HAMILTONIAN FLOW ON AN ENERGY SURFACE:
240 YEARS AFTER THE EULER-MAUPERTUIS PRINCIPLE

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1. Introduction

This paper is concerned with one of the basic formulations of the laws of classical and modern physics, viz. the principle of least action. Generally speaking, this variational principle determines the actual motions of a Hamiltonian system at a given value of the energy by the property that they correspond to the critical points of the "action"-functional, which is a functional defined on curves in configuration space. A short historical review of this principle, which is generally named after Maupertuis, but should be called more correctly the Euler-Maupertuis principle, is presented in section 2.

One of the aims of this paper is to present a new, simpler formulation of this principle. In particular, the important (and confusing) role that is given, even in many modern textbooks, to the parametrization of the curves is reduced to one essential, physical quantity, namely to the transition time of the motion between its endpoints (or to the period if the motion is periodic). This facilitates an investigation into the relation between motions at a prescribed time-interval and those of a prescribed value of the energy. After this new formulation of the action principle in sections 3 and 4, I hope that the reader will not have the feeling that as a reaction to the words Jacobi (1842-1843) "In almost all textbooks, even the best, this principle is presented so that it is impossible to understand", I should have added "I have not chosen to break with tradition" (see Arnold (1978) p.246; see also Abraham & Marsden (1978) p.249).

In section 3 we start with a variational principle in phase space for the trajectories of a Hamiltonian system on a prescribed energy surface. The energy constraint is described in an integrated form, instead of in a pointwise fashion as is usually done. In section 4 this variational principle is reduced to one for a functional J_E defined on curves in the configuration space. In fact, for any such curve x , $J_E(x)$

turns out to be related to the usual Lagrangian functional $K_T(x)$ by a Legendre transformation. For instance, for a natural Hamiltonian which is the sum of kinetic and potential energy V , this functional J_E reads

$$J_E(x) = 2 \int \left\{ \frac{1}{2} \dot{x}^2 \cdot [E - V(x)] \right\}^{\frac{1}{2}} dx.$$

(Note the difference with the usual Jacobi functional, which, for this case, can be written as

$$\int \left\{ \frac{1}{2} \dot{x}^2 \cdot [E - V(x)] \right\}^{\frac{1}{2}} dx.)$$

In the next two sections we show the potential usefulness of this functional J_E by describing the relatively simple proof of the existence of periodic, finite-amplitude motions, and by investigating the relation between the energy and the period. Moreover, although such a periodic motion corresponds to a genuine saddlepoint of J_E , an explicit, analytical description of the saddlepoint is possible, and can be reduced to minimizing the functional J_E on a naturally constrained subset.

Finally, in section 7, we present as generalizations of the ideas presented here, two problems from continuum mechanics, viz. nonlinear wave vibrations and vortex rings in an ideal fluid.

2. The Euler-Maupertuis Principle

In an appendix to his famous work "Methodes Inveniendi Lineas Curvas", which appeared in 1744, Euler describes what we call nowadays the principle of least action. In fact, he shows through a number of examples, that the path in a plane which is followed by a point particle with mass m has the property that the integral

$$m \int \sqrt{v^2} ds$$

is minimal. Here v^2 is the square of the velocity and ds is a line-element of the path under consideration that is transferred precisely with velocity v . From the point of view of modern analytical mechanics, this may be considered to be the formulation of the principle of least action for general conservative systems. Indeed, in all his examples, Euler derives the velocity as function of position from the energy conservation, which can be written as the sum of kinetic and potential energy:

$$\frac{1}{2}mv^2 + V = E,$$

where E is the value of the total energy. Hence, the integral above can be replaced by

$$\int \sqrt{E-V} \, ds,$$

which is precisely the expression considered by Jacobi a century later in his formulation of the principle of least action. In the translation of Carathéodory (1952), Euler describes the basic idea of the principle as follows:

"Da nämlich die Körper, wegen ihrer Trägheit, jeder Änderung ihres Zustandes widerstreben, werden sie den einwirkenden Kräften so wenig wie möglich nachgeben, wenigstens, wenn sie frei beweglich sind. Daraus folgt, dass die wirkliche Bewegung, die Wirkung, die durch die Kräfte hervorgerufen wird, kleiner sein muss, als für irgendeine andere Bewegung, die man dem oder den Körpern aufprägen könnte.

A few months after Euler had written these lines, Maupertuis (1744) presented a paper to the Paris Academy entitled "Accord de différentes lois de la Nature qui avoient jusqu'ici paru incompatibles". In this paper he presents, independent from Euler, a principle of least action. In his words: "le chemin ... est celui par lequel la quantité d'action est la moindre." With Fermat's principle of least time, from which he establishes Snell's law, as the only specific example in the paper, he mainly founds his principle on metaphysical laws, such as "la Nature, dans la production de ses effects, agit toujours par les moyens les plus simples", and "la Nature, dans la production de ses effects, agit toujours par les voies les plus simples".

