m	$1 \cdot 10^{-4}$ m	$1 \cdot 10^{-2}$
Helium (40 bar)	Copper	$6 \cdot 10^{-4}$ m	$1 \cdot 10^{-4}$ m	$2 \cdot 10^{-3}$
Helium (40 bar)	Kapton™	$2 \cdot 10^{-5}$ m	$1 \cdot 10^{-4}$ m	$1 \cdot 10^{-1}$
Air (atmospheric)	Stainless steel	$1 \cdot 10^{-4}$ m	$3 \cdot 10^{-4}$ m	$8 \cdot 10^{-4}$

Table 2.1 – Thermal inertia ratio for three different solid-fluid combinations. $\delta_{\kappa,s}$ is the thermal penetration depth in the solid and $\delta_{\kappa,0}$ is the reference thermal penetration depth in the fluid. The frequency for computing the δ_{κ} 's is 100 Hz and the reference temperature is 293.15 K. Numbers are rounded up to one figure. Data for the solids and fluids is taken from [26].

the model is indeed correct, a satisfying physical explanation of this phenomenon is still required.

For segments with varying cross-sectional area, the acoustic volume flow phasor U_1 is a more convenient quantity, because up to first order in the acoustic Mach number, the volume flow is conserved at the segment interfaces. In terms of the volume flow, the solution of the first order momentum equation is

$$U_1 = S_f \frac{i(1 - f_\nu)}{\langle \rho_m \rangle \omega} \frac{dp_1}{dx}. \quad (2.50)$$

The local velocity in the propagation direction is related to U_1 by

$$u_1 = \frac{\beta}{S_f} U_1, \quad \text{with} \quad \beta = \frac{1 - h_\nu}{1 - f_\nu}. \quad (2.51)$$

In a similar way as the solution for u_1 is obtained, the first order temperature perturbation can be expressed as a function of the acoustic pressure and area-averaged velocity phasor as

$$T_1 = \frac{\alpha_1}{\langle \rho_m \rangle c_p} p_1 + \alpha_2 \frac{i}{\omega} \frac{d\langle \hat{T}_m \rangle}{dx} \langle u_1 \rangle. \quad (2.52)$$

As the acoustic pressure and area-averaged velocity are independent of the position in the cross section, the terms α_1 and α_2 are the only terms responsible for such a spatial dependence. The similarity between Eq. (2.39a) and (2.39c) suggests that the spatial dependence of the acoustic temperature has a shape which closely resembles the shapes of Figure (2.3). An important difference is the boundary condition at the wall. Whereas the velocity boundary condition is zero due to the no-slip boundary condition, the solid's heat equation needs to be taken into account to find the, in general, nonzero temperature oscillation at the wall. However, as was explained in Sec. (2.2), when the thermal mass of the solid is much larger than the thermal mass of the fluid, temperature oscillations in the solid, and hence on the wall, can be neglected.

Table (2.1) shows results of the thermal inertia ratio (Eq. (2.16)) for three gas/solid combinations. For the three metal combinations, temperature oscillations in the solid can be

neglected when the thickness of the metal layer is larger than $\delta_{\kappa,s}$. This assumption does not hold for stack/regenerator material such as Kapton™. For parallel plates, the first order heat equation including the coupling with the solid was solved by Swift [22]. He developed an “ideal stack correction factor”, ϵ_s which accounts for the effect of a small temperature oscillation at the pore boundary. For parallel-plate pores ϵ_s is

$$\epsilon_s = \sqrt{\frac{\langle \kappa_m \rangle \langle \rho_m \rangle c_p}{\kappa_s \rho_s c_s}} \frac{\tanh(\sqrt{i \text{Pr}} s)}{\tanh(\sqrt{2i} \ell / \delta_{\kappa,s})}, \quad (2.53)$$

where ℓ is the stack plates half-width and Pr denotes the Prandtl number

$$\text{Pr} = \text{Pr}_0 \left(\langle \hat{T}_m \rangle / T_0 \right)^{b_\mu - b_\kappa}. \quad (2.54)$$

It is assumed that, for any geometry, ϵ_s can be defined such that to find the temperature distribution by solving the dimensional form of Eq. (2.39c), the $T_1|_{\text{wall}} = 0$ can be used. Hence we assume that for other cross-sectional geometries, ϵ_s can be defined. In that case, α_1 and α_2 are found to be

$$\alpha_1 = 1 - \frac{h_\kappa}{1 + \epsilon_s}, \quad \text{and} \quad \alpha_2 = \frac{1 - \lambda h_\kappa - \text{Pr}(1 - h_\nu)}{(1 - f_\nu)(1 - \text{Pr})}, \quad (2.55)$$

in which λ is

$$\lambda = \frac{1 + \frac{\epsilon_s f_\nu}{f_\kappa}}{1 + \epsilon_s}, \quad (2.56)$$

and $f_\kappa = \langle h_\kappa \rangle$ and h_κ the solution of the differential equation

$$\frac{i r_h^2}{\text{Pr} s^2} \nabla_T^2 h_\kappa + h_\kappa = 0, \quad (2.57)$$

subject to

$$h_\kappa|_{\text{wall}} = 1.$$

The functions f_ν and f_κ are called the viscous and thermal Rott function, respectively. For solids having $\kappa_s \rightarrow \infty$ or $c_s \rightarrow \infty$, the ideal stack correction factor $\epsilon_s \rightarrow 0$. As a consequence $\lambda \rightarrow 1$. For this case, it can be shown that the acoustic temperature oscillation at the wall $T_1|_{\text{wall}} = 0$.

To obtain differential equations for the acoustic pressure and volume flow only, the acoustic temperature phasor T_1 can be eliminated by substituting its expression (Equation 2.52) into the continuity equation. By utilizing the first order equation of state, an equation can be derived for the derivative of the acoustic volume flow phasor:

$$\frac{dU_1}{dx} = \frac{\omega S_f}{i \gamma p_m} \left(1 + \frac{(\gamma - 1) f_\kappa}{1 + \epsilon_s} \right) p_1 + \frac{f_\kappa - f_\nu}{(1 - f_\nu)(1 - \text{Pr})(1 + \epsilon_s)} \frac{1}{\langle \hat{T}_m \rangle} \frac{d\langle \hat{T}_m \rangle}{dx} U_1. \quad (2.58)$$

It is noted that this result is exactly the same as the result of Ward et al. [26] when the mean temperature is assumed to be constant over the cross section.

When the mean temperature distribution and the mean pressure are known, the linear thermoacoustic theory yields two one-dimensional equations

$$\frac{dp_1}{dx} = g(\omega, p_1, U_1, \langle \hat{T}_m \rangle, \text{geometry}), \quad (2.59)$$

$$\frac{dU_1}{dx} = h(\omega, U_1, \langle \hat{T}_m \rangle, \text{geometry}), \quad (2.60)$$

to find the acoustic pressure and volume flow distribution in a thermoacoustic system. Generally, these differential equations need to be solved numerically. However, for the special situation of a homogeneous mean temperature field, analytical solutions exist. Some solutions of value for thermoacoustic systems will be shown.

Isothermal segments

Setting $d\langle T_m \rangle/dx = 0$ in Eq. (2.58) and substituting the result into Eq. (2.50) results in a frequency domain variant of Webster's horn equation [18, p. 360] which includes losses:

$$\frac{1}{S_f} \frac{d}{dx} \left(S_f (1 - f_\nu) \frac{dp_1}{dx} \right) + \frac{\omega}{c_m} \left(1 + \frac{(\gamma - 1)f_\kappa}{1 + \epsilon_s} \right) p_1 = 0, \quad (2.61)$$

where c_m is the small-signal speed of sound

$$c_m = \sqrt{\gamma R_s T_m}. \quad (2.62)$$

If

$$\left| \frac{d(1 - f_\nu)}{dx} \right| \ll \left| \frac{1}{S_f} \frac{dS_f}{dx} \right|, \quad (2.63)$$

which is valid for wide pores and slowly varying cross sections, then a reference value for f_ν, f_κ and ϵ_s can be taken, such that their dependence on x can be neglected. Applying this assumption, Eq. (2.61) reduces to

$$\frac{d^2 p_1}{dx^2} + \frac{1}{S_f} \frac{dS_f}{dx} \frac{dp_1}{dx} + \Gamma_{\text{ref}}^2 p_1 = 0, \quad (2.64)$$

where

$$\Gamma_{\text{ref}} = \frac{\omega}{c_m} \sqrt{\frac{1 + \frac{(\gamma - 1)f_{\kappa, \text{ref}}}{1 + \epsilon_{s, \text{ref}}}}{1 - f_{\nu, \text{ref}}}}, \quad (2.65)$$

is the wave number and subscript ref denotes a reference (position). Hence f_ν, f_κ and ϵ_s need to be evaluated at a certain reference position. Considering viscothermal damping of the acoustic wave, a reference position should be chosen such that the reference shear wave number is somewhere between its extremes.

For conical tubes, the radius varies linearly with the position: $r(x) = r_0 + \alpha x$, where α is the tangent of the expansion angle and r_0 is the radius at $x = 0$. Hence, the cross-sectional area varies as

$$S_f = S_{f,0}(1 + \alpha x)^2. \quad (2.66)$$

Using Eq. (2.66), the acoustic pressure phasor is found as

$$p_1(x) = A \frac{e^{-i\Gamma_{\text{ref}} x}}{r_0 + \alpha x} + B \frac{e^{i\Gamma_{\text{ref}} x}}{r_0 + \alpha x}, \quad (2.67)$$

where A and B are two integration constants that have to be determined from the boundary conditions. For prismatic tubes, $\alpha = 0$, Γ is constant and the solution for p_1 is

$$p_1(x) = A e^{-i\Gamma x} + B e^{i\Gamma x}, \quad (2.68)$$

which is known as the low reduced frequency solution [3] for visco-thermal wave propagation in tubes.

Short regenerators and stacks

Similarly, an estimate of the propagation of waves through a stack can be made. Aben [1, p. 21] has derived a model for this. For a constant cross-sectional area, a small constant mean temperature gradient and a short stack or regenerator ($L \ll |\Gamma^{-1}|$), the following approximate differential equation for the acoustic pressure can be derived

$$\frac{d^2 p_1}{dx^2} - F_{\text{ref}} \frac{dp_1}{dx} + \Gamma_{\text{ref}}^2 p_1 = 0, \quad (2.69)$$

where

$$F_{\text{ref}} = \left[\frac{f_\kappa - f_\nu}{(1 - f_\nu)(1 - \text{Pr})(1 + \epsilon_s)} \frac{1}{\langle \hat{T}_m \rangle} \frac{d\langle \hat{T}_m \rangle}{dx} \right]_{\text{ref}}, \quad (2.70)$$

in which again, a reference value for $f_\kappa, f_\nu, \epsilon_s$, etc. needs to be chosen. A good choice is to evaluate these parameters halfway through the stack, where $\langle \hat{T}_m \rangle = \frac{1}{2}(T_H + T_L)$, then for the acoustic pressure it is found that

$$p_1(x) = \left[A \exp\left(-\sqrt{\frac{1}{4}F_0^2 - \Gamma_0^2} x\right) + B \exp\left(\sqrt{\frac{1}{4}F_0^2 - \Gamma_0^2} x\right) \right] \exp\left(\frac{1}{2}F_0 x\right). \quad (2.71)$$

The acoustic power generation in the stack or regenerator and dissipation in the other segments can be estimated by evaluating [23, p. 106]

$$E_2 = \frac{\omega}{2\pi} \int_{t=0}^{\frac{2\pi}{\omega}} p(t)U(t)dt = \frac{1}{2} \Re[p_1 U_1^*]. \quad (2.72)$$

Discussion

Eqs. (2.67), (2.68), and (2.71), are very useful relations in the preliminary phase of the design of a thermoacoustic system. Acoustic networks can quickly be analyzed by connecting segments, such as a regenerator, a resonator and an inertance/compliance tube. At the segment connections, first order continuity of momentum and energy demands continuity of acoustic pressure and conservation of acoustic volume flow, where the volume flow can be obtained from the acoustic pressure with Eq. (2.50). Boundary conditions are $U = 0$ for a wall and $U_1/S_f = u_p$ for a driving piston, where u_p is the piston velocity phasor. Impedance boundary conditions $p_1/U_1 = Z$ can be utilized to model open end, or acoustic radiation boundary conditions.

For modeling of thermoacoustic engines, the approach is a bit different. A thermoacoustic engine is not “driven” and therefore does not have boundary conditions where the acoustic pressure or volume flow is prescribed other than zero. Moreover, the oscillating frequency is unknown and therefore part of the solution. Instead, thermoacoustic engines are self-excited by the instability of an acoustic eigenmode. Using linear thermoacoustic theory, the stability of acoustic eigenmodes can be determined by finding a nonzero solution of the resulting system of equations, which satisfies all boundary conditions. This is possible for certain ω 's, which are the acoustic eigenfrequencies. As a time response for the acoustic pressure as $p(x, t) = \Re[p_1(x)e^{i\omega t}]$ is assumed, $\Im[\omega] < 0$ yields unbounded growth of the amplitude [9]. In reality, due to nonlinear effects the amplitude saturates at a certain level, called the saturation amplitude. To find this saturation amplitude, nonlinear effects need to be taken into account. A model which includes these nonlinear effects is presented in Sec. (2.6).

As a part of this PhD work, a numerical implementation of the linear thermoacoustic theory was built. With the code, predictions can be made of the stability limit of thermoacoustic engines. One-dimensional thermoacoustic segments are coupled with the boundary conditions as described earlier in this section. A system of equations is created using the transfer matrix method [7, 25]. The implementation is validated using a small atmospheric traveling wave thermoacoustic demonstration device. This work is described in the master thesis of Wilmink [28].

2.6 A nonlinear thermoacoustic model

In this section, a one-dimensional nonlinear thermoacoustic model is derived from the long pore approximation of Sec. (2.4). This model is used in Chapter (5) to simulate the periodic steady-state of an experimental standing wave thermoacoustic engine. The model is based on the work of Watanabe et al. [27], Yuan et al. [29], and Karpov et al. [12]. By area-averaging the equations obtained after the long pore approximation, a one-dimensional model is obtained, which contains source terms for the thermoacoustic exchange of momentum and energy with the surrounding solid. The closure which is proposed, is to derive these exchange terms from linear thermoacoustic theory. These terms will be derived first. In this section, the analysis will be done in dimensional form.

Figure 2.4 – Logarithmic plot of the frequency-dependent part of the viscous resistance coefficient for parallel plate pores vs shear wave number (normalized frequency). The real part is (—) and the imaginary part is (---).

The dimensional form of the first order momentum equation (Eq. (2.39b)) is

$$i\omega\langle\hat{\rho}_m\rangle u_1 + \frac{dp_1}{dx} = \langle\mu_m\rangle \nabla_T^2 u_1. \quad (2.73)$$

Integrated over the cross section and using Gauss's divergence theorem, this becomes

$$i\omega\langle\hat{\rho}_m\rangle U_1 + S_f \frac{dp_1}{dx} = -\mathcal{R}_1, \quad (2.74)$$

where the term on the right-hand side is

$$-\mathcal{R}_1 = \oint_{\Pi} \langle\mu_m\rangle \nabla_T u_1 \cdot \mathbf{n} d\Pi. \quad (2.75)$$

To obtain this term, Gauss's divergence theorem has been used. $-\mathcal{R}_1$ corresponds to the shear force that the wall exerts on the fluid (in $\text{N}\cdot\text{m}^{-1}$). The convention is chosen such that, in the low-frequency limit, $\mathcal{R}$ is a positive number when the shear force counteracts the flow. Comparing Eq. (2.74) with Eq. (2.50) yields an explicit definition for $\mathcal{R}_1$:

$$\mathcal{R}_1 = R_1 U_1 \equiv \frac{\langle\mu_m\rangle}{r_h^2} \frac{is^2 f_v}{1 - f_v} U_1. \quad (2.76)$$

The factor $is^2 f_v/(1 - f_v)$ in R_1 depends on the frequency through the shear wave number.

Figure (2.4) shows a logarithmic plot of this factor as a function of the shear wave number for parallel-plate pores. For low frequency oscillations, the Poiseuille flow limit dictates that the resistance is in phase with the volume flow, and the factor becomes nearly independent of frequency. In Sec. (5.2) limits of $\mathcal{R}_1$ for low frequencies are derived for various pore geometries. Keeping the hydraulic radius constant, for $s \geq 10$, the viscous resistance increases significantly with frequency, as can be seen in Figure (2.4). Hence, higher harmonics are more strongly damped compared to the fundamental frequency. Moreover, for high frequencies the resistance coefficient rotates in the complex plane, resulting in dispersion.

For the first order energy equation, the solution for the acoustic temperature perturbation can be characterized in a similar way. If the total energy equation (Eq. (2.5)) is expanded up to first order, the long pore approximation is applied and the result is integrated over the cross-sectional area, it is found that

$$i\omega c_v S_f (\langle T_m \rangle \langle \rho_1 \rangle + \langle \rho_m \rangle \langle T_1 \rangle) + \frac{d}{dx} (c_p \langle \rho_m \rangle \langle T_m \rangle U_1) = \mathcal{Q}_1, \quad (2.77)$$

where

$$\mathcal{Q}_1 = - \oint_{\Pi} \mathbf{q}_1 \cdot \mathbf{n} d\Pi \quad (2.78)$$

is the heat flow *from* the solid *to* the fluid. If $\langle \rho_1 \rangle$ is eliminated from Eq. (2.77) by using the area-averaged, dimensional form of Eq. (2.39a), and the result is compared with Eq. (2.52) integrated over the cross section, $\mathcal{Q}_1$ is found to be

$$\mathcal{Q}_1 = -S_f H_1 \langle T_1 \rangle - Q_1 \frac{d\langle T_m \rangle}{dx} U_1, \quad (2.79)$$

where H_1 is

$$H_1 = S_f \frac{\langle \kappa_m \rangle \text{Pr}}{r_h^2} \frac{is^2 \frac{f_\kappa}{1+\epsilon_s}}{1 - \frac{f_\kappa}{1+\epsilon_s}}, \quad (2.80)$$

and Q_1 is

$$Q_1 = \frac{\langle \rho_m \rangle c_p}{(1 - \text{Pr})(1 + \epsilon_s)} \left[\frac{f_\kappa}{1 - \frac{f_\kappa}{1+\epsilon_s}} \left(\frac{1 - \lambda f_\kappa - \text{Pr}(1 - f_\nu)}{1 - f_\nu} \right) - \frac{f_\kappa - f_\nu}{1 - f_\nu} \right]. \quad (2.81)$$

Note that for walls with large thermal inertia $\lambda \rightarrow 1$ and $\epsilon_s \rightarrow 0$. For that case Q_1 reduces to

$$Q_1 = \frac{\langle \rho_m \rangle c_p}{1 - \text{Pr}} \left(\frac{f_\nu}{1 - f_\nu} - \text{Pr} \frac{f_\kappa}{1 - f_\kappa} \right), \quad (2.82)$$

which is in agreement with the result of Karpov et al. [12, Eq. (14)].

Now, an explicit form is available for the first order viscous resistance and heat transfer terms. It should be noted that the second term in Eq. (2.79) is a bit strange since it originated from first order enthalpy convection in the x -direction (second term in Eq. (2.39c) on the left-hand side), which does not have anything to do with heat exchange with the wall. However, when a closer look is given to Eq. (2.77), using the result for Q_1 , the first order continuity and state equation, this equation can be rewritten as

$$\langle \rho_m \rangle c_p \left[i\omega S_f \langle T_1 \rangle + \left(1 + \frac{Q_1}{\langle \rho_m \rangle c_p} \right) \frac{d\langle T_m \rangle}{dx} U_1 \right] = i\omega S_f p_1 - S_f H_1 \langle T_1 \rangle. \quad (2.83)$$

For an inviscid fluid, $\text{Pr} = 0$, $f_\nu = 0$, and $\lambda = (1 + \epsilon_s)^{-1}$, resulting in $Q_1 = 0$, so Q_1 originates from the viscous velocity boundary layer. It can be regarded as a correction to the first order convective enthalpy flux derivative resulting from plug flow, which is $\langle \rho_m \rangle c_p \frac{d\langle T_m \rangle}{dx} U_1$.

