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SUMMARY

The polynomial approach to minimax frequency domain optimization of multivariable feedback has been successfully applied to the mixed sensitivity problem to improve robustness and disturbance rejection. The purpose of this paper is to extend the polynomial equalizing approach to what has been called super-optimization, a hierarchical process for minimizing not only the largest, but all the singular values. It is not difficult to find examples where this is of great relevance for further improvement of the quality and robustness of the control.

NOTATION

We begin by introducing some general definitions and notation to be used throughout this paper. For an $m \times n$ matrix A over the complex numbers $\mathbb{C}$ let $[A] = (A^H A)^{1/2}$, where A^H is the conjugate transpose of A . Denote by $s_i(A)$, $i = 1, 2, 3, \dots, n$, the i -th largest singular value of A (counting multiplicities), i.e. the i -th largest eigenvalue of $[A]$. An eigenvector of $[A]$ corresponding to such an eigenvalue is said to be a right singular vector of A . Let $\mathbb{C}^{m \times n}$ be the space of all complex $m \times n$ matrices. Introduce on it the norm

$$|A|_p = \left[\sum_{i=1}^n s_i^p(A) \right]^{1/p}, \quad \text{if } 0 < p < \infty,$$

and

$$|A|_p = |A|_\infty = s_1(A), \quad \text{if } p = \infty.$$

$|A|_\infty$ is nothing but the usual operator norm of A for which we also write $\|A\|$. By $\mathcal{L}_{m \times n}^p$ we mean the space of (equivalence classes of) Lebesgue measurable functions F from the imaginary axis $i\mathbb{R}$ to $\mathbb{C}^{m \times n}$ such that the function $\omega \rightarrow |F(i\omega)|_p$ belongs to $\mathcal{L}^p$. We assume that the following norm is used on $\mathcal{L}_{m \times n}^p$. For $0 < p < \infty$ we define the norm by

$$\|F\|_p = (1/2\pi)^{-1} \left[\int_{-\infty}^{\infty} |F(i\omega)|_p^p d\omega \right]^{1/p},$$

and for $p = \infty$ by

$$\|F\|_p = \|F\|_\infty = \text{ess sup } \{ |F(i\omega)|_\infty : \omega \in \mathbb{R} \} = \text{ess sup } \{ \|F(i\omega)\| : \omega \in \mathbb{R} \}.$$

For F in $\mathcal{L}_{m \times n}^p$ we define $\sigma_i(F)$, $i = 1, 2, \dots, n$ by

$$\sigma_i(F) = \text{ess sup } \{ s_i(F(i\omega)) : \omega \in \mathbb{R} \}.$$

For each positive integer $k \leq n$ we denote by $\sigma(F:k)$ be the k -tuple $\sigma(F:k) = (\sigma_1(F), \dots, \sigma_k(F))$.

1. INTRODUCTION

The mixed sensitivity problem

Consider the multivariable linear feedback control system $\mathcal{P}$ as in Fig. 1, with the plant represented by the transfer matrix H . The plant is controlled by the compensator with transfer matrix G . The signal r is the reference signal, the signal d a disturbance and y the controlled output. V , W_1 and W_2 are rational weighting matrices.

Fig. 1. Feedback system configuration.

A polynomial approach to minimax frequency domain optimization of this system has been treated extensively in Kwakernaak³. Since our purpose is to extend these ideas to what is sometimes called super-optimization, we make the same assumptions about the system as in the earlier work. Thus we require H , V , W_1 , and W_2 to be square $n \times n$ transfer matrices, with H , V , V^{-1} , W_1 and W_2^{-1} proper (i.e., V is biproper). Using coprime factorizations we write

$$H = D^{-1}N, \quad V = D^{-1}M, \quad W_1 = A_1 B_1^{-1}, \quad W_2 = A_2 B_2^{-1},$$

and assume that M has no roots in $\text{Re}(s) \geq 0$, B_1 and B_2 have no roots in $\text{Re}(s) > 0$, D and M are row

reduced and B_1 and A_2 column reduced. Every common left divisor of DB_1 and NB_2 is assumed to be a left divisor of M . Define

$$S := (I + HG)^{-1}, \quad T := G(I + HG)^{-1}, \quad (1.1)$$

where S is the sensitivity matrix and T the input sensitivity matrix of the system. Finally let

$$Z := V^*(S^*W_1^*W_1S + T^*W_2^*W_2T)V, \quad (1.2)$$

where $F^*(s) = F^T(-s)$.

In Kwakernaak³ the following optimization problem is considered. Find a stabilizing compensator G such that

$$\|Z\|_{\infty} := \operatorname{ess\,sup}_{\omega} \|Z(i\omega)\| = \sup_{\omega} \|Z(i\omega)\| \quad (1.3)$$

is minimized. Equivalently, find a stabilizing compensator G that minimizes

$$\begin{vmatrix} W_1SV \\ W_2TV \end{vmatrix}_{\infty}. \quad (1.4)$$

By minimizing (1.3) good disturbance attenuation and robustness may be obtained.