Although, chronologically speaking, Euler has the priority, he gave Maupertuis the full credit of the discovery of this remarkable principle. Carathéodory presents a convincing argument that the reason for this can be found in the philosophical attitude of Euler. He, in contrast to Maupertuis, did not allow metaphysical arguments to intervene with the results of his specific calculations: Euler considered his results to be a posteriori (in the sense that "all" known actual motions did minimize the action), whereas a metaphysical law would determine the actual motion (not from Newton's law but) from the principle of least action, so a priori. Nevertheless, Euler strongly believed in the validity of the principle: "...zweifle ich nicht, dass es möglich sein wird, ihn mit Hilfe einer gesunden Metaphysik besser zu begründen", and, in a letter to Maupertuis: "...j'estime le developpement de cette matiere (= the related principle of minimal potential energy) infiniment plus que les solutions les plus recherchées des problemes particuliers. En effet je suis convaincu que par tout la nature agit selon quelque principe d'un maximum ou minimum..."

This short historical review, initiated by reading the intriguing remarks of Goldstine (1980) on this subject, may suffice to justify the name "Euler-Maupertuis-principle" to the principle of least action (at least, when considered as a variational problem for a functional which is defined on curves in configuration space).

3. An action principle on phase space

In this section we use the Hamiltonian formulation to describe the phase trajectories of a system with Hamiltonian H . We shall assume that H is a "classical" Hamiltonian, i.e. it satisfies the usual smoothness and convexity assumptions, together with an evenness assumption:

$$(H)_0 \quad \begin{aligned} H &= H(x,y) \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}), \\ y \mapsto H(x,y) &\text{ is strictly convex and even, for any } x \in \mathbb{R}^n \end{aligned}$$

Given a value E , we look for trajectories on the energy surface

$$\Sigma_E := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \mid H(x,y) = E\}$$

Σ_E will be assumed to be a regular surface, i.e. $\nabla H(x,y) \neq 0$ for $(x,y) \in \Sigma_E$.

Curves in the phase space $\mathbb{R}^n \times \mathbb{R}^n$ are described with some parameter τ . Depending on the kind of motion one is looking for, they are required to satisfy either $x(0) = x(1)$ and $y(0) = y(1)$ for periodic motions (closed curves) or to connect two points on Σ_E for which the projection onto the configuration space is prescribed: $x(0) = \xi$ and $x(1) = \eta$ for some pair of points ξ, η in this projection.

A phase curve will be called a phase trajectory if after some regular reparametrization (from τ to the physical time $t : t = t(\tau)$) the functions $q(t) := x(\tau(t))$ and $p(t) := y(\tau(t))$ satisfy the usual Hamiltonian equations

$$(3.1) \quad \begin{aligned} \frac{dq}{dt} &= H_p(q,p), \\ -\frac{dp}{dt} &= H_q(q,p). \end{aligned}$$

In order to emphasize that t is the physical time, we shall call $(q(t), p(t))$ the motion in phase space.

Remark. From the remarks made in the introduction, and the references given there, it will be clear why we are so careful in distinguishing the arbitrary parametrization of a curve that is a trajectory from the parametrization with the time t . In

fact, this problem of parametrization distinguishes the prescribed energy problem (for which the transition time, or period, is not known in advance) from the case in which this time is prescribed a priori (but not the energy). Having found a phase trajectory, the transition time (period) can of course be deduced a posteriori.

Let us now turn to the variational formulation. Denote by X the set of curves $x(\tau)$ in configuration space which satisfy $x(0) = x(1)$ (for periodic motions) or $x(0) = \xi$ and $x(1) = \eta$. (We will be somewhat loose at this point in not specifying precise smoothness conditions on the curves.) Then consider the canonical action functional defined on phase curves $(x(\tau), y(\tau))$ for which $x \in X$ as $\int \dot{x}y$, where, here and in the following, a dot denotes differentiation with respect to the parameter τ and $\int$ denotes integration over τ from 0 to 1.

The usual form of the action principle on phase space is to look for critical (= stationary) points of this functional on those phase curves that lie on Σ_E . We shall denote this by

$$(3.2) \quad \text{stat}_{x,y} \left\{ \int \dot{x}y \mid x \in X, H(x(\tau), y(\tau)) = E \right\}.$$

Instead of this, we shall consider the functional on a much larger set of phase curves, viz. those for which the energy constraint is only satisfied in an integrated way:

$$(3.3) \quad \text{stat}_{x,y} \left\{ \int \dot{x}y \mid x \in X, \int H(x, y) = E \right\}.$$

(This formulation has also been used by Rabinowitz (1978) to prove the existence of periodic motions.)