If the continuity equation is area-averaged the result is:

$$S_f \frac{\partial \langle \rho \rangle}{\partial t} + \frac{\partial S_f \langle \rho u \rangle}{\partial x} = 0. \quad (2.84)$$

Similarly the area-averaged x -momentum equation yields

$$S_f \frac{\partial \langle \rho u \rangle}{\partial t} + \frac{\partial S_f \langle \rho u^2 \rangle}{\partial x} + S_f \frac{\partial p}{\partial x} = -\mathcal{R}, \quad (2.85)$$

where

$$\mathcal{R} = - \oint_{\Pi} \mu \nabla_T u \cdot \mathbf{n} d\Pi. \quad (2.86)$$

The area-averaged energy equation results in

$$S_f \frac{\partial}{\partial t} \left(\frac{p}{\gamma - 1} + \frac{1}{2} \langle \rho u^2 \rangle \right) + \frac{\partial}{\partial x} \left[S_f \left(\frac{\gamma}{\gamma - 1} p \langle u \rangle + \frac{1}{2} \langle \rho u^3 \rangle \right) \right] = \mathcal{Q} + \frac{\partial}{\partial x} \left(S_f \left\langle \kappa \frac{\partial T}{\partial x} \right\rangle \right) \quad (2.87)$$

where

$$\mathcal{Q} = \oint_{\Pi} \kappa \nabla_T T \cdot \mathbf{n} d\Pi. \quad (2.88)$$

Finally, the area-averaged state equation yields

$$p = R_s \langle \rho T \rangle. \quad (2.89)$$

Although Eqs. (2.84)-(2.89) are one-dimensional, they contain terms involving area-averaged products of the dependent variables. Moreover, explicit relations for the exchange terms $\mathcal{R}$ and $\mathcal{Q}$ are required. To assist the process of obtaining explicit relations for $\mathcal{R}$ and $\mathcal{Q}$ and the product of area-averaged quantities, one can think of two limits. First, in the limit of an inviscid non-conducting fluid, both $\mathcal{R} \rightarrow 0$ and $\mathcal{Q} \rightarrow 0$ and the area-average of the product of two dependent variables equals the product of the area-averaged. Hence $\langle \rho u \rangle = \langle \rho \rangle \langle u \rangle$, $\langle \rho u^2 \rangle = \langle \rho \rangle \langle u^2 \rangle$, $\dots$. As a result, Eqs. (2.84-2.89) reduce to the quasi one-dimensional Euler equations. Secondly, in the limit of $M \rightarrow 0$, the model should be consistent with the linear thermoacoustic model. This is possible when a special choice for $\mathcal{R}$ and $\mathcal{Q}$ is made. If $\mathcal{R}$ is chosen as:

$$\mathcal{R}(t) = S_f \int_{\omega=-\infty}^{\infty} R_1 \langle \tilde{u} \rangle e^{i\omega t} d\omega, \quad (2.90)$$

where $\langle \tilde{u} \rangle(\omega)$ is the Fourier transform of $\langle u \rangle$:

$$\langle \tilde{u} \rangle(\omega) = \mathcal{F}(\langle u \rangle(t)) = \frac{1}{2\pi} \int_{t=-\infty}^{\infty} \langle u \rangle e^{-i\omega t} dt, \quad (2.91)$$

then in the limit of $M \rightarrow 0$, the momentum equation is consistent with linear thermoacoustic theory. This can be proven by substituting Eq. (2.90) into Eq. (2.85), linearizing for small perturbations and applying the Fourier transform. Note that an asymmetric Fourier transform is used, such that

$$u(t) = \mathcal{F}^{-1}(\tilde{u}(\omega)) \equiv \int_{\omega=-\infty}^{\infty} \tilde{u}(\omega) e^{i\omega t} d\omega. \quad (2.92)$$

Similarly, if $\mathcal{Q}$ is

$$\mathcal{Q}(t) = S_f \int_{\omega=-\infty}^{\infty} \left[H_1 \left(\tilde{T}_w - \langle \tilde{T} \rangle \right) - Q_1 \frac{dT_w}{dx} \langle \tilde{u} \rangle \right] e^{i\omega t} d\omega, \quad (2.93)$$

after linearizing, the energy equation reduces to the result of linear theory. Note that R_1, Q_1 and H_1 depend on the shear wave number, which in turn is proportional to the square root of ω , according to Eq. (2.45). T_w is the time-average of the wall temperature, which is in the limit of $M \rightarrow 0$ and taking into account Eq. (2.38a), equal to $\langle T_m \rangle$. Hence $\tilde{T}_w$ is the Fourier transform of the time-average wall temperature:

$$\tilde{T}_w(\omega) = T_w \delta(\omega), \quad (2.94)$$

where δ is the Dirac delta function.

To make Eqs. (2.84)-(2.89) solvable, the following assumptions are applied:

$$\langle \rho u \rangle \simeq \langle \rho \rangle \langle u \rangle, \quad (2.95a)$$

$$\langle \rho u^2 \rangle \simeq \langle \rho \rangle \langle u \rangle^2, \quad (2.95b)$$

$$\langle \rho u^3 \rangle \simeq \langle \rho \rangle \langle u \rangle^3, \quad (2.95c)$$

$$\left\langle \kappa \frac{\partial T}{\partial x} \right\rangle \simeq \kappa \langle \langle T \rangle \rangle \frac{\partial \langle T \rangle}{\partial x}. \quad (2.95d)$$

as was originally proposed by Watanabe et al. [27]. In this work, it was shown, by comparison with second order thermoacoustic theory (Chapter (3)), that the error introduced by these assumptions is at maximum 20%:

$$\max \left| \frac{\langle \rho u \rangle - \langle \rho \rangle \langle u \rangle}{\langle \rho u \rangle} \right| \simeq 0.2, \quad (2.96)$$

for segments with hydraulic radius of order δ_ν . For tubes with $r_h \gg \delta_\nu$ and for $r_h \ll \delta_\nu$, Watanabe et al. [27] show that the assumption holds well, in view of the approximations already made using the long pore approximation and the linear theory relations for $\mathcal{R}$ and $\mathcal{Q}$. Moreover, in the boundary layer limit, with $dT_w/dx = 0$, this model is consistent with the nonlinear time domain model of Chester [6], who developed a model for resonant oscillations in closed tubes [21].

2.7 Discussion and conclusions

In this chapter, the long pore approximation of the governing fluid dynamic equations has been derived. From this approximation, a linear thermoacoustic theory is derived, which is extensively used in designing thermoacoustic systems [23]. From linear theory, segment models can be derived which describe acoustic wave propagation through, for example, tubes with constant cross section and regenerators and stacks with regular pore shapes.

From linear theory, exchange terms are derived for the exchange of momentum and heat with the surrounding solid. These terms are used to model momentum and heat exchange in the one-dimensional nonlinear thermoacoustic model. This nonlinear thermoacoustic model is consistent with three limiting situations of one-dimensional flow:

- Quasi 1D Euler equations
- Laminar steady pipe flow
- Linear thermoacoustics

The model includes nonlinear effects such as harmonic generation and amplitude saturation of the thermoacoustic instability. Moreover, the dispersive effects of viscous resistance and thermal relaxation are taken into account. In addition, the thermoacoustic effect is modeled, which amplifies the wave in segments with a steep temperature gradient.

This nonlinear model is, however, *not* consistent with the second order heat transfer model of Chapter (3), which is due to the essentially linear exchange terms. These exchange terms do not incorporate the intensification of the time-averaged heat transfer by the presence of the acoustic wave. This shortcoming will be discussed in more detail in Sec. (5.2).

The nonlinear model can be improved by including shape functions for the transverse variation of the dependent variables (except for the pressure). Hence, the velocity, temperature and density are described as

$$\rho(\mathbf{x}, t) = \langle \rho \rangle(x, t) \phi_\rho(x, \mathbf{r}, t), \quad (2.97a)$$

$$T(\mathbf{x}, t) = \langle T \rangle(x, t) \phi_T(x, \mathbf{r}, t), \quad (2.97b)$$

$$u(\mathbf{x}, t) = \langle u \rangle(x, t) \phi_u(x, \mathbf{r}, t), \quad (2.97c)$$

where the area-average of $\phi_{\rho, T, u}$ is set to zero. Using the long pore approximation of the governing equations, and a detailed scaling analysis as is done in Sec. (3.2), approximate relations can be found for the shape functions. This is left as a recommendation for future research.

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A ONE-DIMENSIONAL HEAT TRANSFER MODEL FOR PARALLEL-PLATE THERMOACOUSTIC HEAT EXCHANGERS

A one-dimensional laminar oscillating flow heat transfer model is derived and applied to parallel-plate thermoacoustic heat exchangers. The model can be used to estimate the heat transfer from the solid wall to the acoustic medium, which is required for the heat input/output of thermoacoustic systems. The model is implementable in existing (quasi) 1D thermoacoustic codes, such as DeltaEC. Examples of generated results show good agreement with literature results. The model allows for arbitrary wave phasing; however, it is shown that the wave phasing does not significantly influence the heat transfer.

3.1 Introduction

In a thermoacoustic (TA) engine, acoustic power is generated from heat [15]. This relatively unknown class of heat engines is of commercial interest due to their minimal number of moving parts and high potential efficiency [2]. The engine core consists of a regenerator or stack, placed in between two heat exchangers. The heat exchangers supply and extract the necessary heat required for the engine to execute its thermodynamic cycle. To supply/extract the required heat, a large surface area between solid and fluid is required, leading to relatively small pores for fluid. However, acoustic power loss due to viscous and thermal relaxation becomes more dominant for small pores. An explanation of these dissipation mechanisms can be found in the introductory article by Swift [15]. Furthermore, minor loss phenomenon can dissipate acoustic power at the transition from narrow gaps to wide pores and vice versa [19]. Hence the design of these heat exchangers has a large influence on the performance of the engine.

In thermoacoustic heat exchangers, besides the external heating of heat exchanger solid, the heat transfer from the solid walls of the heat exchanger to the acoustic medium is of importance. When thermal conductivity is assumed to be constant, this “transverse” heat transfer is proportional to the gradient of the mean (time-averaged) temperature at the wall-fluid interface, so the time-averaged temperature profile in the pore has to be determined.

Mozurkewich [9] has derived a model for the time-averaged temperature distribution in a TA stack. In this model, the second order heat equation is solved in terms of eigen solutions, which are defined when the boundary conditions at the plate ends are known. A standing wave phasing was assumed, or more restrictively, a prescribed acoustic field was assumed. Piccolo [12] computed the time-averaged temperature in a parallel-plate heat

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exchanger (in both solid side as well as fluid side) by numerically solving the time-averaged second order energy equation, upon a known (assumed) acoustic solution. In this work, the full solution including the hydrodynamic enthalpy flux in the transverse direction is modeled.

Experimental work has focused on obtaining heat transfer coefficients for different acoustic Reynolds numbers and plate spacing [6, 11]. In these models, the heat transfer has been investigated for standing waves and is expressed in terms of the Nusselt number, or the Colburn j -factor. Since Colburn j -factors predict a Nusselt number of zero in the limit of no acoustic amplitude, these models are less adequate in the low amplitude regime. In fact, the heat transfer stays finite and is governed by conduction. Wakeland and Keolian [18] have measured the TA heat transfer coefficient expressed in the effectiveness, which is the ratio of the actual heat transfer to the maximum possible heat transfer. These derived empirical correlations can be used to compute the heat transfer as a function of the acoustic Reynolds number. In these experimental works, no attention is given to the TA interaction of the fluid with the heat exchanger wall.

The current aim is to develop a theoretical one-dimensional (1D) heat transfer model for heat transfer from the solid walls to the acoustic medium. Attention will be given to the possibility of implementing the model in a (quasi) 1D TA code, such as the publicly available DeltaEC [21]. The model should be valid for arbitrary wave phasing and moderate acoustic amplitude, i.e. from diffusion-dominated to acoustically dominated energy flow. It is invalid when large-scale convection dominates the heat transfer. This is the case when the particle oscillation amplitude has the same, or a larger, order of magnitude as the heat exchanger length. We will show by an order of magnitude estimation, combined with an expansion of the dependent variables in the acoustic Mach number, that the transverse heat transfer in TA heat exchangers is dominated by heat diffusion, so the transverse hydrodynamic enthalpy flux can be neglected. This important result has been found by Piccolo and Pistone [13] and it was implicitly assumed by Mozurkewich [9]. It is used in the subsequent analysis to derive an ordinary differential equation for the cross-sectional averaged and time-averaged fluid temperature for a given wall (i.e. heat exchanger wall) temperature. Viscothermal losses can be computed as a post-processing step.

The most important assumptions and the modeling approach are introduced in Sec. (3.2). In Sec. (3.3), the acoustic solutions are re-derived for the case that the time-averaged transverse temperature is non-constant. The model is solved for a parallel-plate geometry in Sec. (3.4). An application of the model to a literature test case is presented in Sec. (3.5). Finally, we end with the conclusions in Sec. (3.6).

3.2 Modeling assumptions and Mach number expansion

Figure (3.1) gives an overview of the situation for the developed model. The time-averaged wall temperature T_w is assumed to be known. As explained in the work of Mozurkewich [9], the wall heat transfer can be coupled to the fluid, but in this work we limit ourselves to the fluid side heat transfer only. Although the parallel-plate situation is 2D, for future expansion of the model we have chosen to derive the equations for an arbitrary pore shape.

Figure 3.1 – Situation overview for the developed model. An acoustic wave is present in between the heat exchanger plates with plate spacing $2r_h$ and length L . The plate spacing is constant for all plates. A local coordinate frame can be placed in between two plates. The time-averaged wall temperature T_w is assumed to be constant in the z -direction and a prescribed function in the x -direction. Due to heat conduction, the time-average fluid temperature will be non-constant in the y -direction. A typical time-averaged temperature distribution T_m and x -velocity distribution u is depicted.

The derivation is started with the conservation laws for mass, momentum and energy for a Fourier-Newtonian perfect gas, accompanied with the perfect gas law equation of state. Body forces and heat sources are neglected. Additionally, bulk viscosity is neglected. A suitable dimensionless form for these equations was presented in Sec. (2.3) of the previous chapter.

This dimensionless form is helpful for estimating the relative order of magnitude of terms in the equations.

We have assigned numerical values to some of the dimensionless parameters in the equations of Sec. (2.3) by introducing two fictitious TA engines. One is an engine running on air under conditions suitable for easy experimental analysis (atmospheric conditions, low onset temperature), while the other corresponds to a commercial application. The latter engine uses pressurized helium ($p_0 = 40$ bar) as a working fluid. For both engines, an estimation of the dimensionless parameters used in the model is given in Table (3.1). Some of the parameters will be explained and used in the next section.

We assume that the dependent variables can be prescribed by a time-averaged value with a small order of magnitude of the acoustic Mach number, (acoustic) harmonic oscillation [5, 15]:

$$\begin{aligned}\hat{p}(\hat{\mathbf{x}}, \hat{t}) &= \hat{p}_m(\hat{\mathbf{x}}) + M\Re\left[\hat{p}_1(\hat{\mathbf{x}})e^{i\hat{t}}\right] + \dots, \\ \hat{T}(\hat{\mathbf{x}}, \hat{t}) &= \hat{T}_m(\hat{\mathbf{x}}) + M\Re\left[\hat{T}_1(\hat{\mathbf{x}})e^{i\hat{t}}\right] + \dots \text{etc},\end{aligned}\quad (3.1)$$

where the subscript $_1$ corresponds to the phasor for the acoustic variable, subscript $_m$ denotes the time-average, and $\Re[\dots]$ extracts the real part of its argument. The acoustic variables are re-scaled such that $\hat{p}_1$, $\hat{T}_1$, etc. become order unity. The velocity field is split into a dimensionless velocity in the propagation direction ($\hat{u}$) and the velocity in the transverse direction ($\hat{\mathbf{v}}$). 1D wave propagation is considered, so we assume that the pressure is

	Air engine	Helium engine	
Material parameters			
R_s	287	2077	$[\text{J} \cdot \text{kg}^{-1} \cdot \text{K}^{-1}]$
c_{p0}	1003	5195	$[\text{J} \cdot \text{kg}^{-1} \cdot \text{K}^{-1}]$
γ_0	7/5	5/3	[-]
κ_0	0.0256	0.149	$[\text{W} \cdot \text{m}^{-1} \cdot \text{K}^{-1}]$
μ_0	$1.82 \cdot 10^{-5}$	$1.96 \cdot 10^{-5}$	$[\text{Pa} \cdot \text{s}]$
c_0	340	1007	$[\text{m} \cdot \text{s}^{-1}]$
b_μ	0.76	0.68	[-]
b_κ	0.89	0.72	[-]
Operational parameters			
T_0	20	20	$[\text{°C}]$
M	0.04	0.04	[-]
f	100	100	Hz
p_0	$101.3 \cdot 10^3$	$4 \cdot 10^6$	$[\text{Pa}]$
ρ_0	1.21	6.57	$[\text{kg} \cdot \text{m}^{-3}]$
δ_ν	$2.19 \cdot 10^{-4}$	$9.74 \cdot 10^{-5}$	$[\text{m}]$
δ_κ	$2.60 \cdot 10^{-4}$	$1.18 \cdot 10^{-4}$	$[\text{m}]$
L	3	3	$[\text{cm}]$
r_h	0.5	0.5	$[\text{mm}]$
Derived dimensionless quantities			
s_0	3.2	7.3	[-]
ε	$1.67 \cdot 10^{-2}$	$1.67 \cdot 10^{-2}$	[-]
M^2	$1.6 \cdot 10^{-3}$	$1.6 \cdot 10^{-3}$	[-]
k_0	$9.2 \cdot 10^{-4}$	$3.1 \cdot 10^{-4}$	[-]
$M^2 k_0$	$3.7 \cdot 10^{-5}$	$1.2 \cdot 10^{-5}$	[-]
$k_0 \varepsilon / s_0^2$	$1.5 \cdot 10^{-6}$	$9.9 \cdot 10^{-8}$	[-]
$M^2 k_0 / s_0^2$	$1.42 \cdot 10^{-7}$	$9.5 \cdot 10^{-9}$	[-]

Table 3.1 – Derived dimensionless parameters for two fictitious thermoacoustic engines.

constant along the transverse direction. This implies that $\|\hat{\mathbf{v}}\| \ll |\hat{u}|$, and Tjeldeman [17] has shown that for this situation the transverse velocity scales as k_0 compared to the velocity in the propagation direction. Therefore, we include this scaling and redefine the transverse velocity such that $\|\hat{\mathbf{v}}\| \sim 1$. Then the time-harmonic form of the velocity vector is

$$\hat{\mathbf{u}} = \Re \left[(\hat{u}_1(\hat{\mathbf{x}}) \mathbf{e}_x + k_0 \hat{\mathbf{v}}_1(\hat{\mathbf{x}})) e^{i\hat{t}} \right], \quad (3.2)$$

where the same harmonic form is used as for $\hat{p}_1$, etc. The flow is assumed to be laminar and time-averaged mass transport induced by the acoustic wave, so-called *acoustic streaming*, is neglected.

Furthermore, we assume that entrance effects at the start and end of the heat exchanger can be neglected. A condition for the entrance effects to be small is that the particle oscillation amplitude is small compared to the heat exchanger length, i.e. $\xi_1 = u_1/\omega \ll L$. For the chosen reference conditions of Table (3.1), this assumption is violated since $u_1/\omega L \sim M\varepsilon/k_0 \sim 1$. In this scaling, however, it is not taken into account that heat exchangers in thermoacoustic engines are located at high-impedance positions ($z \simeq 10\rho_0 c_0$) in the engine [3]. This reduces the local oscillation amplitude by another factor of 10 for the same reference Mach number, which makes the assumption more plausible.

Second order energy equation

In the case of a non-constant specific heat, the energy equation can be made dimensionless in a similar way as done in Sec. (2.3). For that situation the dimensionless energy equation is

$$\begin{aligned} \hat{c}_p \frac{\partial \hat{T}}{\partial \hat{t}} + M^2 \frac{\partial}{\partial \hat{t}} \left((\gamma_0 - 1) \frac{1}{2} \|\hat{\mathbf{u}}\|^2 \right) - \frac{\gamma_0 - 1}{\gamma_0} \frac{\partial \hat{p}}{\partial \hat{t}} + \\ \frac{M}{k_0} \left(\hat{c}_p \hat{\nabla} \cdot (\hat{\rho} \hat{T} \hat{\mathbf{u}}) + M^2 \hat{\nabla} \cdot \left(\hat{\rho} (\gamma_0 - 1) \frac{1}{2} \|\hat{\mathbf{u}}\|^2 \right) \right) = \\ \frac{1}{s_0^2 \text{Pr}_0} \hat{\nabla} \cdot (\hat{T}^{b_\kappa} \hat{\nabla} \hat{T}) + \frac{M^2 (\gamma_0 - 1)}{s_0^2} \hat{\nabla} \cdot (\hat{\mathbf{x}} \cdot \hat{\mathbf{u}}) \end{aligned} \quad (3.3)$$

The thermal and viscous interaction of the acoustic wave with the solid boundaries is dependent on the time-averaged temperature of the fluid. This time-averaged temperature distribution can be found by solving the energy equation up to second order in the acoustic Mach number. The temperature dependence of the dynamic viscosity and thermal conductivity also has to be taken into account. We use a Taylor expansion of the dimensionless thermal conductivity variation with the temperature:

$$\begin{aligned} \hat{\kappa}(\hat{T}) &= \hat{\kappa}_m + \frac{d\hat{\kappa}}{d\hat{T}}|_{\hat{T}_m} (\hat{T} - \hat{T}_m) + \frac{1}{2} \frac{d^2 \hat{\kappa}}{d\hat{T}^2}|_{\hat{T}_m} (\hat{T} - \hat{T}_m)^2 + \dots \\ &\simeq \hat{T}_m^{b_\kappa} \left(1 + M \hat{T}_m^{-1} b_\kappa \Re [\hat{T}_1 e^{i\hat{t}}] \right). \end{aligned} \quad (3.4)$$

For the dynamic viscosity, a similar expansion is used.