We observe that $\|Z(i\omega)\| = s_1(Z(i\omega))$. Thus we have $\|Z\|_{\infty} = \sigma_1(Z)$. Hence, minimizing (1.3) means that we minimize the least upper bound of the largest singular value of $Z(i\omega)$ as ω is allowed to vary over all frequencies. The remaining singular values of Z play no direct role in the optimization. Thus in general we may expect the optimal compensator to be far from unique, leaving some freedom to select the subdominating singular values. What influence do these have on the disturbance rejection and robustness? Let us look at the following diagonal case.

The diagonal case

Consider n independent systems $\mathcal{P}_i$, $i = 1, 2, \dots, n$, each of the form of Fig 1. Let Z_i be the Z matrix of $\mathcal{P}_i$ and d_i, y_i, q_{1i}, q_{2i} , etc., the signals related to the system $\mathcal{P}_i$. By defining the signal vectors $d = \operatorname{col}(d_1, \dots, d_n)$, $y = \operatorname{col}(y_1, \dots, y_n)$, etc., these n independent systems may be combined to a single system $\mathcal{P}$ of the same form as the subsystems $\mathcal{P}_i$. The Z matrix for the system formed by these n independent systems taken as a whole may be written as the diagonal block matrix

$$Z = \operatorname{diag}(Z_1, Z_2, \dots, Z_n) = \begin{bmatrix} Z_1 & & \\ & Z_2 & \\ & & \ddots \\ & & & Z_n \end{bmatrix}. \quad (1.5)$$

Since the set of eigenvalues of Z is the union of sets of eigenvalues of the individual Z_i 's, it is easy to see that $\sigma_1(Z) = \max(\sigma_1(Z_1), \sigma_1(Z_2), \dots, \sigma_1(Z_n))$. For any system $\mathcal{P}$ as in Fig. 1 we define $\mu(\mathcal{P}) = \inf(\sigma_1(Z) : G \text{ stabilizes the system})$.

Suppose that the link represented by Z_1 is "weaker" than the one represented by Z_2 and that Z_2 is weaker than Z_3 , etc., in the sense that $\mu(\mathcal{P}_1) \geq \mu(\mathcal{P}_2) \geq \dots \geq \mu(\mathcal{P}_n)$. Suppose that G_1 is an optimal compensator for $\mathcal{P}_1$ and for each k , $k = 2, \dots, n$, G_k is any compensator for $\mathcal{P}_k$ such that $\sigma_1(Z_k) \leq \mu(\mathcal{P}_1)$. Then certainly $G = \operatorname{diag}(G_1, G_2, \dots, G_n)$ is an optimal compensator for the whole system $\mathcal{P}$. Thus the compensators G_k , $k = 2, 3, \dots, n$, may be chosen freely as long as $\sigma_1(Z_k) \leq \mu(\mathcal{P}_1)$ for each k . However, we certainly obtain better disturbance rejection and robustness if not only the system $\mathcal{P}_1$ is optimal, but each of the remaining subsystems $\mathcal{P}_k$, $k = 2, \dots, n$, as well. Thus it may be useful to consider a minimization of all the singular values of Z in some hierarchical fashion. This leads to the idea of "super-optimization" of the control system. The problem has been identified by Young, who has steadily contributed to its solution (Young^{5,6}). Other notable works are those of Postlethwaite and Foo⁷, Tsai et al.⁸, and Gu et al.⁹

2. CLASS k OPTIMAL AND SUPER-OPTIMAL COMPENSATORS

Definition. For a given compensator G we say that G or $\mathcal{P}$ (or the corresponding Z in (1.2)) is *class k optimal* ($k = 1, 2, \dots, n$) if for every other stabilizing compensator G'

$$\sigma(Z:k) \leq \sigma(Z':k),$$

with $\leq$ defining ordering in the lexicographical sense. Z' is the transfer matrix (1.2) corresponding to G' . If

$$\sigma(Z:n) \leq \sigma(Z':n)$$

for all stabilizing compensators G' , then we also say that $G(\mathcal{P}, Z)$ is *super-optimal*. We use the notation $\leq$ for lexicographical ordering on the set of all real k -tuples, no matter what k is. $\square$

Later we express G in terms of a solution R , P and Q of two polynomial matrix equations. We then say that the solution is k -optimal or super-optimal if the corresponding compensator has this property.

Class 1 optimization

We recapitulate shortly the method for class 1 optimization given in ³. The optimal compensator is found among those compensators for which Z is of the equalizing type, i.e., for which the matrix Z takes the form

$$Z(i\omega) = \lambda^2 I(i\omega), \quad \omega \in \mathbb{R},$$

for some real constant λ , with I the identity matrix. With the coprime factorizations introduced earlier and factorizing $G = YX^{-1}$ we may write Z as

$$Z = M^* F^{-*} (X^* B_1^{-*} A_1^* A_1 B_1^{-1} X + Y^* B_2^{-*} A_2^* A_2 B_2^{-1} Y) F^{-1} M, \quad (2.1)$$

where $F = DX + NY$. The polynomial matrix F is a generalization of the closed-loop characteristic polynomial of SISO feedback systems. Since $Z = \lambda^2 I$ we must let X have a factor cancelling the zeroes of B_1 . Similarly Y must have a factor that cancels the zeroes of B_2 . One way to accomplish this is to let