Let us verify that (3.3) do indeed produce trajectories of energy E . First note that if $(x(\tau), y(\tau))$ is a critical point, then so is $(x(1-\tau), -y(1-\tau))$ (corresponding to the same phase trajectory but transferred in opposite direction). Therefore we can restrict ourselves to critical points with non-negative value of the action functional. Since Σ_E is a regular energy surface, Lagrange's multiplier rule applies, and, as a consequence, for some multiplier $T \in \mathbb{R}$ a critical point satisfies

$$(3.4) \quad \begin{aligned} \dot{x} &= TH_y(x, y) \\ -\dot{y} &= TH_x(x, y) \end{aligned}$$

(in the case of periodicity conditions for $x : x(0) = x(1)$, there results also $y(0) = y(1)$ as additional natural boundary conditions). For this Hamiltonian system, $H(x(\tau), y(\tau))$ is constant, which constant must necessarily be E as a consequence of

the integrated energy constraint. In other words, (x, y) is a phase curve on Σ_E . Since $H_y(x, y) \cdot y > 0$ for $y \neq 0$ (by convexity), it follows from $\dot{x}y = T \{H_y(x, y) \cdot y = 0$ if and only if $T = 0$, and then $\dot{x} = \dot{y} \equiv 0$, which is either not possible (if $\xi \neq \eta$) or uninteresting. Hence we need only to consider the case $\dot{x}y > 0$. Then $T > 0$, and the simple parameter-scaling

$$(3.5) \quad t = T\tau; \quad q(t) = x(\tau), \quad p(t) = y(\tau)$$

shows that (q, p) satisfies (3.1), i.e. that (q, p) is a motion of the system. This proves part of the following proposition (the other part is even simpler).

Proposition 1. Critical points of (3.3) with $\dot{x}y > 0$ are in an one-to-one correspondence with the motions (up to time-reversal) of the Hamiltonian system with energy E .

4. A new action principle on configuration space

We shall now reduce (3.3) to a variational principle on configuration spaces. To that end consider for a fixed configuration curve $x \in X$ the following set of functions:

$$C_x := \{y \mid \dot{H}(x, y) \leq E\}$$

This set may be empty (iff $\dot{H}(x, 0) > E$); if not empty, it is strictly convex. Now consider the functional defined by

$$(4.1) \quad J_E(x) := \sup_y \{ \dot{x}y \mid y \in C_x \}.$$

(In case C_x is empty, we put $J_E(x) = -\infty$. Note that, since we have not yet specified the precise set of functions x (smoothness, growth properties), this functional may also take the value ∞).

This functional is also given by

$$(4.2) \quad J_E(x) := \sup_y \{ \dot{x}y \mid \dot{H}(x, y) = E \}.$$

Remark. Mathematically speaking, the functional $J_E(x)$ defines the supporting hyperplane of the set C_x orthogonal to $\dot{x}$. For the case that the energy constraint is required to hold in a pointwise fashion, i.e. for (3.2), the same procedure amounts to maximizing for any $\tau \in (0,1)$ the quantity $\dot{x}(\tau)y(\tau)$ on the convex set in $\mathbb{R}^n$: $\{y \in \mathbb{R}^n \mid H(x(\tau), y) \leq E\}$. This then leads to the usual Jacobi- or Jacobi-Finsler-metric and -functional.

We claim that $J_E(x)$ can be expressed in terms of the Lagrangian functional corresponding to H . If L denotes the Legendre transform of H with respect to the y -variable, we can write this transformation in the Fenchel extremizing way as

$$L(x, v) := \sup_{y \in \mathbb{R}^n} \{yv - H(x, y)\},$$

and

$$H(x, y) = \sup_{v \in \mathbb{R}^n} \{yv - L(x, v)\}.$$

Because of conditions $(H)_0$, the Lagrangian satisfies

$$(L)_0 \quad \begin{aligned} &L \in C^1(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}) \\ &v \mapsto L(x, v) \text{ is strictly convex and even, for any } x \in \mathbb{R}^n. \end{aligned}$$

Now consider for any $T > 0$ the Lagrangian functional K_T on the set of configuration curves as

$$(4.3) \quad K_T(x) := T \int_0^1 L(x, \dot{x}/T) dt.$$

Note that for a function $q(t)$ related to $x(\tau)$ via the transformation (3.5), $K_T(x)$ is equal to

$$\int_0^T L(q(t), \frac{dq}{dt}(t)) dt,$$

i.e. the usual Lagrangian functional that would be considered for a problem in which T is prescribed. It is not difficult to verify that for any x , with $\dot{x} \neq 0$, the function

$$(4.4) \quad 0 < T \mapsto K_T(x) \text{ is strictly convex.}$$

Now we can show that for any x , $J_E(x)$ is the Legendre transform of $K_T(x)$. More precisely, we have