Substituting expansions (3.1) and the Taylor expansions for the thermal conductivity and dynamic viscosity into the energy equation (3.3), taking only terms up to second order in the acoustic Mach number into account, time-averaging, neglecting time-averaged mass flux and multiplying with k_0 , we obtain

$$\varepsilon \frac{\partial \hat{e}_x}{\partial \hat{x}} + \hat{\nabla}_T \cdot \hat{e}_T = 0, \quad (3.5)$$

where $\hat{e}_x$ denotes the dimensionless time-averaged second order energy flux in the x -direction, and $\hat{e}_T$ in the transverse direction. $\hat{e}_x$ equals

$$\begin{aligned} \hat{e}_x = & M^2 \hat{\rho}_m \hat{c}_p \frac{1}{2} \Re \left[\hat{T}_1 \hat{u}_1^* \right] \\ & - \frac{k_0 \varepsilon}{s_0^2 \text{Pr}_0} \hat{T}_m^{b_\kappa} \left(\frac{\partial \hat{T}_m}{\partial \hat{x}} + \frac{M^2 b_\kappa}{\hat{T}_m} \frac{1}{2} \Re \left[\hat{T}_1 \frac{\partial \hat{T}_1^*}{\partial x} \right] \right) - \hat{V}_x, \end{aligned} \quad (3.6)$$

where $*$ denotes complex conjugation, $\hat{V}_x$ denotes the dimensionless viscous work flux contribution to the energy

$$\begin{aligned} \hat{V}_x = & \frac{M^2 k_0 (\gamma_0 - 1)}{s_0^2} \hat{T}_m^{b_\mu} \frac{1}{2} \Re \left[k_0 \left(\hat{\nabla}_T \hat{u}_1 \right) \cdot \hat{\mathbf{v}}_1^* + \right. \\ & \left. 2\varepsilon \frac{\partial \hat{u}_1}{\partial \hat{x}} \hat{u}_1^* + \varepsilon k_0^2 \frac{\partial \hat{\mathbf{v}}_1}{\partial \hat{x}} \cdot \hat{\mathbf{v}}_1^* - \frac{2}{3} \left(\hat{\nabla} \cdot \hat{\mathbf{u}}_1 \right) u_1^* \right]. \end{aligned} \quad (3.7)$$

The second order energy flux in transverse direction is

$$\begin{aligned} \hat{e}_T = & M^2 k_0 \hat{\rho}_m \hat{c}_p \frac{1}{2} \Re \left[\hat{T}_1 \mathbf{v}_1^* \right] - \frac{k_0 \varepsilon}{s_0^2 \text{Pr}_0} \hat{T}_m^{b_\kappa} \left(\hat{\nabla}_T \hat{T}_m + \right. \\ & \left. \frac{M^2 b_\kappa}{\hat{T}_m} \frac{1}{2} \Re \left[\hat{T}_1 \hat{\nabla}_T \hat{T}_1^* \right] \right) - \mathbf{V}_T, \end{aligned} \quad (3.8)$$

with the viscous work flux in transverse direction defined as¹

$$\begin{aligned} \hat{\mathbf{V}}_T = & \frac{M^2 k_0 (\gamma_0 - 1)}{s_0^2} \hat{T}_m^{b_\mu} \frac{1}{2} \Re \left[\varepsilon k_0 \frac{\partial \hat{\mathbf{v}}_1}{\partial \hat{x}} \hat{u}_1^* + \right. \\ & \left. \left(\hat{\nabla}_T \hat{u}_1 \right) u_1^* + k_0^2 \left(\hat{\nabla}_T \mathbf{v}_1 + \left(\hat{\nabla}_T \mathbf{v}_1 \right)^T \right) \cdot \mathbf{v}_1^* - \frac{2}{3} k_0 \left(\hat{\nabla} \cdot \hat{\mathbf{u}}_1 \right) \mathbf{v}_1^* \right]. \end{aligned} \quad (3.9)$$

At this point, some simplifying assumptions will be made. For that, we estimated the dimensionless parameters of Eq. (3.5) for two typical heat exchangers used in thermoacoustic engines. The last 6 rows of Table (3.1) give results for the computed dimensionless numbers. We take terms into account up to order $\sim k_0 \varepsilon / s_0^2$. This means that the viscous work flux can be neglected, since this is $\sim M^2 k_0 / s_0^2$.

¹In the original paper [7], the last term of Eq. (3.9) is erroneously missing.

At onset conditions in a thermoacoustic engine (in the limit of $M \rightarrow 0$), the conduction terms are dominating, while for full load, the hydrodynamic enthalpy flux in the propagation direction is the dominating contributor. To be able to model the temperature profile in both situations, the heat conduction in $\hat{x}$ -direction cannot be neglected, despite the fact that it is small compared to the enthalpy flux in the T-direction when the engine runs under operating conditions ($M = 0.04$). Keeping this into account and neglecting all other terms of $\sim M k_0$ and smaller, multiplying all remaining terms by s_0^2/k_0 the energy equation reduces to

$$\begin{aligned} \frac{\partial}{\partial \hat{x}} \left(\frac{M^2 s_0^2 \varepsilon}{k_0} \hat{\rho}_m \hat{c}_p \frac{1}{2} \Re[T_1 u_1^*] - \frac{\varepsilon^2}{\text{Pr}_0} \hat{T}_m^{b_\kappa} \frac{\partial \hat{T}_m}{\partial \hat{x}} \right) - \\ \frac{1}{\text{Pr}_0} \hat{\nabla}_T \cdot \left(\hat{T}_m^{b_\kappa} \hat{\nabla}_T \hat{T}_m \right) = 0. \end{aligned} \quad (3.10)$$

This equation states that the enthalpy flux in the transverse direction is governed by heat conduction, while in the wave propagation direction, the energy flux is a combination of heat conduction and hydrodynamic enthalpy flux. This statement was implicitly used by Mozurkewich [10] and Piccolo and Pistone [13].

Resulting in Eq. (3.10) is nontrivial, since at operating conditions the transverse hydrodynamic enthalpy flux cannot be neglected compared to axial conduction. However, in the case where it becomes that large, it can be neglected compared to the axial hydrodynamic enthalpy flux. Therefore, we require the additional assumption that the hydrodynamic enthalpy flux in transverse direction is always small compared to the $\hat{x}$ -direction. This means

$$k_0/\varepsilon \ll 1. \quad (3.11)$$

Since $k_0/\varepsilon \propto L/\lambda_0$, this corresponds to the short-stack approximation [15]. The scale of axial energy flux depends on the scale of the acoustic amplitude. We introduce a common scale δ for the energy flux in the $\hat{x}$ -direction as

$$\delta = \max \left(\frac{M^2 s_0^2 \varepsilon}{k_0}, \frac{\varepsilon^2}{\text{Pr}_0} \right). \quad (3.12)$$

Since $\delta \ll 1$, a first order perturbation expansion of $\hat{T}_m$ in δ can be used to obtain the mean temperature profile. We write the time-averaged temperature as a spatial average superposed with an $\mathcal{O}(\delta)$ fluctuation in the transverse direction:

$$\hat{T}_m(\hat{x}, \hat{\mathbf{r}}) = \langle \hat{T}_m \rangle(\hat{x}) + \delta \hat{T}_{mT}(\hat{x}, \hat{\mathbf{r}}) + \dots, \quad (3.13)$$

where we add the additional constraint that the $\mathcal{O}(\delta)$ temperature distribution over the cross section has an area-average of zero

$$\frac{1}{\hat{S}_f} \int_{\hat{S}_f} \hat{T}_m(\hat{x}, \hat{\mathbf{r}}) d\hat{S} = \langle \hat{T}_m \rangle(\hat{x}). \quad (3.14)$$

This can only be done when the boundary condition on the time-averaged wall temperature of the pore is symmetric, so no net heat flow crosses the fluid domain in transverse direction.

Substituting expansion (3.13) back in Eq. (3.10), taking terms up to $\mathcal{O}(\delta)$, and dividing by δ yields:

$$\begin{aligned} \frac{\partial}{\partial \hat{x}} \left(\frac{M^2 s_0^2 \varepsilon}{k_0 \delta} \langle \hat{\rho}_m \rangle \hat{c}_p \frac{1}{2} \Re [\hat{T}_1 \hat{u}_1^*] - \frac{\varepsilon^2 \langle \hat{T}_m \rangle^{b_\kappa}}{\text{Pr}_0 \delta} \frac{d \langle \hat{T}_m \rangle}{d \hat{x}} \right) \\ - \frac{\langle \hat{T}_m \rangle^{b_\kappa}}{\text{Pr}_0} \hat{\nabla}_T^2 \hat{T}_m = 0, \end{aligned} \quad (3.15)$$

which is a Poisson equation for the $\mathcal{O}(\delta)$ correction of the time-averaged transverse temperature profile. The boundary condition is that $\hat{T}_m|_{\text{wall}} = \hat{T}_w$. In deriving Eq. (3.15) we have made use of an expansion in δ of the density

$$\hat{\rho}_m = \langle \hat{\rho}_m \rangle + \mathcal{O}(\delta). \quad (3.16)$$

To solve Eq. (3.15), explicit solutions of the first order (acoustic) temperature and velocity profiles have to be known. These solutions are presented in the next section.

3.3 First order equations

When the time-averaged temperature distribution is uniform in the transverse direction, expansions (3.1) up to first order in the acoustic Mach number result in a system of equations which has been solved by Rott [14]. The result is “Rott’s thermoacoustic wave equation” [16]. This is an ordinary differential equation depending on the propagation (x) coordinate only. The transverse viscothermal effects are incorporated in the f_ν and f_κ functions. These functions are, for a fixed frequency, dependent on the cross-sectional geometry and time-averaged temperature. Arnott [1] has shown solutions for arbitrary pore shapes. In the current work, the time-averaged temperature is not assumed to be spatially uniform. However, it is important to see that the variation in transverse direction of the mean temperature field is $\sim \delta$. This means, that on the order of approximation of the first order thermoacoustic equations, this is a very small correction. This is because terms of $\sim k_0$ are already neglected as a result of the low reduced frequency approximation. Hence, for constructing the solution of the first order equations only the area-averaged component of the mean temperature distribution has to be taken into account. Applying this assumption in dimensionless form, the first order continuity equation becomes

$$i \hat{\rho}_1 + \frac{\varepsilon}{k_0} \frac{\partial \langle \hat{\rho}_m \rangle \hat{u}_1}{\partial \hat{x}} + \frac{\langle \hat{\rho}_m \rangle}{k_0} \hat{\nabla}_T \cdot \hat{\mathbf{v}}_1 = 0, \quad (3.17)$$

for the momentum equation in x -direction, neglecting the $\sim \delta$ terms results in

$$i \langle \hat{\rho}_m \rangle \hat{u}_1 - \frac{\langle \hat{T}_m \rangle^{b_\mu}}{s_0^2} \hat{\nabla}_T^2 \hat{u}_1 = - \frac{\varepsilon}{\gamma_0 k_0} \frac{d \hat{p}_1}{d \hat{x}}, \quad (3.18)$$

for the first order energy equation, we obtain

$$\langle \hat{\rho}_m \rangle \hat{c}_p \left(i \hat{T}_1 + \frac{\varepsilon}{k_0} \hat{u}_1 \frac{d \langle \hat{T}_m \rangle}{d \hat{x}} \right) = i \frac{\gamma_0 - 1}{\gamma_0} \hat{p}_1 + \frac{\langle \hat{T}_m \rangle^{b_\kappa}}{s_0^2 \text{Pr}_0} \hat{\nabla}_T^2 \hat{T}_1, \quad (3.19)$$

and the last equation required is the equation of state, in this case the first order ideal gas law

$$\hat{p}_1 = \langle \hat{\rho}_m \rangle \hat{T}_1 + \langle \hat{T}_m \rangle \hat{\rho}_1. \quad (3.20)$$

We conclude that the form of Eqs. (3.17)-(3.20) is similar to the solution obtained for a constant mean temperature in the transverse direction. Sec. (2.5) shows the solutions to Eqs. (3.17)-(3.20) in dimensional form.

Integrating the dimensional form of Eq. (3.15) over the cross-sectional area and applying the divergence theorem for the last term on the left-hand side, the second order enthalpy flow (H_2) in the acoustic medium is found. An expression for arbitrary pore shapes can be deduced:

$$\frac{dH_2}{dx} = \langle \hat{\kappa}_m \rangle \oint_{\Pi} \hat{\nabla}_T T_{mT} \cdot \mathbf{n} d\Pi \equiv q_{in} \quad (3.21)$$

where q_{in} is the *transverse* heat flow in $\text{W} \cdot \text{m}^{-1}$. The second order enthalpy flow H_2 is a combination of heat conduction and thermoacoustic enthalpy flow:

$$H_2 = \frac{1}{2} \Re[p_1 U_1^* \eta] + \left(\frac{\langle \rho_m \rangle c_p |U_1|^2}{\omega S_f} \frac{1}{2} \Im[\zeta] - \langle \kappa_m \rangle S_f \right) \frac{d\langle T_m \rangle}{dx} \quad (3.22)$$

where

$$\eta = 1 - \frac{f_\kappa - f_\nu^*}{(1 - f_\nu^*)(1 + \text{Pr})(1 + \epsilon_s)} \quad ; \quad \zeta = \frac{f_\nu^* + \frac{(f_\kappa - f_\nu^*)\lambda}{1 + \text{Pr}}}{(1 - \text{Pr})|1 - f_\nu|^2}. \quad (3.23)$$

This relation can also be found in the DeltaEC user guide [20], where it also includes the conduction heat flow in the solid. If q_{in} is prescribed, Eq. (3.21) can be used as an equation to solve the mean temperature field. However, generally q_{in} is not known and has to be solved from other boundary conditions, such as the wall temperature of the channel (T_w). For that case, Eq. (3.15) has to be solved for the transverse temperature profile.

3.4 Solution for parallel plates

In this section, the Poisson equation (3.15) is solved to find the solution of the mean temperature profile for a prismatic parallel-plate heat exchanger. For a parallel-plate geometry, h_ν and h_κ are equal to

$$h_\nu = \frac{\cosh(\sqrt{i} sy/r_h)}{\cosh(\sqrt{i} s)} \quad ; \quad h_\kappa = \frac{\cosh(\sqrt{i \text{Pr}} sy/r_h)}{\cosh(\sqrt{i \text{Pr}} s)}. \quad (3.24)$$

Substituting the first order solutions for the acoustic temperature oscillation and velocity in the dimensional form of Eq. (3.15), we obtain

$$\begin{aligned} \langle \kappa_m \rangle \frac{\partial^2 T_{mT}}{\partial y^2} &= \frac{\partial}{\partial x} \left[\frac{1}{S_f} \frac{1}{2} \Re[p_1 U_1^* \alpha_1 \beta^*] \right. \\ &\quad \left. - \frac{d\langle T_m \rangle}{dx} \left(\langle \kappa_m \rangle + \frac{\langle \rho_m \rangle c_p}{\omega S_f^2} |U_1|^2 \frac{1}{2} \Im[\alpha_2 \beta^*] \right) \right], \end{aligned} \quad (3.25)$$

where α_1 , α_2 , and β are as defined in Sec. (2.5).

To find the mean temperature variation over the cross section, we integrate Eq. (3.25) twice along y , apply the boundary condition that $T_m|_{y=r_h} = T_w$, as well as the symmetry condition, stating that the heat flux in the y -direction at the center between two plates is zero: $\frac{\partial T_{mT}}{\partial y}|_{y=0} = 0$. This yields:

$$\begin{aligned} T_{mT} = & \frac{r_h^2}{S_f \langle \kappa_m \rangle} \frac{\partial}{\partial x} \left\{ \frac{1}{2} \Re[\Psi p_1 U_1^*] - \frac{\langle \rho_m \rangle c_p}{\omega S_f} |U_1|^2 \frac{d\langle T_m \rangle}{dx} \frac{1}{2} \Im[\Phi] \right\} \\ & + T_w - \langle T_m \rangle + \frac{1}{2} \frac{r_h^2 (1 - \hat{y}^2)}{\langle \kappa_m \rangle} \frac{d}{dx} \left(\langle \kappa_m \rangle \frac{d\langle T_m \rangle}{dx} \right), \end{aligned} \quad (3.26)$$

where

$$\Psi(\hat{y}) = \int_1^{\hat{y}} \int_0^{\hat{y}} \alpha_1 \beta^* d\hat{y} d\hat{y} \quad ; \quad \Phi(\hat{y}) = \int_1^{\hat{y}} \int_0^{\hat{y}} \alpha_2 \beta^* d\hat{y} d\hat{y}, \quad (3.27)$$

and $\hat{y} = y/r_h$. Assuming the acoustic field is known, Eq. (3.26) cannot yet be solved, as it still contains two unknowns: the mean temperature correction T_{mT} and the area-averaged mean temperature $\langle T_m \rangle$. We can eliminate the mean temperature correction by integrating again, and taking into account that $\int_{\hat{y}=0}^1 T_{mT} d\hat{y} = 0$. We end up with a second order ordinary differential equation for the time-averaged spatially mean temperature in the fluid gap. The result is rearranged and multiplied by $\langle \kappa_m \rangle S_f / r_h^3$:

$$\frac{dJ}{dx} = \frac{\langle \kappa_m \rangle S_f}{r_h^2} (\langle T_m \rangle - T_w), \quad (3.28)$$

where J is defined by

$$J = \frac{1}{2} \Re[\Theta p_1 U_1^*] + \left(\frac{S_f \langle \kappa_m \rangle}{3} - \frac{\langle \rho_m \rangle c_p |U_1|^2}{\omega S_f} \frac{1}{2} \Im[\Lambda] \right) \frac{d\langle T_m \rangle}{dx}. \quad (3.29)$$

The numbers Θ and Λ are defined as:

$$\Theta = \int_0^1 \Psi(\hat{y}) d\hat{y} \quad ; \quad \Lambda = \int_0^1 \Phi(\hat{y}) d\hat{y} \quad (3.30)$$

Note the large similarity in the structure of Eq. (3.21) and (3.29). This similarity allows for an easy implementation of the current model in an existing thermoacoustic code. Θ and Λ are dependent on the shear wave number and the Prandtl number. The evaluation of their integrals is done by using a computer algebra system. The result of this integration is given in Appendix (3.A). As shown in Figure (3.2), Θ and $\Im(\Lambda)$ show some limiting behavior: for small r_h , or low frequencies $\Theta, \Lambda \rightarrow 0$, while for wide pores ($s \rightarrow \infty$), $\Theta \rightarrow -\frac{1}{3}$ and $\Lambda \rightarrow 0$. For wide pores, however, the acoustic power flow, $\frac{1}{2} \Re[p_1 U_1^*]$ stays constant, because f_ν and $f_\kappa \rightarrow 0$. This means that in both limits, Eq. (3.29) can be reduced to

$$\frac{dQ}{dx} = q_{in} = h\Pi(T_w - \langle T_m \rangle) \quad (3.31)$$

Figure 3.2 – Above: real and imaginary part of $-\Theta$ vs. shear wave number. Below: $-\Im(A)$. The results are created for $\text{Pr} = 0.68$.

where $Q = -S_f \langle \kappa_m \rangle \frac{d\langle T_m \rangle}{dx}$ (since we assume prismatic pores, S_f can be put inside the d/dx), and $h = 3 \langle \kappa_m \rangle / r_h$. h is an effective heat transfer coefficient, obtained exactly for the conduction-only problem of high-aspect ratio pores. In the limit of $s \rightarrow \infty$, this approximation is poor because in that limit the transverse heat transport is no longer dominated by heat conduction.

Implementation

A numerical code has been developed to solve the acoustic equations (Eqs. (2.50, 2.58), including the second order ODE governing the mean temperature field (3.29). This code invokes the well-known fourth order Runge-Kutta method [8] for integration of the differential equations. Note that the acoustic field is altered by the change in mean temperature field due to the second term in Eq. (2.58). On the other hand, the mean temperature field is altered by the presence of an acoustic field. A converged solution is found by sequentially updating the mean temperature field in the acoustic equations and updating the acoustic field in the mean temperature equation. This updating is applied after fully integrating the solution for the temperature and acoustic field, respectively. After approximately 4 to 10 iterations, this procedure converges.

Figure 3.3 – Schematic overview of the domain, split into three parts. The left and right duct are unable to exchange heat with the surroundings ($q_{in} = 0$). On the left side, an acoustic impedance boundary z_L is applied and the time-averaged second order enthalpy flow is set to H_L . On the right side, the acoustic velocity phasor is prescribed to be u_R , and the time-averaged temperature is set at $\langle T_m \rangle_R$. The middle part of the domain is a heat exchanging domain where we assume a constant wall temperature.

3.5 Application

In this section, the developed model is applied to a case study. The chosen configuration corresponds to a parallel-plate stack/heat exchanger in a resonator driven by an oscillating piston, as shown in Figure (3.3).

CFD computations on this system were performed by Cao et al. [4]. Their primary obtained result was that the transverse heat flux from the stack plate to the acoustic medium strongly “peaks” at the plate edges. Later on, Mozurkewich [10] was able to reproduce this phenomenon. To demonstrate the validity of the current model the same test case of Cao et al. is used. The developed model allows the investigation of the effect of the wave phasing on the heat transfer.

Model details

The stack plates are assumed to be isothermal, so this corresponds to the initial situation of a driven thermoacoustic refrigerator; no temperature gradient has yet been established. A schematic overview of this case is given in Figure (3.3). The model is built from three segments, the left duct, the heat exchanger and the right duct. Continuity of mean temperature, second order enthalpy flow, acoustic volume flow (U_1) and acoustic pressure (p_1) are assumed at the interfaces between the segments. The mean pressure p_m is set to atmospheric (101.3 kPa). The stack plates have zero thickness, so the porosity $\phi = 1$. The plate spacing is $2r_h$. The mean temperature of the plates is set to $T_w = 20^\circ\text{C}$ and it is assumed that the plates are perfectly conducting, such that $\epsilon_s = 0$. The left and right duct are modeled as inviscid and adiabatic tubes, so in these segments we set $f_\nu = f_\kappa = 0$. This also implies that in these segments the second order time-averaged enthalpy flow (3.21)–(3.22) with $q_{in} = 0$) is independent of x . Helium is chosen as the acoustic medium. For the properties of helium,

the reader is referred to Table (3.1). On the left boundary, zero second order enthalpy flow is assumed ($H_{2L} = 0$). The acoustic boundary conditions are chosen similar to Cao et al. [4]. On the left domain boundary, an impedance boundary condition is chosen corresponding to a wall that is a distance x_L from the left duct: $z_L = iz_0 \cot(2\pi x_L/\lambda)$, where z_0 denotes the characteristic acoustic impedance and λ is the wavelength. Furthermore, on the left domain boundary a zero second order energy flow boundary condition is imposed. Therefore, no heat can enter or leave the domain on this side. At the right boundary, the mean temperature of the fluid is prescribed to be $\langle T_m \rangle = 20^\circ\text{C}$, and $u_1 = c_0 M \sin(2\pi x_R/\lambda)$ as velocity phasor boundary condition is imposed. Various Mach numbers will be used. x_L corresponds to x_{AB} in the work of Cao et al. and x_R to x_{CD} .