Proposition 2. For any configuration curve $x \in X$ it holds:

$$(4.5) \quad J_E(x) = \inf_{T>0} \{TE + K_T(x)\}.$$

Moreover, if the infimum is finite, it is attained at a unique value $T > 0$, which is determined by the extremality relation

$$(4.6) \quad E = -\frac{d}{dT} K_T(x);$$

in that case, the supremum in (4.2) is attained by the function y defined by

$$(4.7) \quad y = L_v(x, \dot{x}/T).$$

Remark. Since (4.5) expresses $J_E(x)$ as the infimum of concave functions, the function

$$(4.8) \quad E \rightarrow J_E(x) \text{ is concave.}$$

As a consequence of (4.4), we also have:

$$(4.9) \quad K_T(x) = \sup_E \{J_E(x) - ET\},$$

and if this supremum is attained, the value of E follows from

$$(4.10) \quad T = \frac{d}{dE} J_E(x).$$

Example. It is instructive to specify these results for the case that H is a natural Hamiltonian (sum of kinetic and potential energy):

$$(4.11) \quad H(x, y) = \frac{1}{2}y^2 + V(x) \quad (\text{where } y^2 := y \cdot y).$$

Then

$$(4.12) \quad K_T(x) = 1/T \left(\frac{1}{2}\dot{x}^2 + T \int V(x), \right.$$

and

$$(4.13) \quad J_E(x) = \begin{cases} 2 \left\{ \frac{1}{2} \dot{x}^2 - [E - \int V(x)] \right\}^{1/2} & \text{if } \int V(x) < E \\ -\infty & \text{if } \int V(x) > E ; \end{cases}$$

the extremality relations (4.6), (4.10) in this case read

$$(4.14) \quad T = \left\{ \frac{\frac{1}{2} \dot{x}^2}{E - \int V(x)} \right\}^{1/2}.$$

As a result of propositions 1 and 2, the physical motion of the system is completely described by critical points of the functional J_E .

Proof of proposition 2.

The proof relies heavily on ideas from convex analysis. Let α denote the value of the maximization problem (4.2), and β the value of the minimization problem (4.5). We have to show $\alpha = \beta$. First $\alpha \leq \beta$. To that end, note that α can also be represented as

$$\alpha = \sup_y \inf_{T \in \mathbb{R}} \left[\int \{ \dot{x}y - TH(x,y) + TE \} \right],$$

which can be estimated as

$$\leq \inf_{T \in \mathbb{R}} \sup_y \left[\int \{ \dot{x}y - TH(x,y) + TE \} \right] \leq \inf_{T \in \mathbb{R}} [TE + \int \{ \sup_{y \in \mathbb{R}^n} [\dot{x}y - TH(x,y)] \}].$$

Now, for $\dot{x} \neq 0$, the supremum in brackets is infinite if $T < 0$. For $T > 0$, the strict convexity of H implies that this supremum is for any $\tau \in (0,1)$ attained at some $y(\tau) \in \mathbb{R}^n$ which satisfies $\dot{x} = TH_y(x,y)$, or, equivalently, (4.7). Hence, for this extremal function y one has

$$(4.15) \quad \frac{\dot{x}}{T} y = L(x, \frac{\dot{x}}{T}) + H(x,y).$$

From this it follows that $\alpha \leq \beta$. To show that in fact $\alpha = \beta$, assume that β is finite. Because of (4.4), the infimum in (4.5) is attained at some $T > 0$. For this T the extremality relation (4.6) holds, i.e.

$$(4.16) \quad E + \int L(x, \dot{x}/T) - 1/T \int L_v(x, \dot{x}/T) \dot{x} = 0.$$

Then, if y is defined by (4.7), the relation (4.15) holds. Integrating this expression, a comparison of the result with (4.16) shows that $\int H(x,y) = E$, and that, moreover,

$$\beta = TE + K_T(x) = \int \dot{x}y,$$

from which, by definition of α , it follows that $\beta < \alpha$. Consequently, $\alpha = \beta$ as was to be shown.

5. Existence of periodic motions

In this section we shall indicate how the functional J_E can be used to prove the existence of periodic motions of the system. The results of section 4 imply that this can be done by proving the existence of critical points of J_E on a suitable set of configuration curves. Note that in the following we shall not restrict ourselves to "small-amplitude" solutions, which may be obtained via some linearization procedure, but, rather, look for finite-amplitude motions of given energy E . Results of this kind were first obtained by Seifert (1948), Berger (1971), Weinstein (1978) and Rabinowitz (1978). In contrast to these references, the proof given here is based on the functional J_E and enables an explicit, analytical description of the motion, which allows an explicit investigation into the relation between the prescribed value of the energy and the resulting period of the motion.

When dealing with variational principles in mathematical physics, it is very rarely that one can obtain interesting solutions by simply maximizing or minimizing the functional. In many cases, the critical points one is interested in correspond to saddle points of the functional (Poincaré (1897), already, was well aware of this fact*)).

The same is true for the functional J_E . Unfortunately, saddlepoints are generally difficult to find and many new topological methods have been invented, especially since the above mentioned paper of Rabinowitz. These topological characterizations are not very constructive in general, and more constructive, analytical methods are presently under investigation. We shall indicate the main ideas of such an explicit, analytical characterization, by way of example, for a simple case only (for full details and more general systems, see van Groesen (1985a)).

*) I am much indebted to prof. J. Mawhin who provided me this reference.

Consider a natural Hamiltonian system of the form (4.11) where the potential V is required to satisfy:

$$(V_0) \quad V \in C^2(\mathbb{R}^n, \mathbb{R}) \text{ and } V \text{ is even and strictly convex.}$$

The evenness of V makes it possible to look for very special periodic motions, viz. normal modes. These are periodic motions for which the configuration trajectory is a symmetric curve through the origin along which the solution oscillates back and forth. The endpoints of this trajectory are "restpoints" at which the velocity vanishes identically. Therefore, the complete motion can be deduced from the motion during one quarter of the period, say for $t \in [0, T]$ (when $4T$ is the period) when q satisfies $q(0) = \frac{dq}{dt}(T) = 0$.

A motion of this kind with energy E corresponds to a critical point of the functional J_E on the space $X := \{x \in W_{1,2}([0,1], \mathbb{R}^n) \mid x(0) = 0\}$, where $W_{1,2}$ is the usual Sobolev space. (The condition at $\tau = 1$, i.e. $\dot{x}(1) = 0$, will appear as a natural boundary condition.)

Guided by the behaviour of the functional J_E , we introduce "polar-coordinates" in X , i.e. with S the unit sphere, we write $X = \mathbb{R}_+ \times S$, and any $x \in X$ can be uniquely written as $x = \rho \xi$, $\rho > 0$, $\xi \in S$. (This idea has been used for the first time by Nehari (1960)). Then, because of the convexity of V , it is not difficult to show that for any $\xi \in S$, the function $\rho \mapsto J_E(\rho \xi)$ has a unique critical point, at which it attains a positive maximum. Letting ξ vary over S , the set of points N_E where this maximum value is attained can be described in either of the two following ways:

$$(5.2) \quad N_E = \{\rho \xi \in X \mid \xi \in S; J_E(\rho \xi) = \max_{\rho > 0} J_E(\rho \xi)\} = \{x \in X \mid E = \int V(x) + \frac{1}{2} \int V'(x)x\}.$$

This observation implies that any non-trivial critical point of J_E on X is a solution of the variational problem

$$\text{stat}_{\xi \in S} \max_{\rho > 0} J_E(\rho \xi),$$

or, stated in another way, that it is a critical point of J_E on the constraint N_E :

$$(5.3) \quad \text{stat}\{J_E(x) \mid x \in N_E\}.$$

The set N_E is (almost by construction) a natural constraint for J_E , i.e. critical points of J_E restricted to N_E are also critical points of J_E on X (the Lagrange mul-

tiplier, entering into the equation because of the constraint N_E , vanishes identically). Then, techniques from functional analysis can be applied to show that J_E can be minimized on N_E , and one obtains:

Theorem 1. The minimization problem

$$(5.4) \quad \inf\{J_E(x) \mid x \in N_E\} \quad (\equiv \inf_{\xi \in S} \max_{\rho > 0} J_E(\rho\xi))$$

has at least one non-trivial solution x , for which $J_E(x) > 0$. The motion $q(t)$ corresponding to this solution via the transformation (3.5), where T is given by (4.14), and periodic continuation, is a normal mode which has energy E and minimal period $4T$.

Remark. The fact that one can show that the motion has minimal period, depends heavily on the explicit extremal characterization (5.4). This fact is also basic for the proof of the result that, under additional, geometrical conditions on the level sets of V , there exist at least n distinct normal modes of energy E (see van Groesen (1983, 1985a); a first result of this kind was obtained by Ekeland & Lasry (1980)).

6. On the energy-period relation

The problem treated in the foregoing section raises a question that is often encountered in many problems from mathematical physics. The question we are hinting at is the investigation of how certain parameters of the problem are related: Given a family of problems, parameterized by some parameter p in a parameter set P , let for any $p \in P$ a solution exist to which there corresponds, physically interesting, parameters λ . In general, λ will depend on p , and $\lambda(p)$ may be multivalued if the solution of the problem with value p is not unique. The interest is then in the specific dependence of λ on p .

For the specific example of the foregoing section, the family of problems is parameterized by the single parameter E , the energy, and an interesting quantity for a solution of the minimization problem (5.4) is the (minimal) period of the corresponding periodic motion. This leads us to investigate the mapping $E \rightarrow T(E)$. Unfortunately, this problem is much harder than it may look at first sight (and than is sometimes suggested). The reason for this is that since the solution is not known explicitly, the extremality relation (4.14) is not very useful, and one has to find more global characterizations of T in terms of E . For the problem at hand we shall describe some of the results that were obtained recently (see van Groesen (1985b) for a complete account).