Enthalpy flow

Figure (3.4) shows a graph of the second order time-averaged enthalpy flow H_2 (top), and its derivative (bottom) for different acoustic Mach numbers at a scaled plate spacing of $\delta_{\kappa 0}/r_h = 0.2$. In the figure, large peaks can be observed for low amplitudes at the ends of the heat exchanger/stack. From Cao et al. [4], it is found that the transverse heat flux magnitude scales with the particle oscillation amplitude. Therefore, this scaling is used to normalize the transverse heat flux magnitude. Because the acoustic power is zero on the left side, the segment should be isothermal. It is clear that the heat exchange concentrates at the ends of the plates.

Heat transfer coefficient vs wave phasing

For investigating heat transfer, we change the domain configuration for the more interesting heat exchanger length of 3 cm. The wave phasing is changed by applying a different acoustic impedance boundary condition. To establish a traveling wave, the left impedance boundary condition is set to $z_L = -\rho_0 c_0$. Now, we choose a higher wall temperature boundary condition T_w , moreover we change the left boundary condition from zero enthalpy flow ($H_2 = 0$) to a mean temperature boundary condition, such that $\langle T_m \rangle_L = \langle T_m \rangle_R = T_0$. This way, focus is put on the heat transfer characteristics, rather than the thermoacoustic effects. We define a heat transfer coefficient

$$h = \frac{q_{\text{in}}}{\Pi(T_w - \langle T_m \rangle)}, \quad (3.32)$$

where Π denotes the wetted perimeter, equal to S/r_h for parallel-plate pores. The heat transfer coefficient is normalized with h_0 , which is the exact heat transfer coefficient for the conduction-only problem. T_w is set to 100°C . It is found that the heat transfer coefficient results are approximately independent of T_w . Figure (3.5) shows this heat transfer coefficient for different plate spacing $\delta_{\kappa 0}/r_h$. As shown in this figure, the heat transfer coefficient does not depend much on the wave phasing. For relatively small hydraulic radius, the heat transfer is not augmented by the presence of an acoustic field. Above $M \simeq 0.04$, the heat transfer coefficient does not significantly increase anymore with increasing amplitude. The strange behavior for the case of $\delta_{\kappa 0}/r_h = 0.2$ results because of the crossing

Figure 3.4 – Top: Second order enthalpy flow (H_2) scaled with the magnitude of the enthalpy flow at the right boundary $|H_R|$. Bottom: derivative of enthalpy flow scaled with the particle displacement amplitude. Both plots are for the case of $\delta_{\kappa 0}/r_h = 0.2$.

Figure 3.5 – Heat transfer coefficient h for Mach numbers 0.01 and 0.04 and different relative plate spacing: above: $\delta_{\kappa 0}/r_h = 0.05$ middle: $\delta_{\kappa 0}/r_h = 0.1$, below: $\delta_{\kappa 0}/r_h = 0.2$. “TW” denotes traveling wave phasing and “SW” standing wave phasing. Results are generated for helium ($Pr_0 = 0.68$).

Figure 3.6 – Centerline temperature heat transfer coefficient h_m for the standing wave case and different M , Above: $\delta_{\kappa 0}/r_h = 0.05$. Below: $\delta_{\kappa 0}/r_h = 0.2$. $\text{Pr} = 0.68$ (helium).

of $\langle T_m \rangle - T_w$ through zero. Therefore, it should be stressed that a simple heat transfer coefficient model fails in the cases where thermoacoustic effects cannot be neglected.

By computing the heat transfer coefficient based on the mean temperature at the mid-point between two plates

$$h_m = \frac{q_{\text{in}}}{\Pi(T_w - T_m(y=0))}, \quad (3.33)$$

we can compare results with the simple heat transfer coefficient model of Mozurkewich [10]:

$$h_m \simeq 0.61\kappa/\delta_{\kappa}. \quad (3.34)$$

Figure (3.6) shows results for the centerline heat transfer coefficient vs position in the heat exchanger. A good agreement is found for the high-amplitude case of $M = 0.04$. At low amplitudes the result differs because conduction becomes more important for the heat transfer. For high amplitudes and wide plate spacing, a good comparison is made with the

current model. Again the jump for the case $\delta_{\kappa 0}/r_h = 0.2$ is due to a crossing of zero of $T_w - \langle T_m \rangle(y = 0)$ in between the grid-points.

3.6 Conclusions

A one-dimensional thermoacoustic heat transfer model has been developed. This model can be used for arbitrary wave phasing and can easily be implemented in a 1D second order thermoacoustic code. The model is solved and tested for a parallel-plate geometry. A comparison is made with the work of Mozurkewich for the centerline temperature-based heat transfer coefficients for parallel-plate heat. It is shown that in the range of applicability of the model, a good agreement is found between the two models. The effects of wave phasing on heat transfer have been studied; however, for the modeling cases considered in this work, the influence of heat transfer on the wave phasing is not important.

3.A The Θ and Λ terms

This appendix shows the expressions for Θ and $\Im[\Lambda]$. It should be stressed that for f_ν , the *parallel-plate* variant that should be used is

$$f_\nu = \frac{\tanh(\sqrt{i}s)}{\sqrt{i}s}. \quad (3.35)$$

For Λ , only the imaginary part is of importance, therefore we do not show some of the terms that contribute to the real part of Λ . This results in a shorter expression. The expression found for Θ is

$$\begin{aligned} \Theta = & \frac{1}{f_\nu^* - 1} \left\{ \frac{1}{3} - \frac{si + \sqrt{-i}}{s^3} + \frac{si\sqrt{\text{Pr}} - \sqrt{i}}{s^3 \text{Pr}^{\frac{3}{2}}(1 + \epsilon_s)} \dots \right. \\ & + \frac{is(1 - \text{Pr}^2) + \sqrt{-i}(1 - 3\text{Pr}) \tanh(s\sqrt{-i})}{s^3(\text{Pr} + 1)^3(1 + \epsilon_s)} \dots \\ & + \frac{2\sqrt{\text{Pr}} \tanh(s\sqrt{i\text{Pr}}) \left(\frac{1}{2}\sqrt{i}(\text{Pr} - 3) + s(1 + \text{Pr}) \tanh(s\sqrt{-i}) \right)}{s^3(\text{Pr} + 1)^3(1 + \epsilon_s)} \dots \\ & \left. + \frac{2\sqrt{i}}{s^3 \text{Pr}^{\frac{3}{2}}(e^{2s\sqrt{i\text{Pr}}} + 1)(1 + \epsilon_s)} + \frac{2\sqrt{-i}}{s^3(e^{2s\sqrt{-i}} + 1)} \right\}, \end{aligned} \quad (3.36)$$

and for $\Im[\Lambda]$ is

$$\begin{aligned} \Im[\Lambda] = & \frac{1}{(1 - \text{Pr})|1 - f_\nu|^2} \Im \left[\frac{2\text{Pr}\sqrt{i}}{s^3(e^{2s\sqrt{i}} + 1)} - \frac{2\lambda\sqrt{i}\text{Pr}^{-\frac{3}{2}}}{s^3(e^{2s\sqrt{i\text{Pr}}} + 1)} \dots \right. \\ & - \frac{\lambda\sqrt{\text{Pr}} \tanh(s\sqrt{i\text{Pr}}) [\sqrt{i}(\text{Pr} - 3) + 2s(1 + \text{Pr}) \tanh(s\sqrt{-i})]}{s^3(\text{Pr} + 1)^3} \dots \\ & + \frac{\lambda\sqrt{-i}(3\text{Pr} - 1) \tanh(s\sqrt{-i}) + i\lambda s(\text{Pr}^2 - 1)}{s^3(\text{Pr} + 1)^3} \dots \\ & \left. + \frac{\sqrt{i} \left(\lambda \text{Pr}^{-\frac{3}{2}} - \text{Pr} \right) + is \left(1 - \frac{\lambda}{\text{Pr}} \right) + \sqrt{i}(\text{Pr} - 1) \tan(s\sqrt{i})}{s^3} \right]. \end{aligned} \quad (3.37)$$

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A FREQUENCY DOMAIN METHOD FOR SOLVING PERIODIC NONLINEAR ACOUSTIC PROBLEMS

A frequency domain method is proposed for solving periodic nonlinear acoustic problems. A significant reduction in computational cost can be achieved in comparison with time integration methods, especially when spectra are decaying rapidly. In the future, this method will be used to model thermoacoustic devices. This paper discusses the theory and a validation application, in which we solve a one-dimensional plane-propagating wave problem. Results are found to be in good agreement with the analytical solution for this problem. Moreover, the computation time is in the order of a few seconds.

4.1 Introduction

In this chapter, we propose a method to solve nonlinear acoustic problems, with an emphasis on the modeling of thermoacoustic systems. Thermoacoustic (TA) systems convert energy from heat to sound and vice versa [16]. Modeling of TA systems is difficult due to their inherent nonlinear nature and the large spread in length scales involved [5]. Therefore, reduced order methods based on additional assumptions are used, such as the well-established linearized thermoacoustic theory [13, 15], which is valid for low amplitudes. Extending the linear theory to the nonlinear regime is found to be a difficult task [7], but is regarded as necessary when one is interested in steady-state quantities of TA systems, such as acoustic power and efficiency.

In many cases, the initial transient regime of TA systems is not important, and the interest is primarily on the periodic steady-state, as this is the situation to design for. Hence, it can be beneficial to make use of the periodic behavior of the flow and directly solve the periodic steady-state, thereby avoiding the need to resolve the long initial transient start-up phase of TA systems.

Time domain models require the solution to be integrated over time through many cycles before the periodic steady-state is reached. This is especially the case for self-excited systems, such as thermoacoustic engines. Thermoacoustic engines generally start from a condition of linear instability, followed by an exponential growth in amplitude. Amplitude saturation by nonlinear effects, such as thermoacoustic streaming, harmonics generation and flow minor losses, can easily take up to 100 oscillations [5], counted from the initial conditions. Using a time domain model, first this transient regime has to be resolved, and time steps are restricted by the stability of the numerical scheme. However, once periodicity is reached, only a few harmonics are required to describe the periodic steady-state up to an acceptable accuracy [4].

In this paper, we propose a method that can directly solve the periodic steady-state of nonlinear acoustic problems, termed the nonlinear frequency domain (NLFD) method. This method is based on the harmonic balance method and the NLFD method as described by

Hamilton et al. [6]. The main advantage of the current method compared to the method of Hamilton et al. is the reduction in computational cost. This reduction is accomplished by computing the nonlinear terms in the time domain using a discrete Fourier transform. Harmonic-balance-based methods have been successfully applied to aerodynamic [12], electrical [10], aero-elastic [9], and turbomachinery [2, 3] problems. It will be shown that these methods can be very useful to reduce the computational cost of simulating periodic nonlinear acoustic and thermoacoustic systems. For thermoacoustic systems, there is an additional advantage. When source terms are defined in frequency domain, there is no need to transform these terms to time domain anymore.

In Sec. (4.2), the NLFD method is discussed. Some of its characteristics, which are of interest for modeling of thermoacoustic systems, are highlighted. In Sec. (4.3), the method will be applied to derive the NLFD form of the one-dimensional Euler equations. As an alternative to solving these equations with pseudo-time integration, in Sec. (4.3) we propose a direct solution strategy using the Newton-Raphson method. In Sec. (4.4), the NLFD form of the Euler equations is used to solve the classical nonlinear acoustics problem of a plane propagating wave. Results are compared with the exact solution for this problem. We will briefly discuss the extension of the model to thermoacoustic systems, but this will be discussed in detail in Chapter (5).

4.2 NLFD Theory

This section will explain the theory behind the NLFD method. A more heuristic introduction to the harmonic balance method and derived methods is given by Hall et al. [4]. We will explain the method by applying it to a general conservation equation

$$\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial x} F(\xi) = 0, \quad (4.1)$$

where t is time and F is a real-valued nonlinear (flux) function of the state variable (ξ). It is noted that the Euler equations described in Sec. (4.3) can also be written in this form. We assume that the solution is both real and periodic in time, with a fundamental angular frequency (ω), so the solution can be written as a Fourier series

$$\xi(x, t) = \Re \left[\sum_{n=0}^{\infty} \tilde{\xi}_n(x) e^{in\omega t} \right], \quad (4.2)$$

where $\tilde{\xi}_0 \in \mathbb{R}$ and $\tilde{\xi}_n \in \mathbb{C}$ for $n > 0$. $\Re[\dots]$ denotes the real part of its argument. This is also done for the flux function. We comply with the notation used by Rott and Swift [15], by using a subscript corresponding to the harmonic number, and $e^{+i\omega t}$ notation. It is assumed that the spectrum of ξ is decaying, so a good approximation will be to limit n to N_f harmonics. This results in $2N_f + 1$ degrees of freedom: one stationary term, and real and imaginary part for each nonzero frequency. We substitute this truncated Fourier series into the differential Eq. (4.1). Since the function F is nonlinear, a truncated Fourier series can result in harmonics higher than the originally assumed number of harmonics (N_f). We

assume that their amplitude is small enough to be neglected. Hence, we are left with only the originally assumed number of harmonics:

$$\Re \left[i\omega \sum_{n=0}^{N_f} n \tilde{\xi}_n(x) e^{in\omega t} \right] + \frac{d}{dx} \left(\Re \left[\sum_{n=0}^{N_f} \tilde{F}_n(x) e^{in\omega t} \right] \right) = 0, \quad (4.3)$$

where $\tilde{F}_n$ are the Fourier coefficients of the function $F(x, t)$. Eq. (4.3) is only one equation including $2N_f + 1$ unknowns. However, we can make use of the orthogonality of sinusoids. Multiplying Eq. (4.3) by $e^{-im\omega t}$ and integrating over one time period, only the terms of harmonic m are left in the equation:

$$i\omega m \tilde{\xi}_m + \frac{d}{dx} (\tilde{F}_m) = 0 \quad 0 \leq m \leq N_f. \quad (4.4)$$

The Fourier coefficients of F can be obtained by evaluating the integrals

$$\tilde{F}_m(x) = \begin{cases} \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} F \left(\Re \left[\sum_{n=0}^{N_f} \tilde{\xi}_n(x) e^{in\omega t} \right] \right) dt & ; \quad m = 0 \\ \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} F \left(\Re \left[\sum_{n=0}^{N_f} \tilde{\xi}_n(x) e^{in\omega t} \right] \right) e^{-im\omega t} dt & ; \quad m > 0 \end{cases}. \quad (4.5)$$

Now we have obtained $2N_f + 2$ coupled equations. However, since we have assumed that $\tilde{F}_0$ is real, the equation for the imaginary part of $\tilde{F}_0$ is redundant, so we are left with $2N_f + 1$ equations. In the frequency domain, nonlinear effects result in the generation of (sub)harmonics. For example, the product of two cosines generates a time-average term and a cosine of double the frequency: $\cos^2(\omega t) = \frac{1}{2}(1 + \cos(2\omega t))$. Suppose $\kappa(t) = \xi(t)\eta(t)$, and ξ, η are truncated Fourier series, then we can find the k^{th} nonzero frequency of κ by

$$\begin{aligned} \tilde{\kappa}_k &= \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} \xi(t)\eta(t) e^{-ik\omega t} dt = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} \Re \left[\sum_{n=0}^{N_f} \tilde{\xi}_n(x) e^{in\omega t} \right] \Re \left[\sum_{m=0}^{N_f} \tilde{\eta}_m(x) e^{im\omega t} \right] e^{-ik\omega t} dt \\ &= \frac{1}{2} \sum_{n=0}^k [\tilde{\xi}_n \tilde{\eta}_{k-n}] + \frac{1}{2} \sum_{n=0}^{N_f-k} [\tilde{\xi}_{n+k} \tilde{\eta}_n^* + \tilde{\xi}_n^* \tilde{\eta}_{n+k}], \quad 0 < k \leq N_f \end{aligned} \quad (4.6)$$

and for the time-average (zero-frequency) component we find

$$\tilde{\kappa}_0 = \tilde{\eta}_0 \tilde{\xi}_0 + \frac{1}{2} \Re \left[\sum_{n=1}^{N_f} \tilde{\xi}_n \tilde{\eta}_n^* \right]. \quad (4.7)$$

In order to find the truncated vector of Fourier components $\tilde{\kappa}$, $\mathcal{O}(N_f^2)$ multiplication operations are required. Hence, for a product of three truncated Fourier series, $\mathcal{O}(N_f^3)$

operations are required, and so on. This can result in a high computational cost. Moreover, in many nonlinear operations, the Fourier integral cannot be evaluated analytically. For example, for the division of two truncated series, no analytical solution exists. To circumvent both problems, the solution of Maple et al. [11] is adopted. The idea is to generate $2N_f + 1$ time instances (“levels”) from the truncated Fourier series

$$\xi(t_m) \equiv \xi_m = \Re \left[\sum_{n=0}^{N_f} \tilde{\xi}_n e^{in\omega t_m} \right], \quad (4.8)$$

where $t_m = m\Delta t$, $0 \leq m < 2N_f + 1$ and

$$\Delta t = \frac{2\pi}{\omega(2N_f + 1)}. \quad (4.9)$$

By introducing the vector of (truncated) Fourier coefficients $\tilde{\xi}$ and the vector of time levels ξ , the vector of Fourier coefficients of - for example - the product of two quantities ($\kappa = \xi\eta$) can be computed by

$$\tilde{\kappa} = \mathcal{F}(\eta \circ \xi), \quad (4.10)$$

where $\circ$ denotes the elementwise product “Hadamard product” of two vectors¹. It can also be written as $\text{diag}(\eta) \cdot \xi$, where $\text{diag}()$ creates a square matrix with the components of its argument on the diagonal, and $\cdot$ is the matrix-vector product sign, resulting in a vector. $\mathcal{F}$ is the discrete Fourier transform (DFT) operator which generates the vector of truncated Fourier coefficients from the vector of time instances. In this form, it is termed the nonlinear frequency domain method. Applying this method to Eq. (4.4), we obtain

$$\mathbf{d} \circ \tilde{\xi} + \frac{d}{dx}(\mathcal{F}(\mathbf{F}(\mathcal{F}^{-1}\xi))) = \mathbf{0}, \quad (4.11)$$

where $\mathbf{F}$ is the nonlinear operator F , applied elementwise to the vector of time instances, and $\mathbf{d}$ is the time-derivative vector $\{ 0 \quad i\omega \quad 2i\omega \quad 3i\omega \quad \dots \}^T$. The approximate Fourier coefficients of ξ can be obtained by

$$\tilde{\xi}_k = [\mathcal{F}(\xi)]_k = \begin{cases} \frac{1}{2N_f+1} \sum_{n=0}^{2N_f} \xi_n & ; k = 0 \\ \frac{2}{2N_f+1} \sum_{n=0}^{2N_f} \xi_n e^{-ik\omega t_n} & ; k > 0 \end{cases}, \quad (4.12)$$

such that the time levels can be found similar to Eq. (4.8).

Taking the inverse DFT of Eq. (4.11) yields the time-spectral form. The difference between the time-spectral form and the NLFD form lies in a similarity transformation [4]. The NLFD form is chosen, because the boundary conditions and source terms in TA systems are stated in frequency domain (see Chapter (2) and (5)).

¹For Matlab users: the $\circ$ operation

To summarize, the approach for nonlinear operations is as follows. First, we perform an inverse DFT, which results in a vector of time instances. Secondly, the nonlinear operation is applied to each time instance individually. Finally, the result is transformed back to the frequency domain. Using this method, the Fourier integrals are not evaluated exactly, as would have been done with, for example, Eq. (4.6). Instead the midpoint rule is used to approximate the integral. However, it can be verified that by applying this method, for example, the product of two sinusoids $\Re[\xi_n e^{in\omega t}] \Re[\eta_n e^{in\omega t}]$ is computed exactly, provided that $N_f \geq 2n$. This is important, because in thermoacoustics we are often interested in the time-average effect of oscillating quantities.

When the product of two sinusoids results in harmonics at frequencies higher than N_f , aliasing occurs, resulting in spurious generation of low-frequency content. To reduce such errors, the number of harmonics should be chosen such that the energy in the higher harmonics is low compared to the energy in the lower harmonics. This can be checked afterwards.