The idea is to investigate the value of the constrained minimization problem as a function of E :

$$(6.1) \quad j(E) := \inf\{J_E(x) \mid x \in N_E\}, \quad E > 0.$$

Using the explicit mini-max formulation, it is not difficult to show that

$$(6.2) \quad 0 < E \rightarrow j(E) \text{ is continuous and monotonically increasing.}$$

As a consequence, j is differentiable almost everywhere on $\mathbb{R}_+$. It is instructive to verify that if in some neighbourhood of a specific value $\hat{E} > 0$ there exists a family of solutions $x(E)$ of (6.1) which depend continuously differentiable on E , then the derivative $\frac{dj}{dE}(E)$ is precisely the period T given by (4.14). More generally, writing j'_+ and j'_- for the right- and left-hand side derivative of j , the following result holds.

Proposition 3. Let x be any solution of the constrained minimization problem (6.1). Then the period T of this solution as given by (4.14) satisfies

$$(6.3) \quad j'_+(E) < T < j'_-(E).$$

Note that from (6.3) it follows that at every point of differentiability, all solutions of (6.1) have one and the same period $T = j'(E)$.

In order to illustrate the typical dependence of the period T on the energy E , consider the special case of a positively homogeneous potential V of degree $\alpha > 1$:

$$(6.4) \quad V(\lambda x) = \lambda^\alpha V(x), \quad \lambda > 0, \quad x \in \mathbb{R}^n.$$

In that case the function j is homogeneous too:

$$(6.5) \quad j(E) = E^\sigma j(1), \quad \text{with } \sigma = \frac{\alpha+2}{2\alpha},$$

and from proposition 3 it follows that the period $T = T(E)$ satisfies

$$(6.6) \quad T(E) = E^{\sigma-1} T(1).$$

In order to obtain results for more general classes of potentials, it is convenient to restrict ourselves to functions V that satisfy in addition to $(V)_0$ another condition on the growth of V on rays through the origin in $\mathbb{R}^n$. By way of example we shall consider "super-quadratic" growth, for which it is required that V satisfies

$$(V)_1 \quad \begin{aligned} &V'(x) \cdot x > \theta V(x) \text{ for some } \theta > 2, \text{ and } |x| \text{ sufficiently large, and} \\ &V''(x) \cdot x \cdot x - V'(x) \cdot x > 0 \text{ for all } x \in \mathbb{R}^n \setminus \{0\}. \end{aligned}$$

[Upon integrating the first inequality, it follows that $V(x) > a|x|^\theta + b$ for some constants $a > 0$ and b , which declares the name super-quadratic. Note that a homogeneous potential V satisfies this condition precisely if its degree is larger than 2].

Along with normal modes of prescribed energy E , we now consider specific normal modes of prescribed period $4T$. Therefore we look for critical points of the functional K_T defined on the space X introduced in section 5. Again, K_T is not bounded from above nor from below (because of $(V)_1$), but critical points that are saddle points can be obtained by considering

$$(6.7) \quad k(T) := \inf_{\xi \in S} \sup_{\rho > 0} K_T(\rho\xi) \equiv \inf\{K_T(x) \mid x \in N_T\},$$

where, again, N_T is a natural constraint, now given by

$$(6.8) \quad N_T = \{x \in X \mid 1/T \cdot \int \dot{x}^2 = T \cdot \int V'(x) \cdot x\}.$$

Then it can be shown (van Groesen (1984)):

Theorem 2. There exists a number $T^* > 0$ such that $\forall T \in (0, T^*)$ $k(T)$ is strictly positive and the constrained minimization problem (6.7) has at least one solution; the normal mode corresponding to this solution via the transformation (3.5) has minimal period $4T$.

In section 4 it has been shown that the functionals $J_E(x)$ and $L_T(x)$, for fixed configuration curve x , are related to each other by a Legendre transformation. Now it turns out that the same result holds for the functions $E \mapsto j(E)$ and $T \mapsto k(T)$:

Proposition 4. The functions j and k , defined by (6.1) and (6.7) respectively, are related through a Legendre transformation:

$$(6.9) \quad j(E) = \inf_{T \in (0, T^*)} \{TE + k(T)\}, \quad E > 0$$

and

$$(6.10) \quad k(T) = \sup_{E \geq 0} \{j(E) - TE\}, \quad T \in (0, T^*).$$

Consequently, $j(E)$ is a concave function on $\mathbb{R}_+$ and $k(T)$ is convex on $(0, T^*)$.

It follows from this result that $T(E)$, being a subset of the (concave) subdifferential of j at E , which consists for almost all E of a single point, is non-increasing: $T(E_1) \geq T(E_2)$ for $E_1 < E_2$. We conjecture that, for the stringent conditions on V as required above, the functions j and k are in fact smooth (and hence strictly concave and convex respectively), so that $T(E)$ is a single valued, monotonically decreasing function on E .

For the special class of homogeneous potentials (6.4) with $\alpha > 2$, the proposition (and conjecture) are easily proved since an explicit expression for $k(T)$ can be found: $T^* = \infty$ in this case, and

$$(6.11) \quad k(T) = T^\mu k(1), \text{ with } \mu = \frac{\alpha+2}{\alpha-2}, \text{ (thus } \frac{1}{\sigma} - \frac{1}{\mu} = 1), \text{ and } k(1) > 0.$$

7. Generalizations

It is clear that the variational results of sections 3 and 4 are not restricted to the special problem under consideration. Before presenting some specific generalizations to problems from continuum mechanics, consider the following formulation of what may be called a

Generalized action principle

Let X and Y be two topological spaces and $\Lambda : X \rightarrow Y^*$ a continuous, linear map from X into Y^* , the dual of Y . For $F : X \times Y \rightarrow \mathbb{R}$ a continuously differentiable functional, consider for given $E \in \mathbb{R}$ the variational problem:

$$(7.1) \quad \text{stat}_{x \in X, y \in Y} \{ \langle \Lambda x, y \rangle \mid F(x, y) = E \},$$

If the mapping $y \mapsto F(x, y)$ is assumed to be strictly convex and coercive, for any $x \in X$, then the set $C_x := \{y \in Y \mid F(x, y) \leq E\}$, if nonempty, is a convex, closed (and hence weakly closed) and bounded subset of Y . If ∂C_x is a regular level set of F , i.e. if $(F_x, F_y) \neq (0, 0)$ for $F(x, y) = E$, then the functional $J_E(x)$ defined by the maximization problem

$$(7.2) \quad J_E(x) := \sup_{y \in Y} \{ \langle Ax, y \rangle \mid F(x, y) = E \},$$

can also be described as

$$(7.3) \quad J_E(x) = \inf_{T \in \mathbb{R}} \{ TE + TG(x, Ax/T) \},$$

where $G(x, \cdot)$ is the convex conjugate (i.e. Legendre transform) of $F(x, \cdot)$.

In particular, if $F(x, y) = f(x) + g(y)$ for some functionals f and g , then

$$(7.4) \quad J_E(x) = \inf_{T \in \mathbb{R}} \{ T[E - f(x)] + Th(Ax/T) \},$$

where h is the convex conjugate of g .

So, for these more general systems, critical points of the functional J_E given by (7.3) provide critical points of (7.1). Of course, existence and character of the critical points of J_E depends also on the behaviour of the functional $x \mapsto F(x, y)$, and this investigation may differ from the analysis of section 5.

Remark. In case the mapping $y \mapsto F(x, y)$ is not even for all $x \in X$, the minimization problem

$$(7.5) \quad \inf_{y \in Y} \{ \langle Ax, y \rangle \mid F(x, y) = E \}$$

defines a functional on X which differs (apart from sign) from J_E above. This functional can be characterized in terms of G in a way analogous to (7.3). Its critical points may provide critical points of (7.1) which differ from those found by an investigation of J_E .

The first specific example deals with

Nonlinear Wave Vibrations.

Consider on a bounded domain $\Omega \subset \mathbb{R}^n$ the following nonlinear wave equation with Dirichlet boundary conditions

$$(7.6) \quad \begin{aligned} -u_{tt} + \Delta u &= f(x, u), & x \in \Omega, \\ u(x, t) &= 0, & x \in \partial\Omega, \end{aligned}$$

where f is a given function. During the past few years there has been some progress in proving, for a restricted set of functions f , the existence of periodic solutions of prescribed period T (see Brezis (1983) for a survey). No results concerning

periodic solutions of prescribed energy are known up to now. We shall present a variational formulation in the spirit of this paper which may be useful in obtaining results for the prescribed energy case.

Let, for simplicity, the function $u \rightarrow f(x,u)$ be odd. Then we may consider special periodic solutions, which are the generalization of the normal mode vibrations of section 5, viz. standing waves. Define $F(x,u) = \int_0^u f(x,s)ds$, and suppose that f satisfies certain growth conditions which guarantee that the functional $\int_{\Omega} F(x,u)$ is continuous for $u \in W_{1,2}(\Omega, \mathbb{R}) \equiv H$. Then, a standing wave of energy E , i.e. for which

$$\int_{\Omega} \left\{ \frac{1}{2} u_t^2 + \frac{1}{2} (\nabla u)^2 + F(x,u) \right\} dx = E, \quad \text{for all } t \in \mathbb{R},$$

is related, via continuation and the (time)-scaling $t = T\tau$, $u(x,t) = v(x,\tau)$ to the critical points $v(x,\tau)$ of

$$(7.7) \quad \text{stat} \left\{ \int_{\Omega} \frac{1}{2} v_{\tau}^2 \cdot \left[E - \int_{\Omega} \left\{ \frac{1}{2} (\nabla v)^2 + F(x,v) \right\} \right] \mid v \in W_{1,2}([0,1], H), v(\cdot, 0) = 0 \right\},$$

where $\int_{\Omega}$ denotes $\int_0^1 d\tau \int_{\Omega} dx$; the resulting period of the vibration is $4T$ where $T > 0$ is given by

$$(7.8) \quad T^2 = \int_{\Omega} \frac{1}{2} v_{\tau}^2 \cdot \left[E - \int_{\Omega} \left\{ \frac{1}{2} (\nabla v)^2 + F(x,v) \right\} \right]^{-1}.$$

For (7.7), a natural constraint as in section 5 can be defined. The variational problem (7.7) is currently under investigation.