4.3 Application: Euler equations

In this section, the NLFD theory is applied to the Euler equations. We assume calorically perfect gas and isentropic changes in state, so the energy equation can be reduced to a barotropic state equation. To later account for the expansion of the model to include thermoacoustic interaction terms, we solve for the dependent variables of pressure (p), density (ρ), velocity (u) and temperature (T). Then, the equations to solve are

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0, \quad (4.13)$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial (p + \rho u^2)}{\partial x} = 0, \quad (4.14)$$

$$p - p_0 \left(\frac{\rho}{\rho_0} \right)^\gamma = 0, \quad (4.15)$$

$$p - \rho R_s T = 0, \quad (4.16)$$

where t is time [s], x is the axial position [m], and R_s is the specific gas constant [J/kg·K]. Subscript 0 denotes a reference condition and γ is the specific heat ratio. It should be noted that the temperature can be found as a post-processing step. For thermoacoustics this is not possible anymore, because a full energy equation needs to be solved. To simplify the numerical implementation, we have chosen to store the Fourier coefficients of each dependent variable as a vector of real numbers of length $N_s = 2N_f + 1$:

$$\tilde{\xi} = \left\{ \xi_0 \quad \Re(\xi_1) \quad \Im(\xi_1) \quad \Re(\xi_2) \quad \dots \right\}^T. \quad (4.17)$$

Using this storage vector, we can perform forward and backward DFT using standard linear algebra routines. We redefine $\mathcal{F}$ and $\mathcal{F}^{-1}$ to be matrices that perform the forward and

inverse DFTs respectively. The forward $\mathcal{F}$ becomes

$$\mathcal{F} = \frac{2}{N_s} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \dots \\ 1 & c(l) & c(2l) & c(3l) & c(4l) & \dots \\ 0 & s(l) & s(2l) & s(3l) & s(4l) & \dots \\ 1 & c(2l) & c(4l) & c(6l) & c(8l) & \dots \\ 0 & s(2l) & s(4l) & s(6l) & s(8l) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad (4.18)$$

where $c(x) = \cos(x)$, $s(x) = \sin(-x)$, and $l = 2\pi/N_s$. A similar matrix can be found for $\mathcal{F}^{-1}$

$$\mathcal{F}^{-1} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & \dots \\ 1 & c(l) & s(l) & c(2l) & s(2l) & \dots \\ 1 & c(2l) & s(2l) & c(4l) & s(4l) & \dots \\ 1 & c(3l) & s(3l) & c(6l) & s(6l) & \dots \\ 1 & c(4l) & s(4l) & c(8l) & s(8l) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad (4.19)$$

which can be readily derived from Eq. (4.8). Next, we apply the NLFD method to Eqs. (4.13)-(4.16). After transforming to the frequency domain and applying the forward-backward transform to the nonlinear operations, we obtain the following semi-discrete form for the continuity equation

$$\mathbf{D} \cdot \tilde{\rho} + \frac{d}{dx} \mathcal{F} \cdot [\mathcal{F}^{-1} \cdot \tilde{\rho} \circ \mathcal{F}^{-1} \cdot \tilde{\mathbf{u}}] = 0, \quad (4.20)$$

where $\mathbf{D}$ is a skew-symmetric matrix form of the time-derivative vector $\mathbf{d}$,

$$\mathbf{D} = \omega \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & -1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & -2 & \dots \\ 0 & 0 & 0 & 2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (4.21)$$

Similar forms are obtained for momentum, isentropic state and perfect gas as expressed in Eqs. (4.14)-(4.16). These are

$$\mathbf{D} \cdot \mathcal{F} \cdot (\mathcal{F}^{-1} \tilde{\rho} \circ \mathcal{F}^{-1} \cdot \tilde{\mathbf{u}}) + \dots \quad (4.22)$$

$$\frac{d}{dx} [\tilde{\mathbf{p}} + \mathcal{F} \cdot (\mathcal{F}^{-1} \cdot \rho \circ (\mathcal{F}^{-1} \cdot \tilde{\mathbf{u}})^2)] = \mathbf{0},$$

$$\tilde{\mathbf{p}} - p_0 \mathcal{F} \cdot \left(\frac{\mathcal{F}^{-1} \cdot \tilde{\rho}}{\rho_0} \right)^\gamma = \mathbf{0}, \quad (4.23)$$

$$\tilde{\mathbf{p}} - R_s \mathcal{F} \cdot (\mathcal{F}^{-1} \cdot \tilde{\rho}) \circ (\mathcal{F}^{-1} \cdot \tilde{\mathbf{T}}) = \mathbf{0}. \quad (4.24)$$

It should be noted that the exponents $(2, \gamma)$ are applied elementwise to each vector component.

Eq. (4.20) and Eqs. (4.22)-(4.24) are defined in the frequency domain. This eases the implementation of viscous drag and TA source terms. For example, we take the linearized form of the frequency domain viscous drag operator as derived by Watanabe et al. [17]:

$$\mathcal{D}(\tilde{u}) = i\omega\rho_0 \frac{f_\nu}{1 - f_\nu} \tilde{u}, \quad (4.25)$$

where f_ν is the viscous Rott function and ρ_0 is the mean density. This operator is straightforward to add to the left-hand side of Eq. (4.22), without any problems related to the time-frequency domain conversion, possibly resulting in spurious instabilities of the higher acoustic modes [19]. The extension of the model for TA systems is elaborated in Chapter (5).

Solution method

For the spatial discretization, (second order) central finite differences are used on an equidistant grid. A stable spatial discretization was found by using a staggered grid. Details about the fully discretized forms of each equation are given in Appendix (4.B).

Since the fully discretized forms of Eqs. (4.20), and (4.22)-(4.24) are nonlinear, an iterative solution method has to be chosen. As proposed by Maple et al. [11], a pseudo-time can be introduced, with which the solution can be driven to its steady-state in this pseudo-time. The pseudo-time method is applicable for flow problems with small oscillatory components. However, in nonlinear acoustics the oscillatory component is large and nonlinearities are generally treated as small. That is why perturbation methods can be used to find nonlinear corrections to the linear solution. For this reason we state that a direct solution searching method is more appropriate. We propose solving the equations using the Newton-Raphson method. For this method, the Jacobian matrix is required, which we evaluate analytically.

We introduce an operator $L_{k,i}$ for equation k on node i , where $k = c, m, e, s$ are the continuity, momentum, isentropic and state equations, respectively. For example, the fully discretized continuity equation operator for an equidistant grid is

$$\begin{aligned} \mathbf{L}_{c,i} &= 2\Delta x \mathbf{D} \cdot \tilde{\rho}_i + \mathcal{F} \cdot [(\mathcal{F}^{-1} \cdot \tilde{\rho}_{i+1}) \circ (\mathcal{F}^{-1} \cdot \tilde{\mathbf{u}}_{i+1})] \\ &- \mathcal{F} \cdot [(\mathcal{F}^{-1} \cdot \tilde{\rho}_{i-1}) \circ (\mathcal{F}^{-1} \cdot \tilde{\mathbf{u}}_{i-1})]. \end{aligned} \quad (4.26)$$

Then the Jacobian terms of each operator can be computed. For the example of Eq. (4.26):

$$\frac{\partial \mathbf{L}_{c,i}}{\partial \tilde{\rho}_i} = 2\Delta x \mathbf{D}, \quad (4.27a)$$

$$\frac{\partial \mathbf{L}_{c,i}}{\partial \tilde{\mathbf{U}}_i} = [\mathbf{0}], \quad (4.27b)$$

$$\frac{\partial \mathbf{L}_{c,i}}{\partial \tilde{\rho}_{i+1}} = \mathcal{F} \cdot \text{diag}(\mathcal{F}^{-1} \cdot \tilde{\mathbf{u}}_{i+1}) \cdot \mathcal{F}^{-1}, \quad (4.27c)$$

$$\frac{\partial \mathbf{L}_{c,i}}{\partial \tilde{\mathbf{u}}_{i+1}} = \mathcal{F} \cdot \text{diag}(\mathcal{F}^{-1} \cdot \tilde{\rho}_{i+1}) \cdot \mathcal{F}^{-1}, \quad (4.27d)$$

$$\frac{\partial \mathbf{L}_{c,i}}{\partial \tilde{\rho}_{i-1}} = \mathcal{F} \cdot \text{diag}(\mathcal{F}^{-1} \cdot \tilde{\mathbf{u}}_{i-1}) \cdot \mathcal{F}^{-1}, \quad (4.27e)$$

$$\frac{\partial \mathbf{L}_{c,i}}{\partial \tilde{\mathbf{u}}_{i-1}} = \mathcal{F} \cdot \text{diag}(\mathcal{F}^{-1} \cdot \tilde{\rho}_{i-1}) \cdot \mathcal{F}^{-1}, \quad (4.27f)$$

where $[\mathbf{0}]$ denotes the null matrix. Once all the operators have been computed, a global operator is created by appending all the local operators of all equations to one vector:

$$\mathbf{L}(\mathbf{x}) = \left\{ \mathbf{L}_{c,0} \quad \mathbf{L}_{m,0} \quad \dots \quad \mathbf{L}_{m,N_{gp}-1} \quad \mathbf{L}_{e,N_{gp}-1} \quad \mathbf{L}_{s,N_{gp}-1} \right\}^T, \quad (4.28)$$

where $(\mathbf{x})$ is the total stacked solution vector and N_{gp} is the number of grid-points. When the global Jacobian matrix $\mathbf{J}$ is assembled from all sub-Jacobians, an update of the global solution vector is obtained by

$$\delta \mathbf{x} = \mathbf{J}^{-1} \cdot \mathbf{L}(\mathbf{x}). \quad (4.29)$$

It should be noted that, when starting this solution method under the initial conditions of a quiescent field, stopping after the first iteration results in the (discretized) linearized acoustic solution. The solution is converged for a solution vector $\mathbf{x}_{\text{conv}}$ when the Euclidean norm of $\mathbf{L}(\mathbf{x}_{\text{conv}})$ is below a convergence criterion, which we take as 10^{-10} . For the case described in Sec. (4.4), just 5 iterations are sufficient to achieve convergence.

4.4 Nonlinear plane wave propagation

To test and verify the NLFD Euler equations implementation, the method is applied to the case of an inviscid plane propagating wave problem in a semi-infinite prismatic tube. A schematic overview of the situation is given in Figure (4.1). The fluid is air, assumed to be a calorically perfect gas at standard temperature and pressure (STP) conditions ($p_0 = 101.325$ Pa, $T_0 = 293.15$ K). At the left boundary ($x = 0$), the velocity is prescribed: $u(x = 0, t) \equiv u_0(t) = u_a \sin(\omega t)$. Since this excitation is periodic, we expect the response in the tube to also be periodic. The tube extends infinitely, so no reflections can come back, and hence we expect a propagating wave. For finite amplitude disturbances, the wave profile will distort. The physical condition allows for two situations, depending on the initial velocity amplitude (u_a). For weak sound waves, wave steepening occurs until the spatial gradients

Figure 4.1 – Schematic overview of the plane propagating wave problem. At the left boundary ($x = 0$), a velocity is assumed. For the numerical solution, the computational domain stops at the right boundary ($x = L$), where a non-reflective boundary condition is applied, such that a pure traveling wave is generated in the domain. A piece of the numerical grid is shown as well.

become high enough, such that viscous effects and thermal conduction can no longer be neglected. Then, these dissipation mechanisms will dampen out the wave. In the case of strong sound waves, periodic shock fronts will be formed, which travel at the undisturbed speed of sound [8, p. 385], c_0 . Eventually, these shock fronts will also dampen out.

In the model, no physical dissipation is taken into account. Therefore if the domain is large enough a shock wave will appear. Hence, the validity of the model is for distances smaller than the distance at which a discontinuity will form.

An analytical solution for this plane propagating wave problem is available, namely the Fubini-Ghiron solution [1]. Some details of the analytical solution are presented in Appendix (4.A). Details about the numerical solution are discussed in Sec. (4.4). In Sec. (4.4), results will be compared and discussed.

Numerical solution

To create the numerical model, a finite domain is required. Hence, the domain is cut off at a distance $x = L$. At the end of the domain, we employ a non-reflecting boundary condition (Eq. (4.40)). Filling in the amplitude vectors, we find that for the right boundary condition

$$\tilde{\mathbf{p}}(x = L) \equiv \tilde{\mathbf{p}}_L = p_0 \mathcal{F} \cdot \left(\mathbf{1} + \frac{\gamma - 1}{2} \frac{\mathcal{F}^{-1} \cdot \tilde{\mathbf{u}}_L}{c_0} \right)^{\frac{2\gamma}{\gamma-1}}, \quad (4.30)$$

where $\mathbf{1} = \begin{Bmatrix} 1 & 1 & \dots \end{Bmatrix}^T$. Upon linearizing Eq. (4.30) by letting ($\tilde{\mathbf{u}}_L \rightarrow \mathbf{0}$) we find

$$\tilde{\mathbf{p}}_L = p_0 \mathcal{F} \cdot \mathbf{1} + z_0 \tilde{\mathbf{u}}_L, \quad (4.31)$$

where $\mathcal{F} \cdot \mathbf{1} = \begin{Bmatrix} 1 & 0 & 0 & \dots \end{Bmatrix}^T$. Eq. (4.31) is the exact linear acoustics non-reflective boundary condition. To determine the absolute pressure in the domain, we have to apply a pressure boundary condition on the left side. However, at this boundary the velocity is prescribed. We convert the velocity boundary condition to a pressure boundary condition, by again using Eq. (4.40). A purely sinusoidal velocity will result in a slightly deformed pressure profile. We obtain the first four pressure phasors for the left numerical boundary condition from the analytical solution, as is explained in Appendix (4.A). The starting guess solution for the Newton iterations is a quiescent homogeneous field at STP.

Results

We describe results for a test case having a base frequency of 100 Hz and where the acoustic Mach number

$$M_a = \frac{u_a}{c_0} \quad (4.32)$$

is set to 0.028, corresponding to a velocity u_a of $4 \cdot 10^3 / z_0 \simeq 9.7 \text{ m} \cdot \text{s}^{-1}$, with z_0 the characteristic acoustic impedance. This corresponds to a shock distance ($x_\perp$) of 16.1 m. The numerical domain size is set to $0.75x_\perp$. Two cases have been computed, for a number of harmonics of 2 and 4. In both cases, the number of grid-points is chosen such that ~ 15 grid-points fit in one wavelength of the highest frequency that is taken into account. A grid study has been done to check the grid-independency of the solution. For $N_f = 4$, the total degrees of freedom to solve for are

$$\underbrace{423}_{N_{\text{gp}}} \times \underbrace{(2 \times 4 + 1)}_{N_s} \times \underbrace{4}_{\rho, u, T, p} = 15228, \quad (4.33)$$

for which the solution time is 2 s.

Figure (4.2) shows the first four pressure phasors of both the analytical and numerical solution with $N_f = 4$. The results agree very well, except for harmonic number 4. When the fourth harmonic requires better resolution, more harmonics should be added to the numerical solution. For the first harmonic, the relative difference between the analytical and the numerical solution at $x = L$ is only 1.4%. Figure (4.3) shows the normalized reconstructed pressure time profile at the right side of the domain. The numerical solutions are shown for both $N_f = 2$ and $N_f = 4$. As shown, the numerical profile converges to the analytical profile when the number of harmonics is increased. If we set $N_f = 15$, the numerical curve lies on top of the analytical curve.

Discussion

In the current NLFD implementation, all degrees of freedom need to be stored in memory during the solution process. Depending on the amount of harmonics taken into account, a drawback of the NLFD method can be the memory usage. For example, at distances closer to $x_\perp$, the spectrum of the analytical solution shows less decay with increasing frequency. For that case, we need to include more harmonics. Yet when the minimum number of grid-points in the smallest wavelength is constrained, increasing the number of harmonics also entails augmenting grid-points; hence, the number of degrees of freedom scales with N_f^2 . Due to this principle, the required memory usage can quickly increase. A possible solution is to vary the number of harmonics per grid-point [11], but this has the disadvantage of needing a much more complicated computer code. Thermoacoustic system sizes are generally either $\sim \lambda_0$, or $< \lambda_0$ [16] (when using a mechanical resonator). This means that the current validation test case already has a relatively high number of grid-points, compared to the number used for modeling a TA system. Therefore we do not expect to encounter memory problems when modeling thermoacoustic systems.

Figure 4.2 – Analytically and numerically computed pressure phasors for the simple wave of Sec. (4.4). On the abscissa, the distance is normalized using the shock distance ($x_{\perp}$). On the ordinates, the value of the pressure phasors of the first four harmonics is normalized by $z_0 u_a$. The Mach number M_a of the sinusoidal velocity on the left side is 0.028. For the numerical result, a non-reflective boundary condition is used on the right side of the domain.

Figure 4.3 – Normalized pressure time profile at the end of the numerical domain ($x = L$). To emphasize the developed degree of nonlinearity, the linear acoustics solution (denoted by Linear sol.) is plotted as well.

The NLFD method can also be applied to the full three-dimensional compressible Navier-Stokes equations [12]. In that case, we expect that by using the NLFD method, a significant decrease in computational cost can be achieved. A problem can be that all harmonics - or time instances - have to be kept in memory during the solution procedure, as opposed to solving the system in time domain. One possible disadvantage is that this method is limited to periodic systems, however, not all TA systems behave this way [18]. Nevertheless, commercially feasible systems should behave periodically.

4.5 Conclusions

The nonlinear frequency domain method is explained and applied to the 1D Euler equations. These equations are used to solve the classical nonlinear acoustic problem of finite amplitude plane wave propagation. The numerical results agree with the analytical results, showing that the method has been implemented correctly. A comparison of the pressure profile in time has been shown in which the wave profile is significantly distorted. For this case, the numerical solution converges to the analytical solution when the number of harmonics increases.

In comparison with the traditional nonlinear frequency domain method [6], a reduction in computational cost is accomplished by computing the nonlinear terms in time domain and afterwards transforming back to frequency domain. The ability to apply frequency domain source terms provides several advantages when thermoacoustic systems are to be modeled. Moreover, the computational cost of the method for modeling TA systems is expected to be much smaller than for time-accurate methods since we can, with only a few Newton iterations, directly solve its periodic steady-state.

For modeling TA systems, several model adjustments are required. As was explained, the thermoacoustic coupling terms need to be included. Secondly, for the situation of resonators having a varying cross section, the 1D Euler equations have to be replaced with their quasi 1D variants [17]. Different segments need to be coupled, e.g. a heat exchanger to a stack and to a resonator tube. In the case of thermoacoustic engines, the fundamental frequency is not known, so the solution algorithm needs to be adjusted for an unknown fundamental frequency. These subjects will be addressed in Chapter (5).

4.A Analytical solution for nonlinear plane wave propagation

In this appendix, the analytical solution for a nonlinear plane propagating wave is described. A more elaborated derivation is given in, for example the book by Landau and Lifshitz [8, p. 394]. For a so-called simple wave, u is constant along lines with slope $\frac{dx}{dt} = c + u$, or following [14, p. 265]

$$\frac{\partial u}{\partial t} + \left(c_0 + \frac{\gamma + 1}{2} u \right) \frac{\partial u}{\partial x} = 0. \quad (4.34)$$

This has the implicit solution

$$u(x, t) = f \left(x - \left(c_0 + \frac{\gamma + 1}{2} u \right) t \right), \quad (4.35)$$

where f is some function to be determined by the initial/boundary conditions. Filling in the boundary condition for u_L , the implicit solution reads:

$$u(x, t) = u_a \sin \left(\omega t - \frac{\omega}{c_0 + \frac{\gamma + 1}{2} u} x \right). \quad (4.36)$$

Applying a first order Taylor expansion of the denominator, the solution can be made explicit by expanding in a Fourier series in space. One finally finds

$$u(x, t) = u_a \sum_{n=1}^{\infty} \frac{2J_n(nx/x_{\perp})}{nx/x_{\perp}} \sin[n(\omega t - k_0 x)], \quad (4.37)$$

where J_n is the Bessel function of first kind and n^{th} order, k_0 is the wave number [rad/m] corresponding to the base excitation

$$k_0 = \frac{\omega}{c_0}, \quad (4.38)$$

and $x_{\perp}$ the shock-distance:

$$x_{\perp} = \frac{2c_0}{(\gamma + 1)u_a k_0}. \quad (4.39)$$

Eq. (4.37) is known as the Fubini-Ghiron solution [1]. Once the solution for the velocity is known, the pressure can be computed from the local speed of sound, which is only a function of the local velocity:

$$p = p_0 \left(1 + \frac{\gamma - 1}{2} \frac{u}{c_0} \right)^{\frac{2\gamma}{\gamma - 1}}. \quad (4.40)$$

To obtain pressure phasors, we use the same approach as for the numerical solution; that is, generating a time level vector for u , and substituting in the NLFD form of Eq. (4.40), which is

$$\tilde{\mathbf{p}} = p_0 \mathcal{F} \cdot \left(\mathbf{1} + \frac{\gamma - 1}{2} \frac{\mathcal{F}^{-1} \cdot \tilde{\mathbf{u}}}{c_0} \right)^{\frac{2\gamma}{\gamma - 1}}. \quad (4.41)$$

For the analytical solution, we use 101 time instances, corresponding to $N_f = 50$. The first four harmonics obtained at $x = 0$ are used as boundary conditions for the numerical solution of Sec. (4.4).