A second specific example belongs to the class of

Vorticity problems.

Several significant problems of modern mathematical physics are "confinement"-problems for the solenoidal part of a vectorfield that satisfies certain nonlinear equations. These are free boundary problems, since the domain to which the solenoidal part is confined is not known in advance. Specific examples are the Lundquist equations for plasma fusion, the Ginzberg-Landau (-Abrikosov) theory for superconductivity in type II materials and the quark confinement problem (MIT "bag"-model, and instanton solutions of Atiyah, Hitchin and Singer). (See also the contributions in section C of the Maxwell book edited by Berger (1984)).

A more classical problem which exhibits many of the relevant features of these problems are vortex rings in an ideal fluid. The most interesting global results for this problem have been obtained by Fraenkel and Berger (1974). We shall present a variational formulation of this particular fluid motion in the spirit of this paper.

Axially symmetric, steady vortex flow with constant velocity $U (> 0)$ in the z -direction can be described by introducing the Stokes stream function $\Psi(r, z)$ for the Eulerian velocity $\underline{v}$ (in a cylindrical coordinate system, moving with the vortex ring):

$$\underline{v} = \text{curl}\left(\frac{\Psi}{r}\underline{e}_\theta\right) = \frac{-1}{r} \frac{\partial \Psi}{\partial z}\underline{e}_r + \frac{1}{r} \frac{\partial \Psi}{\partial r}\underline{e}_z \equiv L\Psi.$$

The vorticity $\underline{\omega} = \omega \underline{e}_\theta$ is then given by

$$(7.9) \quad \omega/r = L^* L\Psi \equiv -\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial z^2}$$

(here L^* is the L_2 -adjoint of (with measure $d\mu = 2\pi r dr dz$) of L). Euler's equation implies that ω/r is constant along streamlines, i.e. for some vorticity function f and some $\lambda \in \mathbb{R}$:

$$(7.10) \quad \omega/r = \lambda^2 f(\Psi).$$

If ϕ denotes the stream function in the laboratory frame, and $\phi = \frac{1}{2}Ur^2 + k$, ($k > 0$) the one that corresponds to uniform flow in z -direction, we have $\Psi = \phi - \phi$, and require $\phi = 0$ at $r = 0$ and $\nabla\phi/r \rightarrow 0$ for $r^2 + z^2 \rightarrow \infty$. With these boundary conditions, (7.9) and (7.10) provide the relevant equations for ϕ :

$$(7.11) \quad L^* L\phi = \lambda^2 f(\phi - \phi)$$

In order to obtain a free boundary problem, it is assumed that f vanishes identically for $\Psi < 0$, and is strictly increasing for $\Psi > 0$. (The free boundary, i.e. the boundary of the set to which the vorticity is confined, is then given by the zero-level of Ψ). Fraenkel and Berger obtained their existence result by maximizing the "potential energy" $\int F(\Psi) d\mu$, where $F' = f$ on level sets of the kinetic energy $\frac{1}{2} \int |L\phi|^2 d\mu$ (so λ enters as a (positive) multiplier).

A formulation of the problem in which the total energy is prescribed is obtained by starting from the variational problem

$$(7.12) \quad \text{stat} \left\{ \int \underline{p} \cdot L\phi d\mu \mid \left\{ \frac{1}{2} \underline{p}^2 + F(\phi - \phi) \right\} d\mu = E \right\}.$$

Taking, for fixed ψ , the supremum over $\underline{p}$ in (7.12) leads to the square root of the functional

$$(7.13) \quad 2 \int \{ |L\psi|^2 d\mu \cdot [E - \int F(\psi-\phi) d\mu];$$

critical points of this functional satisfy (7.11) for a specific value of λ^2 that can be expressed in terms of E and ψ as before.

It is also possible to take for fixed $\underline{p}$ the supremum over ψ in (7.12). Introducing $\zeta = L^* \underline{p}$, and a mapping K which is the positive square root of the inverse of $L^* L$, there results the functional

$$(7.14) \quad \int \zeta \cdot \phi d\mu + \inf_{\lambda} \lambda [E - \int \{ \frac{1}{2} |K\zeta|^2 + G(\zeta/\lambda) \} d\mu]$$

where G is the convex conjugate of F .

For specific choices of the function F , this formulation of the problem is related to formulations investigated by Friedman (1982). (See van Groesen and van der Beek (1985) for more details.)

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