4.B Fully discretized form of the Euler equations

In this appendix, we show the operators for the fully discretized form of the Euler equations. A stable numerical scheme is found by using a staggered grid. In Figure (4.1), the grid is schematically shown. The discrete points for the pressure are shifted half a grid-point with the points for velocity, density and temperature. For this grid, the NLFD continuity equation operator for grid-point i becomes

$$\begin{aligned} \mathbf{L}_{c,i} = & \mathbf{D} \cdot \tilde{\rho}_i + \frac{1}{2\Delta x} \mathcal{F} \cdot (\mathcal{F}^{-1} \cdot \tilde{\rho}_{i+1} \circ \mathcal{F}^{-1} \cdot \tilde{\mathbf{u}}_{i+1}) \\ & - \frac{1}{2\Delta x} \mathcal{F} \cdot (\mathcal{F}^{-1} \cdot \tilde{\rho}_{i-1} \circ \mathcal{F}^{-1} \cdot \tilde{\mathbf{u}}_{i-1}) \end{aligned} \quad (4.42)$$

where Δx is the grid spacing. The fully discretized momentum equation operator is

$$\begin{aligned} \mathbf{L}_{m,i} = & \mathbf{D} \cdot \mathcal{F} \cdot (\mathcal{F}^{-1} \cdot \tilde{\rho}_i \circ \mathcal{F}^{-1} \cdot \tilde{\mathbf{u}}_i) + \frac{1}{2\Delta x} \mathcal{F} \cdot [\mathcal{F}^{-1} \cdot \tilde{\rho}_{i+1} \circ (\mathcal{F}^{-1} \cdot \tilde{\mathbf{u}}_{i+1})^2] \\ & - \frac{1}{2\Delta x} \mathcal{F} \cdot [\mathcal{F}^{-1} \cdot \tilde{\rho}_{i-1} \circ (\mathcal{F}^{-1} \cdot \tilde{\mathbf{u}}_{i-1})^2] + \frac{1}{\Delta x} (\tilde{\mathbf{p}}_{i+\frac{1}{2}} - \tilde{\mathbf{p}}_{i-\frac{1}{2}}). \end{aligned} \quad (4.43)$$

The fully discretized isentropic state equation is

$$L_{e,i} = \frac{1}{2} (\tilde{\mathbf{p}}_{i+\frac{1}{2}} - \tilde{\mathbf{p}}_{i-\frac{1}{2}}) - p_0 \mathcal{F} \cdot (\mathcal{F}^{-1} \cdot \tilde{\rho}_i / \rho_0)^\gamma. \quad (4.44)$$

And finally, the state equation:

$$L_{e,i} = \frac{1}{2} (\tilde{\mathbf{p}}_{i+\frac{1}{2}} - \tilde{\mathbf{p}}_{i-\frac{1}{2}}) - R_s \mathcal{F} \cdot (\mathcal{F}^{-1} \cdot \tilde{\rho}_i \circ \mathcal{F}^{-1} \cdot \tilde{\mathbf{T}}_i). \quad (4.45)$$

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MODELING OF THERMOACOUSTIC SYSTEMS USING THE NONLINEAR FREQUENCY DOMAIN METHOD

The design of thermoacoustic devices requires accurate yet fast computational models. In this paper such a model is presented, which is fast because it directly solves the periodic steady-state of a thermoacoustic device. The model is defined in the frequency domain, which eases the implementation of phase delays due to viscous resistance and thermoacoustic heat exchange. As a demonstration, the model is used to solve an experimental standing wave thermoacoustic engine. The obtained results agree with experimental results, as well as with results from a nonlinear time domain model from the literature.

5.1 Introduction

When designing thermoacoustic (TA) systems, fast computational models are required for quick feedback on design decisions. However, fast computational models are often based on some crude simplifications of the underlying physics. Hence, generally a trade-off has to be made between the accuracy of the physics and the computational cost. On the one hand, we have well-established one-dimensional linear TA theory [20, 23], which is fast to solve, but only valid for low amplitude, while on the other hand, multi-dimensional computational fluid dynamics (CFD) simulations offer deep insight into the physical processes of a TA system, but these simulations demand significantly more time to solve.

From a computational time perspective, simulating a complete TA system with CFD is considered too costly and therefore not suitable for optimization purposes. Hence, additional assumptions are often applied. In the work of Hamilton et al. [10], the Courant number restriction on the time step is loosened by neglecting the momentum equation in the direction perpendicular to the wave propagation direction. In other multi-dimensional modeling efforts, a low Mach number analysis [11, 28] has been applied and/or only a sub-domain of the full system [4] is taken into account. For the interaction with the part of the system not modeled in detail, typically a linear acoustic model is used, or linear acoustic boundary conditions are applied. Consequently, nonlinear effects are neglected in that part of the system which is not modeled.

The main focus of this paper is the development of a nonlinear one-dimensional TA model. The research on such a model has been started by Watanabe et al. [27] and Yuan et al. [29]. With this time domain model, they were able to simulate a thermoacoustic system starting from initial conditions, up to the periodic steady-state. The onset of a self-excited

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system was triggered with an initial pressure disturbance. Initially, their model suffered from spurious instabilities, caused by unstable linear eigenmodes which are introduced by the viscous resistance and heat transfer models. To suppress these instabilities, artificial diffusion was added. Later, Karpov and Prosperetti [14] solved this problem with better models for resistance and heat transfer, which they call the “exchange terms” as they are responsible for exchanging mass and momentum with the surrounding solid. Still, their exchange terms are only in agreement with linear theory for the first and second harmonic of the solution’s spectrum. The frequencies of these harmonics needed to be determined before starting the simulation, which makes their model, in some sense, dependent on global (tunable) parameters.

A more recent work discusses the application of the method of characteristics [19] to model the experimental standing wave thermoacoustic engine of Swift. [22]. Here, heat transfer and viscous resistance is computed using semi-empirical models.

The model presented in this work is based on the quasi one-dimensional model for thermoacoustic devices used by Karpov et al. [14]. Computational time is diminished by directly solving the periodic steady-state. This is facilitated by a reformulation in the frequency domain using the nonlinear frequency domain (NLFD) method [8, 15]. Chapter (4) has shown that the NLFD method is able to solve the periodic steady-state of a one-dimensional acoustic system with a computational time of ~ 1 s. With the NLFD method, all physical quantities are assumed to be periodic in time, hence, they can be described by a Fourier series. Then, instead of doing a time integration, all Fourier coefficients are simultaneously solved. Using the Fourier coefficients, the periodic time domain response can be reconstructed by applying an inverse discrete Fourier transform. An additional advantage of a frequency domain formulation is that the frequency domain exchange models of linear TA theory can be used in unmodified form, which makes the model in exact agreement with linear thermoacoustic theory. A drawback of the method is that the initial transients of a system cannot be modeled. However, in most cases the interest lies primarily on the performance of a TA system in steady-state.

In Sec. (5.2), the model is explained, including the application of the NLFD method. Furthermore, a frequency domain form is derived for the exchange terms. Then, in Sec. (5.3), the numerical method used to solve the model of Sec. (5.2) will be described. Notably, several problems accompanying the NLFD method, which are related to solving for closed self-excited systems (TA prime movers), are treated. Then, in Sec (5.4), the model will be applied to solve the steady-state of an experimental standing wave TA engine [3]. The results are discussed, and finally in Sec. (5.6) conclusions are drawn.

5.2 Model

Governing equations

As shown in Sec. (2.6), a quasi one-dimensional nonlinear continuity, momentum, and energy equation can be derived from the governing equations. For convenience, the model is

restated here:

$$\frac{\partial \rho}{\partial t} + \frac{1}{S_f} \frac{\partial}{\partial x} (S_f \rho u) = 0, \quad (5.1a)$$

$$\frac{\partial}{\partial t} (\rho u) + \frac{1}{S_f} \frac{\partial}{\partial x} (S_f \rho u^2) + \frac{\partial p}{\partial x} = -\mathcal{R}_u, \quad (5.1b)$$

$$\frac{\partial}{\partial t} \left(\frac{p}{\gamma - 1} + \frac{1}{2} \rho u^2 \right) + \frac{1}{S_f} \frac{\partial}{\partial x} \left[u S_f \left(\frac{\gamma}{\gamma - 1} p + \frac{1}{2} \rho u^2 \right) \right] = \mathcal{H}_u - \mathcal{Q}_u \frac{dT_w}{dx}, \quad (5.1c)$$

accompanied with the perfect gas equation of state

$$p = \rho R_s T, \quad (5.1d)$$

where x is the wave propagation direction, t is time, ρ density, u (axial) velocity, p pressure, T temperature, γ the ratio of specific heats, R_s the specific gas constant and S_f is the fluid cross-sectional area. So $S_f = \phi S$, where ϕ is the porosity and S the total cross-sectional area. $\mathcal{R}_u$ is the resistive drag force [$\text{N} \cdot \text{m}^{-3}$] in the wave propagation direction due to shear stress from the fluid acting on the wall. It is by definition

$$\mathcal{R}_u \equiv -\frac{1}{S_f} \oint_{\Pi} (\underline{\tau} \cdot \mathbf{n})_x d\Pi, \quad (5.2)$$

where $\underline{\tau}$ is the viscous stress tensor and $\mathbf{n}$ is the unit normal pointing transversely from the fluid in the direction of the solid. The transverse direction is perpendicular to the wave propagation direction. Π is the “wetted perimeter”, the contact line between solid and fluid. The two terms on the right side of Eq. (5.1c) together describe heat flow from the solid to the fluid:

$$\mathcal{H}_u - \mathcal{Q}_u \frac{dT_w}{dx} \equiv -\frac{1}{S_f} \oint_{\Pi} (\mathbf{q} \cdot \mathbf{n}) d\Pi, \quad (5.3)$$

where $\mathbf{q}$ is the heat flux vector.

Unless otherwise stated, all quantities in Eqs. (5.1) are area-averaged, and for brevity the $\langle \dots \rangle$ notation of Chapter (2) is dropped. Chapter (2) provides overview of the major assumptions behind this model. In the energy equation, two sources are defined for heat exchange with the wall. The source $\mathcal{H}_u$, is due to transverse conduction as a result of a difference between the wall temperature and the fluid area-averaged temperature. The source $\mathcal{Q}_u \frac{dT_w}{dx}$ is due to convective enthalpy flow, originating from the velocity and temperature boundary layer close to the wall. In Sec. (5.2), the momentum and energy exchange terms $\mathcal{R}_u$, $\mathcal{H}_u$ and $\mathcal{Q}_u$ are treated in detail. We assume that the temperature of the wall is constant in time, but a known function of space: $T_w = T_w(x)$.

Application of the NLFD method

We assume a periodic steady state, so all dependent variables can be described by a Fourier series. The basis of the nonlinear frequency domain (NLFD) method [15] is to substitute a

truncated Fourier series for each dependent variable in Eqs. (5.1) and to solve for the Fourier coefficients. To explain this method, we take the density as the example dependent variable. The truncated Fourier series for the density is

$$\rho(x, t) = \tilde{\rho}_{c,0}(x) + \Re \left[\sum_{n=1}^{N_f} \tilde{\rho}_{c,n}(x) e^{in\omega t} \right], \quad (5.4)$$

where ω is the fundamental radial frequency, n the harmonic index, N_f is the total number of harmonics taken into account, $\Re[\dots]$ denotes the real part of its argument, and $\tilde{\rho}_{c,i}$ with $0 \leq i \leq N_f$ is the Fourier coefficient (phasor) for harmonic i . We store all Fourier coefficients in a real-valued vector $\tilde{\rho}$ as

$$\tilde{\rho} = \left\{ \tilde{\rho}_{c,0} \quad \Re[\tilde{\rho}_{c,1}] \quad \Im[\tilde{\rho}_{c,1}] \quad \Re[\tilde{\rho}_{c,2}] \quad \Im[\tilde{\rho}_{c,2}] \quad \cdots \quad \Im[\tilde{\rho}_{c,N_f}] \right\}^T, \quad (5.5)$$

where $\Im[\dots]$ denotes the imaginary part of its argument. The imaginary part of the time-averaged Fourier component ($\tilde{\rho}_{c,0}$) is zero and is therefore not taken into account in the storage vector. From the vector of density Fourier coefficients, a vector of time instances ρ can be constructed by applying the inverse discrete Fourier transform (DFT):

$$\rho_m \equiv \rho(t_m) = \tilde{\rho}_0 + \Re \left[\sum_{n=1}^{N_f} (\tilde{\rho}_{2n-1} + i\tilde{\rho}_{2n}) e^{inm\omega\Delta t} \right], \quad 0 \leq m < 2N_f + 1, \quad (5.6)$$

where $t_m = m\Delta t$ and Δt is the time step between each time instance, which can be computed from the fundamental frequency and the number of harmonics with

$$\Delta t = \frac{2\pi}{\omega(2N_f + 1)}. \quad (5.7)$$

The operation of Eq. (5.6) applied to all time instance simultaneously is written as

$$\rho = \mathcal{F}^{-1} \cdot \tilde{\rho}, \quad (5.8)$$

where $\mathcal{F}$ is referred to as the discrete Fourier transform matrix, and hence $\mathcal{F}^{-1}$ the backward discrete Fourier transform matrix (see Chapter (4)). For the continuity equation, we also need the time derivative of the density. For the periodic function of Eq. (5.4), the time derivative can be decomposed in terms of the time derivatives of the individual complex exponentials. So the vector of the time derivatives of the density can be evaluated as

$$\frac{\partial}{\partial t} \rho = \mathcal{F}^{-1} \cdot \omega \tilde{\mathbf{D}} \cdot \tilde{\rho}, \quad (5.9)$$

where $\check{\mathbf{D}}$ is the “time-derivative matrix”, scaled with ω :

$$\omega\check{\mathbf{D}} = \omega \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & -1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & -2 & \dots \\ 0 & 0 & 0 & 2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (5.10)$$

Thus, the time derivative of the density in the frequency domain is

$$\widehat{\frac{\partial \rho}{\partial t}} = \mathcal{F} \cdot \omega \mathcal{F}^{-1} \cdot \check{\mathbf{D}} \cdot \tilde{\rho} = \omega \check{\mathbf{D}} \cdot \tilde{\rho}. \quad (5.11)$$

To compute the spatial derivative of the mass flux, the nonlinear product $m = \rho u$ is required. This is done by first applying the backward discrete Fourier transform to the density and velocity vectors of Fourier coefficients, then computing the product of density and velocity at each time instance, and finally transforming back into the frequency domain. Accordingly, the Fourier coefficients of the mass flux are computed as

$$\tilde{m} = \mathcal{F} \cdot ((\mathcal{F}^{-1} \cdot \tilde{\rho}) \circ (\mathcal{F}^{-1} \cdot \tilde{u})), \quad (5.12)$$

where the operator $\circ$ denotes the elementwise product of two vectors as defined in Chapter (4), yielding a vector.

We apply this method to Eqs (5.1a)-(5.1d). As dependent variables the frequency domain storage vectors for cross-sectional area averaged density ($\tilde{\rho}(x)$), volume flow ($\tilde{U}(x)$), temperature ($\tilde{T}(x)$) and pressure ($\tilde{p}(x)$) are chosen. The continuity, momentum and energy equation are multiplied with the fluid cross-sectional area. For the continuity equation, we find

$$S_f \omega \check{\mathbf{D}} \cdot \tilde{\rho} + \frac{d}{dx} \mathcal{F} \cdot (\rho \circ U) = 0. \quad (5.13)$$

For brevity, the inverse DFT inside the parentheses is not shown. Therefore, for the time domain vectors ρ and U , one should read $\mathcal{F}^{-1} \cdot \tilde{\rho}$ and $\mathcal{F}^{-1} \cdot \tilde{U}$, respectively. The volume flow can be related to the velocity by

$$U \equiv u S_f. \quad (5.14)$$

For the NLFD momentum equation we find

$$\omega \check{\mathbf{D}} \cdot \mathcal{F} \cdot (\rho \circ U) + \frac{d}{dx} \mathcal{F} \cdot \left(\frac{\rho \circ U^2}{S_f} \right) + S_f \frac{d\tilde{p}}{dx} = -\tilde{\mathcal{R}}_U, \quad (5.15)$$

and for the NLFD form of the energy equation

$$\omega \tilde{\mathbf{D}} \cdot \left(\frac{S_f \tilde{\mathbf{P}}}{\gamma - 1} + \mathcal{F} \cdot \frac{1}{2S_f} (\boldsymbol{\rho} \circ \mathbf{U}^2) \right) + \frac{d}{dx} \mathcal{F} \cdot \left(S_f \boldsymbol{\kappa} \circ \frac{dT}{dx} \right) + \frac{d}{dx} \mathcal{F} \cdot \left(\mathbf{U} \circ \left(\frac{\gamma}{\gamma - 1} \mathbf{p} + \frac{1}{2S_f^2} (\boldsymbol{\rho} \circ \mathbf{U}^2) \right) \right) = \tilde{\mathcal{H}}_U - \tilde{\mathcal{Q}}_U \frac{dT_w}{dx}, \quad (5.16)$$

where κ is the thermal conductivity, and hence $\boldsymbol{\kappa}$ the vector of time instances for the thermal conductivity. Eq. (5.16) is not the full NLFD-equivalent variant of Eq. (5.1c), because we have added an extra term corresponding to axial conduction. As stated by Karpov et al. [14], this term is only important near walls. The reason for adding this term is explained in Appendix (5.A). The last equation to solve for is the NLFD perfect gas equation of state:

$$\tilde{\mathbf{p}} = R_s \mathcal{F} \cdot (\boldsymbol{\rho} \circ \mathbf{T}). \quad (5.17)$$

In Eqs. (5.15) and (5.16), viscous resistance and heat transfer terms are redefined such that $\tilde{\mathcal{R}}_U = S_f \tilde{\mathcal{R}}_u$, $\tilde{\mathcal{H}}_U = S_f \tilde{\mathcal{H}}_u$, and $\tilde{\mathcal{Q}}_U = S_f \tilde{\mathcal{Q}}_u$, where the exchange terms with lower case u are defined as the Fourier transform of the resistance and heat transfer operators of Eqs. (5.1a)-(5.1d).

The exchange terms

We assume laminar, fully developed - oscillating - flow, so the resistance $\tilde{\mathcal{R}}$ is a linear function of the volume flow. A closed form laminar resistance coefficient $\tilde{\mathbf{R}}$ in frequency domain is derived from the linearized momentum equation, as is explained by Watanabe et al. [27]. Applying the NLFD method to this model results in

$$\tilde{\mathcal{R}}_U = \tilde{\mathbf{R}} \diamond \tilde{\mathbf{U}}, \quad (5.18)$$

where $\tilde{\mathbf{R}}$ is the frequency domain resistance coefficient. The symbol $\diamond$ denotes the product in the frequency domain. Since the real and imaginary part of the value at each frequency is stacked in the form of Eq. (5.5), an equivalent matrix-vector product is generated. For example, the $\tilde{\mathbf{R}}$ this matrix is

$$M(\tilde{\mathbf{R}}) = \begin{bmatrix} \tilde{R}_0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \Re(\tilde{R}_1) & \Im(-\tilde{R}_1) & 0 & 0 & \dots \\ 0 & \Im(\tilde{R}_1) & \Re(\tilde{R}_1) & 0 & 0 & \dots \\ 0 & 0 & 0 & \Re(\tilde{R}_2) & \Im(-\tilde{R}_2) & \dots \\ 0 & 0 & 0 & \Im(\tilde{R}_2) & \Re(\tilde{R}_2) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (5.19)$$

Then, the frequency domain product $\tilde{\mathbf{R}} \diamond \tilde{\mathbf{U}}$ is computed as $M(\tilde{\mathbf{R}}) \cdot \tilde{\mathbf{U}}$, where the operator $\cdot$ is the matrix-vector product. The components of $\tilde{\mathbf{R}}$ are defined such that

$$\tilde{R}_{2n-1} + i\tilde{R}_{2n} = \frac{\tilde{\mu}_0}{r_h^2} \frac{is_n^2 f_{\nu,n}}{1 - f_{\nu,n}} \quad \text{for } n > 0, \quad (5.20)$$

where $\tilde{\mu}_0$ is the time-averaged dynamic viscosity, $f_{\nu,n} = f_{\nu,n}(s_n)$ is the cross-sectional geometry dependent viscous Rott function [20, 23] evaluated at shear wave number s_n . s_n is the shear wave number for harmonic n , given as

$$s_n = r_h \sqrt{\frac{\tilde{\rho}_0 \omega n}{\tilde{\mu}_0}}, \quad (5.21)$$

where r_h is the hydraulic radius. It is defined as the cross-sectional area divided by the wetted perimeter:

$$r_h = \frac{S_f}{\Pi}. \quad (5.22)$$

The shear wave number can be related to the viscous penetration depth for harmonic n as

$$s_n \equiv \sqrt{2} \frac{r_h}{\delta_{\nu,n}}. \quad (5.23)$$

If one would use this definition of the viscous resistance in the time domain, computing the viscous drag would require the computation of the convolution integral of the inverse Fourier transform of the frequency domain resistance coefficient with the volume flow. However, performing this integration over all cycles necessary to reach steady-state is completely impractical [27]. Moreover, for simple geometries, such as circular tubes and parallel plates, no closed form solution exists for the inverse Fourier transform of the resistance coefficient. These are important reasons for choosing a frequency domain formulation rather than a time domain formulation.

In the drag model it is assumed that nonlinear effects in the viscous boundary layer are small, such that the boundary layer flow can be regarded as a superposition of linear boundary layer velocity profiles. Hence, the drag coefficient is based on the following transverse velocity distribution:

$$u(x, \mathbf{r}, t) = \Re \left[\sum_{n=0}^{N_f} \langle \tilde{u}_n \rangle(x) \frac{1 - h_{\nu,n}(x, \mathbf{r})}{1 - f_{\nu,n}(x)} e^{in\omega t} \right], \quad (5.24)$$

where $\mathbf{r}$ is the transverse coordinate, $\langle \dots \rangle$ denotes area-averaging and $h_{\nu,n}$ is the transverse velocity shape function, from which the Rott functions are derived:

$$\langle h_{\nu,n} \rangle = f_{\nu,n}. \quad (5.25)$$

For example, for parallel-plate pores, $h_{\nu,n}$ is

$$h_{\nu,n}(x, y) = \frac{\cosh\left(s_n \sqrt{i} \frac{y}{r_h}\right)}{\cosh\left(s_n \sqrt{i}\right)}, \quad (5.26)$$

	f_ν	$\tilde{R}_0$	$\tilde{H}_0$	$\tilde{Q}_0$
Parallel-plate pores	$\frac{\tanh(\sqrt{i} s_n)}{\sqrt{i} s_n}$	$\frac{3\tilde{\mu}_0}{r_h^2}$	$\frac{3\tilde{\kappa}_0}{r_h^2}$	$\frac{1}{5}\tilde{\rho}_0 c_p$
Cylindrical pores	$\frac{J_1(2i\sqrt{i} s_n)}{i\sqrt{i} J_0(2i\sqrt{i} s_n)}$	$\frac{2\tilde{\mu}_0}{r_h^2}$	$\frac{2\tilde{\kappa}_0}{r_h^2}$	$\frac{1}{3}\tilde{\rho}_0 c_p$
Boundary layer approximation	$\frac{-i\sqrt{i}}{s_n}$	0	0	$-\tilde{\rho}_0 c_p$

Table 5.1 – Derived low-frequency limits for the thermoacoustic exchange terms.

where y is the distance from the center line between two plates.

Eq. (5.24) implies that the transverse “modes” of the axial velocity distribution do not mix, hence in the boundary layer the linear (viscous) effects dominate.

As $\tilde{R}_n$ becomes singular for $n = 0$, we evaluate $\tilde{R}_0$ by computing

$$\tilde{R}_0 = \lim_{n\omega \rightarrow 0} \frac{\tilde{\mu}_0}{r_h^2} \frac{is_n^2 f_{\nu,n}}{1 - f_{\nu,n}}. \quad (5.27)$$

This procedure is done for the viscous Rott functions for the earlier mentioned parallel-plate pores, for cylindrical tubes, for which the viscothermal Rott functions are defined in terms of Bessel functions [1, 24, p. 88], and for the boundary layer approximation. In Table (5.1), the computed limits are listed.

For the heat exchange terms, $\tilde{\mathcal{H}}_U$ and $\tilde{\mathcal{Q}}_U$, the same procedure is followed as for the viscous resistance. For $\tilde{\mathcal{H}}_U$ we obtain

$$\tilde{\mathcal{H}}_U = S_f \tilde{\mathbf{H}} \diamond (T_w \tilde{\mathbf{1}} - \tilde{\mathbf{T}}), \quad (5.28)$$

where

$$\tilde{H}_{2n-1} + i\tilde{H}_{2n} = \frac{\text{Pr} \tilde{\kappa}_0}{r_h^2} \frac{is_n^2 f_{\kappa,n}}{1 - f_{\kappa,n}}, \quad (5.29)$$

where Pr is the Prandtl number and $f_{\kappa,n}$ is the thermal Rott function. The thermal Rott function can be evaluated as the viscous Rott function at shear wave number $\sqrt{\text{Pr}} s_n$:

$$f_{\kappa,n} = f_\kappa(s_n, \text{Pr}) = f_\nu(\sqrt{\text{Pr}} s_n). \quad (5.30)$$

The vector $\tilde{\mathbf{1}}$ is

$$\tilde{\mathbf{1}} \equiv \begin{Bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{Bmatrix} = \mathcal{F} \cdot \mathbf{1} \equiv \mathcal{F} \cdot \begin{Bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{Bmatrix}, \quad (5.31)$$

so it only acts on the time-averaged part of the heat transfer. Similarly to Eq. (5.28), for $\tilde{\mathcal{Q}}$ we find

$$\tilde{\mathcal{Q}}_U = \tilde{\mathcal{Q}} \diamond \tilde{\mathbf{U}}, \quad (5.32)$$

where

$$\tilde{Q}_{2n-1} + i\tilde{Q}_{2n} = \frac{\tilde{\rho}_0 c_p}{1 - \text{Pr}} \left(\frac{f_{\nu,n}}{1 - f_{\nu,n}} - \text{Pr} \frac{f_{\kappa,n}}{1 - f_{\kappa,n}} \right). \quad (5.33)$$

In Eq. (5.33), c_p is the specific heat at constant pressure. The results for $\tilde{H}_0$ and $\tilde{Q}_0$ are also listed in Table (5.1). Note that if we linearize Eqs. (5.13) and (5.15)-(5.16) with Eqs. (5.18), (5.28) and (5.33) for the thermoacoustic exchange terms and neglect axial conduction, the model reduces to the exact linearized model of Swift and Rott, except that multiple non-interacting harmonics can be computed simultaneously. Likewise, for an inviscid non-conducting gas, $\tilde{\mathcal{R}}_U$, $\tilde{\mathcal{H}}_U$ and $\tilde{\mathcal{Q}}_U$ all reduce to the null vector and we obtain the quasi one-dimensional Euler equations for a calorically perfect gas.

It is striking that the boundary layer approximation of $\tilde{Q}_0$ in Table (5.1) has a different sign than the limit for parallel-plate and cylindrical pores. Nevertheless, this solution is in contradiction with its underlying assumption. This assumption states that the boundary layer thickness is small compared to the typical cross-sectional length scale, yet the boundary layer thickness for fully developed steady flow is infinite by definition, because as $\omega \rightarrow 0$, $\delta_{\nu,\kappa} \rightarrow \infty$. Therefore, this boundary layer approximation limit of Table (5.1) is nonphysical and should not be used to model time-averaged interaction with the wall. For the case of Sec. (5.4), the cylindrical resonator and hot end are modeled using the boundary layer approximation for $n > 0$, while for $n = 0$, the cylindrical pore limit is used.

A problem with the heat exchange terms is that the model for wall heat exchange significantly underestimates the time-averaged heat transfer between the solid and the fluid. This is because as a nonlinear effect, the presence of an acoustic wave affects the time-averaged transverse temperature profile, which in effect influences the time-averaged heat transfer. In Chapter (3), this effect is modeled in the framework of second order thermoacoustic theory. Additionally, empirical models have been developed to compute time-averaged heat transfer [18]. The latter uses a clear distinction between oscillating heat transfer by the acoustic wave and induced time-averaged heat transfer. The oscillating heat transfer is modeled using linear thermoacoustic theory, and time-averaged heat transfer with an empirical model. The empirical model needs statistical input from the acoustic field, such as the root mean square (RMS) acoustic Reynolds number. A fully nonlinear model does not allow such a distinction, and the model presented in this section does not take such nonlinear heat transfer effects into account. Inclusion of these effects requires a complete revision of the current thermoacoustic heat exchange terms. At this point, this is left for future research.

5.3 Numerical implementation

In this section, the numerical method used to solve the model of Sec. (5.2) will be explained. For the spatial discretization, a staggered cell-centered finite volume method is used [16]. For constant grid spacing this results in second order spatial discretization. The volume flow is evaluated at the cell walls and density, temperature and pressure at the vertices.

For self-excited thermoacoustic systems, the fundamental frequency is unknown and part of the solution. This introduces an extra degree of freedom for the numerical system, which requires solving an extra equation. We add the additional equation that fixes the

phase of the first harmonic at one of the walls of the system. For example, for the left wall we add

$$\Im(\tilde{p}_1(x=0)) = 0 \quad (5.34)$$

to the system of equations. The details of this implementation are explained in Appendix (5.B).

In the system of equations, one time-averaged continuity equation is replaced by the global conservation of mass equation

$$\sum_{j=0}^{N_{\text{cells}}-1} \tilde{\rho}_{0,j} V_{f,j} = m_p, \quad (5.35)$$

where j is the cell index, N_{cells} is the total number of cells in the system, $V_{f,j}$ is the local volume of a cell and m_p the prescribed (fluid) mass in the system, which is computed as

$$m_p = \frac{p_{\text{ref}}}{RT_{\text{ref}}} \sum_{j=0}^{N_{\text{cells}}-1} V_{f,j}, \quad (5.36)$$

where p_{ref} is the reference pressure (which is set when the system is initially filled with gas), and T_{ref} is the reference temperature. Eq. (5.35) guarantees that the total mass in the system is conserved. The necessity of this replacement is explained in Appendix (5.A).

The inclusion of thermal conduction in the model results in thermal boundary layers of thickness $\sim \delta_\kappa$ at the tube ends where the temperature is prescribed. One such temperature boundary condition is always present, as at least one Dirichlet boundary condition for the temperature is needed to make the solution unique. The small thermal boundary layer is responsible for so-called thermal hysteresis (thermal relaxation) [26], which dissipates acoustic power. To resolve the acoustic temperature in the boundary layers, the grid spacing needs to be $< \delta_\kappa$. Therefore, a local grid refinement is made at the expected location of boundary layers. This boundary grid stretches exponentially to the inner segment (standard) grid spacing. This way, boundary layer effects can be resolved while the number of grid-points stays manageable.

The computations are done on a standard desktop computer with *Intel i7 860* cores running at a clock frequency of 2.80 GHz with 8192 kB of cache size. A C++ program has been written, which runs serially and uses the *Armadillo* linear algebra package [21] for computing the Jacobian terms. For solving the linear system at each Newton iteration, the direct *SparseLU* Solver of the *Eigen* linear algebra package [7] is used. Typical solution times range from 1 to 5 minutes, which is fast compared to computation times of ~ 10 h when using the method of characteristics [19].

5.4 Application: standing wave TA engine

In this section we introduce the application of the NLFD method to solve the periodic steady-state of a standing wave thermoacoustic engine. As the application test case, the standing wave engine of Atchley [3] is chosen, because we have both experimental data of the linear

Figure 5.1 – Schematic overview of the standing wave engine of Atchley. The picture is not at scale. The wall temperature profile is shown in the graph below. On the right side of the stack, T_w is set to T_H .

and nonlinear aspects of this device. Moreover, using the results of this test case allows us to compare them with earlier numerical results, presented by Karpov et al. [14] and Yuan et al. [29].

Figure (5.1) shows a schematic overview of the geometry of this standing wave engine. The geometry consists of discrete “segments”: the resonator, cold heat exchanger, stack, hot heat exchanger and the hot duct. The geometrical and modeling details for each segment are listed in Table (5.2). The total cross-sectional area of each segment is the same and is based on the inner diameter of the tube, which is 38.2 mm. One geometrical detail has been left out of the model. In the experimental setup, the cold heat exchanger consists of two identical structures. However, in our case this is modeled as a single heat exchanger with twice the original length. The wall temperature distribution T_w is prescribed. It is constant everywhere except in the stack, where T_w is linear in x . For the fluid temperature, the resonator segment has an adiabatic wall boundary condition, while the hot end has an isothermal wall at T_H . On both sides, the volume flow is set to zero as a result of the no-slip condition. The working gas is helium and the system is filled to an elevated pressure of 3.76 atm. Due to heating and the presence of thermoacoustic oscillations, the mean pressure increases to a steady-state value which is 0.1 – 0.2 atm higher.

At the segment interfaces, the geometry has discrete jumps in cross-sectional area and hydraulic radius. These jumps are softened using a smooth transition of all geometric parameters. However, the viscothermal Rott functions are not smoothed. Smoothing the geometries together has several disadvantages. Firstly, the solution becomes dependent on the chosen transition length. Secondly, flow “minor losses” [25] due to sudden expansions and contractions cannot be modeled. An improved approach (beyond the current scope) would be to impose conservative jump conditions at the segment interfaces. This would also allow for “branching” different segments, as is required in (traveling wave) TA systems with side resonators.

The amplitude of the acoustic oscillations in the system is determined by the strength of the thermoacoustic effect, which is controlled by the wall temperature gradient in the stack.

Segment	Length (mm)	r_h (mm)	ϕ	Shape	BL
Resonator	879.7	9.55	1.00	Circ. tube	Yes
Cold HX	20.4	0.51	0.70	Par. plates	No
Stack	35.0	0.385	0.73	Par. plates	No
Hot HX	7.62	0.51	0.70	Par. plates	No
Hot end	473.0	9.55	1.00	Circ. tube	Yes

Table 5.2 – Geometrical and modeling details of the standing wave engine of Atchley. The column “BL” denotes if for that segment the boundary layer approximation for the viscothermal Rott functions is used. For Circ. tube geometries, $r_h = 1/2R$, where R is the tube radius. For Par. plates, $r_h = y_0$, which is half the plate spacing.

This temperature gradient is in turn determined by the temperature difference between the cold and the hot side: $\Delta T = T_H - T_0$, where T_H is the wall temperature of the hot end, and T_0 is the ambient temperature, which is set to 293.15 K.

As a measure of grid convergence, the amplitude of the first harmonic of the pressure is used. For all shown results, this pressure amplitude does not change more than 1.5% with a doubling of the number of grid-points. For the low-amplitude cases, the number of harmonics in the simulations is set to $N_f = 6$, which is sufficient to find the amplitude of the first harmonic to have negligible dependence on N_f . With $N_f = 6$, the number of grid-points is 1059, which results in a total of 55068 degrees of freedom for the solution.

5.5 Results

Using this nonlinear model, the onset temperature difference can be determined approximately. Below onset, the numerical method does not converge because as discussed in Appendix (5.B), the fundamental frequency becomes undefined when the oscillation amplitude goes to zero. We use the following procedure to determine the onset temperature difference. First, a temperature difference is chosen which is higher than the onset temperature. Then, the temperature difference is lowered in fixed steps of 1K, until the solution does not converge. The last value for which the solution does converge is the onset temperature.

A close agreement has been found in the onset temperature difference between our computational results and the experimental results. For the onset temperature difference we find a value of 324 K. This value is only 1K off from the experimentally obtained value of 325 K. Close to the onset temperature, the amplitude dependence on ΔT is strong. This was also found by Karpov et al. [14].

Figure (5.2) shows the spatial distribution of the pressure harmonics at $\Delta T = 325$ K. From these distributions, it is clear that the resulting time response of the pressure strongly depends on the axial position. The shapes of the harmonic profiles closely resemble the li-

Figure 5.2 – Spatial distribution of the real (—) and imaginary (---) part of the acoustic pressure harmonics in the engine running at the lowest temperature difference of 325 K. The influence of the engine core can be seen at $x \simeq 0.9\text{m}$, where the curvature in the profiles is relatively high.

near acoustic eigenmodes of a tube closed at both sides. The relative phases are determined by the nonlinear coupling. The resulting amplitudes of the higher harmonics are determined by the eigenfrequencies and quality factors of the linear acoustic modes. Specifically, the amount of *dissonance* [12] is of importance: the difference between the acoustic eigenfrequencies and the higher harmonics of the fundamental (driven) tone. To suppress this cascade energy transfer to higher harmonics, more dissonance could be created by shaping the cross-sectional area of the resonator [5, 9]. For $x \geq 0.9\text{ m}$, the influence of the heat exchangers and stack causes these profiles to deviate from “pure” cosines.

For higher temperature differences - and hence higher drive ratios - the shape of the pressure phasors does not change significantly. However, the higher harmonics become more prominent in relation to the fundamental tone, hence nonlinear distortion increases.

Figures (5.3) and (5.4) show the acoustic pressure as a function of time on the left side of the resonator for a stack temperature difference of $\Delta T = 325\text{ K}$ and $\Delta T = 368\text{ K}$. The first temperature difference is slightly above onset and the profile is nearly sinusoidal. In

Figure 5.3 – Solution of the pressure waveform at $x = 0$ for $\Delta T = 325$ K and $N_f = 6$.

the second case, the drive ratio has been increased to approximately 10%. The fundamental frequency of the $\Delta T = 368$ K is 520 Hz, which is a bit higher than the fundamental of the $\Delta T = 325$ K case of 516 Hz. This is due to the higher average speed of sound in the system. Figure (5.4) can directly be compared with Figure (11) of Karpov et al. Some small wrinkles are found in the wave valleys; these are also found in the experimental results [2], and in the results of the time domain model of Karpov et al. [14]. Increasing N_f does not visibly change the wave profile. Moreover, the wrinkles in the wave valleys do not disappear.

A typical quantity of interest in a thermoacoustic system is the acoustic power flow, or the cycle-average of the work flow

$$E = \overline{pU}, \quad (5.37)$$

where the overline means time-averaging. This quantity can be computed from the frequency domain data by

$$E = p_0 U_0 + \sum_{n=1}^{N_f} E_n = p_0 U_0 + \frac{1}{2} \Re \left[\sum_{n=1}^{N_f} p_{c,n} U_{c,n}^* \right]. \quad (5.38)$$

In Eq. (5.38), E_n is the acoustic power flow of the n^{th} harmonic. Figure (5.5) shows the acoustic power in the first three harmonics at $\Delta T = 325$ K. For all i , the power is zero at both ends of the tube. It can be seen that both the fundamental and second harmonic gain acoustic power in the stack. Yet the second harmonic loses nearly all its gained power in the cold heat exchanger by viscothermal losses. All harmonics higher than the second transport net power from the resonator to the stack and are therefore dissipative. For this situation,

Figure 5.4 – Solution of the pressure waveform at $x = 0$ for $\Delta T = 368$ K and $N_f = 6$.

linear losses are dominant, since the power in the higher harmonics is negligible compared to the fundamental, as would be expected close to the onset temperature difference. This is no longer true in the high drive ratio/high temperature case of Figure (5.6), where the higher harmonics - especially the second - have a visible effect on the fundamental.

At the high drive ratio, the heat load in the heat exchangers is relatively low, because the transverse temperature difference between solid and fluid in the heat exchangers and stack is small compared to the axial temperature difference of 368 K. Therefore, the underestimation of the time-averaged heat transfer with the model of Sec. (5.2) does not have a significant influence on the computed saturation amplitude. For a traveling wave engine running at similar pressure amplitude, the wave would have 20 to 100 times more acoustic power, and therefore the heat load would be approximately that same factor higher. In that case, the overestimation time-average temperature difference in the heat exchangers would have a significant effect on the saturation amplitude.

For the high drive ratio, the peak RMS Reynolds number is $\sim 3 \cdot 10^4$. With a fundamental shear wave number of 273, this results in “weakly turbulent flow” according to Figure (7.4) of the book by Swift [24, p. 161]. Thus, the laminar flow drag model fails. In order to obtain results closer to the experimental results, a turbulent flow drag model is required, which is beyond the scope of this research.

The computed amplitude for the high temperature case is about 40% higher than the result from the time domain model of Karpov et al. [14], due to several differences between the models. First, since the wall temperature is fixed in this model, the wall temperature gradient is not decreased by thermoacoustic streaming, i.e. the wall acts as a perfect heat source/sink. Secondly, in the model of Karpov et al., heat transfer is lowered by a factor

Figure 5.5 – Acoustic power of first three harmonics as a function of position in the tube slightly above onset ($\Delta T = 325\text{K}$). From the left, the first vertical line is located at the left side of the cold HX. The second line is at the right side of the hot HX.

$K_h = 25/35$, to compensate for the fact that not all plates “participate in the heat transport along the stack.” We have not included this factor in our model. Finally, their approximate drag and heat transfer model overestimates damping for the higher frequencies, as can be seen from the drag curve (Figure (1) in the paper by Karpov et al. [14]). The combined differences bring their model closer to the experimental results and this model closer to linear thermoacoustic theory.

5.6 Conclusions

The periodic steady-state of a standing wave thermoacoustic engine has been numerically solved using a quasi one-dimensional thermoacoustic model. The periodic steady-state is found in frequency domain by applying the nonlinear frequency domain method to this

Figure 5.6 – Acoustic power of first three harmonics as a function of position in the tube at $\Delta T = 368\text{K}$. The vertical lines are the same as in Figure (5.5).

model. Several difficulties are treated related to solving self-excited thermoacoustic systems with the nonlinear frequency domain method, among them a method to obtain the unknown fundamental frequency of the periodic steady-state. The low-amplitude limit of this model is in exact agreement with linear thermoacoustic theory. Nonlinear effects appear in the results in the form of the generation of higher harmonics. It has been found that taking only 6 harmonics into account is enough to accurately describe nonlinear effects in the experimental setup and to reach global convergence. The results are in good agreement with a time domain model from the literature, which shows a similar qualitative agreement with numerical experiments. The computation times of the developed computer code are small enough such that the model can be used in future optimization studies.

5.A Mass balance

When a time domain model for a closed system is solved using a conservative integration scheme, global quantities such as mass and momentum are conserved as the solution progresses in time. If the NLFD equivalent of this time domain model is solved, this is no longer true as the solution does not follow physical states during the solution process. It results in a numerical problem, which can be explained by assuming a steady-state system. This can be done using the NLFD method by setting $N_f = 0$. For $N_f = 0$ all storage vectors have length one, so they contain only the time-average of the state. The model we consider is a piece of prismatic tube of length L with an isothermal wall on both sides, as shown in Figure (5.7). On the left side, the temperature is kept at the reference temperature T_0 . On the right side, the tube is heated to a temperature T_H . In the domain, we assume adiabatic tube walls and we neglect viscothermal effects. The problem is one-dimensional and we consider a calorically perfect gas with constant thermal conductivity. Then, the continuity equation states

$$u \frac{d\rho}{dx} + \rho \frac{du}{dx} = 0. \quad (5.39)$$

This equation can be integrated to obtain

$$u(x) = \frac{\rho_{\text{ref}}}{\rho(x)} u_{\text{ref}}. \quad (5.40)$$

So if we set $u_{\text{ref}} = u(0) \equiv 0$, the solution we find is that the velocity is zero at all positions in the tube: the tube is quiescent. If we use this result for the momentum equation, we find

$$\frac{dp}{dx} = 0, \quad (5.41)$$

for which we find the solution $p = \text{constant}$ in the domain. Next, the energy equation reduces to

$$\frac{d^2 T}{dx^2} = 0. \quad (5.42)$$

Using the left and right boundary condition, we find the linear temperature distribution $T = T_0 + (T_H - T_0)x/L$. Finally, using the perfect gas equation of state, the density can be

Figure 5.7 – Example situation of a closed system with adiabatic walls on the side. At $x = 0$, and $x = L$, two isothermal walls are present. A cell-centered grid is shown, where the discrete density variables lie in the middle of the cells.

determined up to the same constant: $\rho = \text{constant}/R_s T$. Since we do not have a boundary condition where the pressure or the density is prescribed, the value of the pressure and the density cannot be determined. Moreover, we have too many boundary conditions for the velocity: when the velocity is zero at one boundary, it *has* to be zero at the other boundary due to Eq. (5.40). For the numerical method, this situation results in a singular Jacobian matrix.

Nevertheless, the pressure in the system can uniquely be determined by applying knowledge of the global state of the system. For example, if we know the amount of mass in the tube, we obtain the global equation

$$\int_{V_f} \rho dV = m_p, \quad (5.43)$$

where V_f is the total fluid volume in the tube and m_p the amount of fluid mass in the tube. Then we can obtain the pressure as

$$p = \frac{(T_H - T_0)}{\ln\left(\frac{T_H}{T_0}\right)} \frac{R_s m_p}{V_f}. \quad (5.44)$$

For the numerical method, we apply a discretization of the governing equations, which results in a nonlinear system of algebraic equations. Adding Eq. (5.43) to this system yields an over-determined system. Therefore, the time-averaged continuity equation for one grid-point is replaced by the equation describing the global conservation of mass. For a finite volume scheme, we replace one local DC continuity equation with Eq. (5.35). Since closed thermoacoustic systems are often filled at room temperature, a beneficial effect of this global equation is the fact that we obtain the DC pressure increase in the system under operating conditions. For systems with large resonators, this increase in mean pressure is small. Yet for thermoacoustic systems which have a high heated volume with respect to the total volume, such as engines with mechanical resonators, the mean pressure can increase significantly.

At this point, we are in the position to say something about the inclusion of axial conduction in Eq. (5.16). If we had neglected conduction, Eq. (5.42) would reduce to an identity and the system would be completely under-determined. Inclusion of the conduction term solves this problem. Moreover, it enables us to resolve the thermal boundary layer adjacent to an isothermal wall.

5.B The fundamental frequency

A TA engine is a self-excited system, therefore the fundamental frequency is unknown and part of the solution. This appendix describes the numerical implementation used to solve for unknown fundamental frequency. As explained in Chapter (4), a global residual operator $\mathbf{L}$ is introduced, which combines the governing equations for all degrees of freedom. A state vector $\mathbf{x}$ is the solution if

$$\mathbf{L}(\mathbf{x}) = \mathbf{0}. \quad (5.45)$$

This solution is searched for using the Newton-Raphson method, where we analytically evaluate the Jacobian matrix $\frac{d\mathbf{L}}{d\mathbf{x}}$.

Since for a self-excited system no boundary conditions are given for the phase, this residual operator is invariant to a shift in time. So if a global time shift is applied to one solution, this yields another valid solution. A unique solution is found by fixing the phase. Therefore, an extra equation is added to the system, following the original idea of Nakhla et al. [17], as developed by Gilbert et al. [6]. To obtain one extra equation, the phase of the solution is fixed by adding an extra equation. We call this extra equation the timing constraint. This timing constraint can be set to any dependent variable. As explained in Sec. (5.3), we add the equation setting the imaginary part of the amplitude of the first harmonic of the pressure at the left wall equal to zero, as represented by Eq. (5.34). Note that the relative phase of higher harmonics with the fundamental tone is determined by the nonlinear coupling.

We define a new system of equations in which the fundamental frequency is added as unknown and the timing constraint as an equation. So the augmented solution vector is

$$\mathbf{y} = (\mathbf{x}, \omega), \quad (5.46)$$

and the augmented residual is

$$\mathbf{M} = (\mathbf{L}(\mathbf{x}), \Im(\tilde{p}_1(x=0))). \quad (5.47)$$

To search for the solution, Newton iterations are done using the Jacobian of the augmented residual operator $\mathbf{M}$:

$$\delta\mathbf{y} = -\frac{d\mathbf{M}}{d\mathbf{y}}^{-1} \cdot \mathbf{M}, \quad (5.48)$$

where

$$\frac{d\mathbf{M}}{d\mathbf{y}} = \begin{bmatrix} \frac{\partial \mathbf{L}}{\partial \mathbf{x}} & \frac{\partial \mathbf{L}}{\partial \omega} \\ \frac{\partial \Im(\tilde{p}_1(x=0))}{\partial \mathbf{x}} & 0 \end{bmatrix}. \quad (5.49)$$

In Eq. (5.49), $\frac{\partial \Im(\tilde{p}_1(x=0))}{\partial \mathbf{x}}$ is a single row which is zero everywhere, but is one at the global degree of freedom number corresponding to $\Im(\tilde{p}_1(x=0))$. The column $\frac{\partial \mathbf{L}}{\partial \omega}$, is the sensitivity of the residual to a change in frequency. For brevity we only show the semi-discrete form of these sensitivities. These sensitivities can be derived from the governing equations. For the continuity equation this sensitivity is

$$\frac{\partial \mathbf{L}_c}{\partial \omega} = S_f \check{\mathbf{D}} \cdot \tilde{\rho}. \quad (5.50)$$

For the momentum and energy equation, however, we neglect the sensitivity to the operators $\check{\mathbf{D}}$, $\mathcal{H}$ and $\check{\mathbf{Q}}$, so for the momentum equation we use

$$\frac{\partial \mathbf{L}_m}{\partial \omega} \simeq \check{\mathbf{D}} \cdot \mathcal{F} \cdot (\rho \circ \mathbf{U}), \quad (5.51)$$

and for the energy equation

$$\frac{\partial \mathbf{L}_e}{\partial \omega} \simeq \check{\mathbf{D}} \cdot \left(\frac{S_f}{\gamma - 1} \tilde{\mathbf{p}} + \frac{1}{2S_f} \mathcal{F} \cdot (\rho \circ \mathbf{U}^2) \right), \quad (5.52)$$

and finally for the equation of state

$$\frac{\partial \mathbf{L}_s}{\partial \omega} = \mathbf{0}. \quad (5.53)$$

In most cases it is only the fundamental mode which is unstable [13]. The frequency of the instability can be found using linear theory. For the application of Sec. (5.4), a close enough guess for the numerical solution is

$$\omega \simeq \frac{\pi c_0}{L}, \quad (5.54)$$

where c_0 is the ambient undisturbed speed of sound and L is the length of the engine.

It should be noted that for self-excited systems the quiescent situation cannot be used as an initial guess for the solver, since a quiescent solution makes the augmented Jacobian matrix singular. This is because at quiescent conditions, the solution is no longer dependent on the fundamental frequency. Therefore, as a starting guess, a “driven” solution is used. For the application of Sec. (5.4) the starting guess is the solution of this system driven at the left boundary with frequency as defined in Eq. (5.54) and unit amplitude.

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CONCLUSIONS AND RECOMMENDATIONS

In this chapter, the conclusions from individual chapters are summarized and put in a more general perspective. Based on the findings of this research, recommendations are given for further research.

6.1 Conclusions

As described in Chapter (1), the main objective of the research described in this thesis is the development of mathematical and numerical modeling techniques for thermoacoustic systems, both to better understand the behavior, and to predict and optimize the performance when designing new TA systems. To this end, a second order one-dimensional thermoacoustic heat transfer model for heat exchangers has been developed. Besides that, a nonlinear one-dimensional thermoacoustic mode has been developed which can be used to directly solve the periodic steady-state of a thermoacoustic system. The conclusions drawn based on the findings of this research are described in this chapter.

Chapter (2) introduced the theory used in the subsequent chapters. From first principles, a simplified approximate nonlinear model is derived which is valid for long pores. This model is subsequently used to derive the linear thermoacoustic model of Swift and Rott. Moreover, a nonlinear model for TA systems is derived from this long pore approximation. This model is used as the basis for Chapter (5). For low oscillation amplitude, this nonlinear model is asymptotically in agreement with the linear TA theory. Moreover, in the limit of zero thermal conductivity and zero viscosity, the model reduces to the quasi one-dimensional Euler equations.

As described in Chapter (3), a one-dimensional thermoacoustic heat transfer model has been developed for parallel-plate thermoacoustic heat exchangers. The model is valid in the limit of low amplitude oscillations, particularly in the case where the particle displacement amplitude is smaller than the heat exchanger length. The model takes the interaction of heat transfer with the acoustic wave into account and it is shown that the common approach of a simple “time-averaged heat transfer proportional to time-averaged temperature difference” does not suffice to describe the heat transfer in compressible oscillating flows.

To compare results with the literature, an effective heat transfer coefficient has been computed for some test cases and the results showed good agreement with literature results. It is also shown that the wave phasing (traveling wave/standing wave) does not have a large influence on heat transfer. From this it can be learned that experimental heat transfer results obtained with a standing wave are also representative for traveling waves. As the developed model is one-dimensional, contrary to theoretical models from the literature, this model allows for easy implementation in an existing one-dimensional thermoacoustic code.

In Chapter (4), a method is explained to directly compute the periodic steady-state of acoustic systems. The method is called the nonlinear frequency domain (NLFD) method

and is based on a Fourier series expansion of the time response of the dependent variables in a system. To test the implementation of this method in a computer code, the problem of an inviscid non-conducting nonlinear plane propagating wave is simulated. The results are compared with analytical results, showing a correct implementation of the method. Moreover, it is shown that this method is also suitable for modeling periodic nonlinear (thermo)acoustic systems.

In Chapter (5), the use of the NLFD method to simulate thermoacoustic systems is discussed. The NLFD method is applied to a quasi one-dimensional thermoacoustic model. The developed computer code is used to simulate a standing wave thermoacoustic engine from the literature. The results are in good agreement with a time domain model from the literature, which shows a similar qualitative agreement with numerical experiments. The required computation time on the order of minutes is short compared with reported computation times of multiple hours in other research efforts. This makes the use of this code viable for optimization loops. Using this nonlinear TA model, several physical mechanisms can be captured which could not be captured with the linear TA model:

- **Gedeon streaming** The sources for Gedeon streaming are included in the model, as this model naturally allows resolving the DC flow in the system.
- **High amplitude motion** The cascade harmonic generation process can be fully resolved, as the motion is not assumed to be sinusoidal anymore, but merely periodic. Moreover, the effects of high-amplitude convective enthalpy transport are resolved. This means that the effect of gas parcels which move larger distances than the segment distance can be included.
- **Alternator drift** As a result of the acoustic wave, the time-averaged pressure also varies throughout a thermoacoustic system. As a result, a net force can act on the linear alternator piston. This results in a “drift” of the linear alternator from its equilibrium position. Due to this drift, the stroke of the alternator is reduced, which results in a lower maximum power output. When the radiation pressure is correctly determined during the design, measures can be taken to account for this effect.

6.2 Recommendations for further research

In this section a list of subjects is given which require further research. The last subject discussed is the application of the nonlinear model in different research fields.

An improved nonlinear thermoacoustic heat transfer model

As discussed in Chapter (2), the nonlinear model of Chapter (5) is inconsistent with the heat transfer model of Chapter (3). Up to second order, this nonlinear coupling between time-averaged heat exchange and the acoustic wave is taken into account in the heat transfer model of Chapter (3). However, the thermal interaction of the acoustic wave with the wall is assumed to be linear in the nonlinear model. As a result, in the nonlinear model for a given temperature difference between the solid and the fluid, the time-averaged heat transfer is

underestimated. A recommendation for further research is to revise the heat transfer model of Chapter (5) such that it incorporates these nonlinear effects.

The nonlinear frequency domain method for CFD

As the NLFD method decreases computational cost for the one-dimensional thermoacoustic model, a speed-up can also be expected when the method is applied to full three-dimensional models. The time spectral method, the time domain variant of the NLFD method, has shown significant reductions in the computational costs for modeling of periodic flow in turbomachinery. Hence, reductions in computational cost are also expected when modeling TA systems. Moreover, when a frequency domain implementation is used, the application of impedance boundary conditions becomes much more straightforward. Therefore, the application of the NLFD method for CFD modeling of TA systems should be investigated.

Towards nonlinear modeling of a thermoacoustic Stirling engine generator

In Chapter (5), we have shown that using the NLFD method, thermoacoustic systems can efficiently be modeled. However, for modeling of a thermoacoustic Stirling engine generator, several additions and modifications have to be made to the computer code. These recommendations all refer to the one-dimensional nonlinear code:

- **Turbulent flow** The current nonlinear model provides models for segments in which the flow is laminar. In the case of high amplitude, the flow can become turbulent and the laminar model fails. For these situations, an oscillating flow turbulence model should be implemented in the code.
- **Solid heat equation** The model of Chapter (5) uses a prescribed solid wall temperature distribution. In stacks and regenerators, however, this temperature is the result of the balance in heat between solid and fluid. Hence, a heat equation needs to be implemented for the temperature of the solid.
- **Porous material regenerators** In TA Stirling engines, the regenerator material used does not have a regular pore shape. For example, stacked metal wire screens and/or porous metal fibre materials with a random structure are used. For these types of materials, additional models need to be included to compute flow resistance, heat transfer and conduction losses. The literature provides several models for (randomly) porous regenerator materials.
- **Piston boundaries** To allow mechanical power extraction from the fluid domain, moving wall boundary conditions need to be implemented.
- **Branching of segments** TA Stirling systems have a looped topology. This means that segments need to be “branched” as is similarly done in DeltaEC. Currently, this branching of segments is not included. Moreover, the coupling of different segments is currently done by “smoothing” one geometry to another. As this method introduces tunable parameters (i.e. the smoothing length) in the model, this is not ideal. An

improved approach would be to impose conservative jump conditions at the segment interfaces.

- **Jet pumps** To model the effect of jet pumps in TA systems, segments with flow minor losses need to be implemented. This would also allow us to compare the jet pump pressure drop predicted from sinusoidal flow with the nonlinear flow results. To this end, transitional jumps which include minor losses can be modeled using lumped minor loss coefficients.
- **Streaming losses** Heat leak due to acoustic streaming is an important loss mechanism which is hard to account for with a one-dimensional model. A simplified account for Rayleigh streaming and turbulent flow heat losses in thermal buffer tubes can be made by using a higher effective thermal conductivity.

Other applications of the nonlinear model

When all recommendations previously defined are implemented in an improved computer code, other Stirling type machines can also be modeled. Examples are pulse tube cryocoolers and free-piston Stirling engines. Moreover, other applications where nonlinear viscothermal wave propagation is involved can be modeled. This is the case in some musical wind instruments. For example, in a clarinet driven at high amplitude, the acoustic field in the tube can become strongly nonlinear. For such “machines,” the applicability of the presented modeling techniques is well worth investigating.

LIST OF SYMBOLS

Roman Symbols

ℓ	Stack plates half-width	[m]
$\mathbf{q}$	Heat flux	[W·m ⁻²]
$\mathbf{e}_x$	Unit vector in x -direction	[-]
$\mathbf{I}$	Identity tensor	[-]
$\mathbf{n}$	Normal vector pointing from the solid into the fluid	[-]
$\mathbf{r}$	Transverse position vector	[-]
$\mathbf{u}$	Velocity vector	[m·s ⁻¹]
$\mathbf{x}$	Position vector	[m]
Pr	Prandtl number	[-]
b_κ	Thermal conductivity exponent	[-]
b_μ	Viscosity exponent	[-]
c_0	Reference speed of sound	[m·s ⁻¹]
c_m	Small-signal speed of sound at $T = T_m$	[m·s ⁻¹]
c_p	Specific heat at constant pressure	[J·kg ⁻¹ ·K ⁻¹]
c_s	Specific heat of the solid	[J·kg ⁻¹ ·K]
c_v	Specific heat at constant density	[J·kg ⁻¹ ·K ⁻¹]
D	Time derivative matrix	[s ⁻¹]
E	Total energy per unit mass	[J·kg ⁻¹]
e	Euler's number	[-]
e	Internal energy per unit mass	[J·kg ⁻¹]
f	Frequency	[Hz]
f_κ	Thermal Rott function	[-]
f_ν	Viscous Rott function	[-]
H	Total enthalpy per unit mass	[J·kg ⁻¹]
h	Specific static enthalpy	[J·kg ⁻¹]
H_2	Second order time-averaged acoustic enthalpy flow	[W]
h_κ	Viscothermal shape function for the temperature	[-]
h_ν	Viscothermal shape function for the velocity	[-]

i	Imaginary unit	[-]
J	Bessel function of the first kind	
k	Reduced frequency	[-]
L	Length	[m]
M	Reference acoustic Mach number	[-]
N	Number of	[-]
p	Pressure	[Pa]
Q	Heat	[J]
R	Viscous resistance coefficient	[N·s·m ⁻⁴]
r_h	Hydraulic radius	[m]
R_s	Specific gas constant	[J/kg/K]
S	Cross-sectional area	[m ²]
S	Entropy	[J·K ⁻¹]
s	Shear wave number	[-]
T	Temperature	[K]
t	Time	[s]
U	Volume flow	[m ³ ·s ⁻¹]
u	Velocity in wave propagation direction	[m·s ⁻¹]
V	Volume	[m ³]
v	Specific volume	[m ³ ·kg ⁻¹]
W	Work	[J]
Z	Volume flow impedance	[Pa·s·m ⁻³]
z	Acoustic impedance	[Pa·s·m ⁻¹]

Calligraphic Symbols

$\mathcal{F}$	Fourier transform operator	
$\mathcal{H}$	Transverse heat transfer term 1	[W·m ⁻¹]
$\mathcal{O}$	Order (in expansion)	
$\mathcal{Q}$	Transverse heat transfer term 2	[W·K ⁻¹]
$\mathcal{R}$	Viscous resistance	[N·m ⁻¹]

Decorators

$\langle \dots \rangle$	Area-averaged	
$\overline{\dots}$	Time-average	
$\widehat{\dots}$	Dimensionless	
$\widetilde{\dots}$	Fourier transformed	

Greek Symbols

δ_κ	Thermal penetration depth	[m]
δ_ν	Viscous penetration depth	[m]
ϵ_s	Ideal stack correction factor	[-]
η	Thermal efficiency	[-]
Γ	(Lossy) wave number	[rad·m ⁻¹]
γ	Ratio of specific heats	[-]
κ	Thermal conductivity	[W/m/K]
λ	Second ideal-stack correction factor	[-]
λ	Wavelength	[m]
μ	Dynamic viscosity	[Pa·s]
ω	(Fundamental) angular frequency	[rad·s ⁻¹]
ϕ	Phase angle	[rad]
ϕ	Porosity	[-]
Π	Wetted perimeter	[m]
π	Ratio a circle's circumference to its diameter	[-]
$\underline{\tau}$	Viscous stress tensor	[Pa]
ε	Aspect ratio	[-]
ξ	Dummy variable	[· · ·]
ξ	Particle displacement amplitude	[m]

Miscellaneous Operators

$(\dots)^T$	Transpose	
$(\dots)_T$	In transverse direction	
$\cdot$	Inner product	
$\circ$	Elementwise product of two vectors	
Δ	Difference	
$\diamond$	Frequency domain product of two vectors	
$\Im$	Imaginary part	
$\text{diag}(\dots)$	Matrix with argument vector components on diagonal	
∇	Gradient	[m ⁻¹]
∇^2	Laplacian	[m ⁻²]
$\otimes$	Dyadic product	
$\Re$	Real part	
$\sim$	Same order of magnitude	

$\simeq$ Approximately equal to

Often used Abbreviations/Acronyms

1D	One-dimensional
2D	Two-dimensional
3D	Three-dimensional
CFD	Computational Fluid Dynamics
DC	Direct current
DFT	Discrete Fourier transform
Eq(s).	Equation(s)
HX	Heat exchanger
NLFD	Nonlinear frequency domain
ODE	Ordinary differential equation
RMS	Root mean squared
Sec(s).	Section(s)
STP	Standard temperature and pressure
TA	Thermoacoustic
TASEG	Thermoacoustic Stirling engine generator
TASHE	Thermoacoustic Stirling heat engine
TBT	Thermal buffer tube

Subscripts

f	Fluid
H	Hot
m	Mean
s	Solid
w	Wall
0	Reference
in	Input
out	Output
ref	Reference
T	Transverse
wall	At the wall

RESUMÉ

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Publications

- J.A. de Jong, Y.H. Wijnant and A. de Boer. "A one-dimensional heat transfer model for parallel-plate thermoacoustic heat exchangers", *The Journal of the Acoustical Society of America* 135.3 (2014), pp. 1149 - 1158.
- J.A. de Jong, Y.H. Wijnant, D. Wilcox and A. de Boer. "Modeling of thermoacoustic systems using the nonlinear frequency domain method", *The Journal of the Acoustical Society of America*, in press.
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Presentations

- J.A. de Jong, Y.H. Wijnant and A. de Boer. “A one-dimensional parallel-plate thermoacoustic heat exchanger model”, *Second international workshop on Thermoacoustics 05/23/2014, Sendai, Japan*.

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A: *“Mag ik u hartelijk bedanken voor deze fijne voorstelling?”*

B: *“Gaat uw gang”*

A: *“Ik wil u hartelijk bedanken voor deze fijne voorstelling.”*

B: *“Gaat uw gang”*

A: *“Hartelijk bedankt voor deze fijne voorstelling”*

B: *“Geen dank”*

A: *“Geen dank? En ik had het nog zo gevraagd!”*

B: *“Ja, d'r is niks meer aan te doen”*

A: *“O, nou, u wordt bedankt!”*

B: *“Ja, dat idee kreeg ik al!”*

Uit: Herman Finkers, Het meisje van de slijterij, track 16: Gebarentaal

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