$$X = B_1 P, \quad Y = B_2 Q, \quad (2.2)$$

where P and Q are polynomial matrices to be determined. In fact, it turns out in ³ that this is the proper way to achieve cancellation. Using this factorization of X and Y , we may rewrite Z as

$$Z = M^* F^{-*} (P^* A_1^* A_1 P + Q^* A_2^* A_2 Q) F^{-1} M. \quad (2.3)$$

Given P and Q we may use spectral factorization to find a polynomial matrix R such that $\det(R)$ has no roots in the right-half plane, and

$$R^* R = P^* A_1^* A_1 P + Q^* A_2^* A_2 Q.$$

Suppose that F may be factorized as $F = \lambda^{-1} M R$. Inserting this expression for F into (2.3) we see that Z indeed has the form $Z = \lambda^2 I$. Since $F = D B_1 P + N B_2 Q$ this suggests that we try to solve the equations

$$\lambda^{-1} M R = D B_1 P + N B_2 Q, \quad (2.4a)$$

$$R^* R = P^* A_1^* A_1 P + Q^* A_2^* A_2 Q, \quad (2.4b)$$

for polynomial matrices P , Q , and R . The compensator required to achieve the prescribed Z then is given by

$$G = YX^{-1} = B_2 Q P^{-1} B_1^{-1}. \quad (2.5)$$

We observe that (2.4a) is linear in the unknowns P , Q and R , while (2.4b) is quadratic. A numerical solution technique based on the Newton-Raphson method has been described in ³ for solving the equations (2.4) for an equalizing, class 1 optimal solution. The procedure is the following.

For $|\lambda| = \infty$ we may find a solution R_∞ , P_∞ , Q_∞ of (2.4) such that $\det(R_\infty)$ has all its roots in the left-half plane. By gradually decreasing $|\lambda|$ and solving (2.4) we reach a point $\lambda = \mu_1$ where one or more of the roots of $\det(R)$ cross from the left- to the right-half plane (possibly via infinity). For this λ we may determine a common factor of the solution R , P , Q . By cancelling the common factor we obtain a solution R , P , Q of reduced degree, i.e. a solution such that $\deg(\det(R)) < \deg(\det(R_\infty))$. Denote this reduced solution by R_1 , P_1 , Q_1 , and observe that $\det(R_1)$ has all its roots in the left-half plane. This reduced solution is class 1 optimal. Of course $\sigma_1(Z) = \mu_1^2$ for the corresponding Z .

A sufficient condition for class 1 optimality

In proving that the reduced solution P_1 , Q_1 , R_1 of (2.4) is class 1 optimal, P_1 , Q_1 , R_1 are used to construct a real rational matrix function Φ that satisfies the following lemma proved in ³.

Lemma 2.1. Let Φ be non-trivial, real rational, para-Hermitian, strictly proper matrix function such that $\Phi(i\omega) \geq 0$ for all real ω . Suppose that

$$\int_{-\infty}^{\infty} \text{trace}(Z(i\omega)\Phi(i\omega)) d\omega$$

is minimized by some compensator G and that the corresponding Z satisfies $Z(i\omega) = \lambda^2 I$ for all ω , with λ a real constant. Then G also minimizes $\|Z\|_\infty$ (i.e., is class 1 optimal). $\square$

We generalize this result to a criterion that may be used to prove class k optimality ($k = 1, 2, \dots, n$). The result is given under slightly more general conditions.

A sufficient condition for the minimality of $c_k(Z)$

Consider the problem of minimizing $\sigma_k(Z)$ as Z varies over some family Z . No attention is paid to the size of the other singular values of Z . We develop a

sufficient condition for the minimality of $\sigma_k(Z)$, which later on, in the process of finding super-optimal solutions, will be used repeatedly on smaller and smaller subspaces of Z . First we need some notation.

Let $Z \in \mathcal{L}_{n \times n}^\infty$ and use singular value decomposition to write $Z(i\omega) = U_1(i\omega)\Lambda(i\omega)U_2(i\omega)$, where $U_1(i\omega)$ and $U_2(i\omega)$ are $n \times n$ unitary matrices and $\Lambda(i\omega) = \text{diag}(s_1(Z(i\omega)), s_2(Z(i\omega)), \dots, s_n(Z(i\omega)))$. Let $c_1(i\omega), c_2(i\omega), \dots, c_n(i\omega)$ be the orthonormal columns of $U_2^*(i\omega)$. It is easy to see that $c_1(i\omega)$ is a right singular vector of $Z(i\omega)$ associated with the singular value $s_1(Z(i\omega))$, i.e. an eigenvector of $[Z(i\omega)]$ associated with $s_1(Z(i\omega))$. For $k, 1 \leq k \leq n$, define $\mathcal{V}_k(Z(i\omega))$ to be the space spanned by the column vectors $c_k(i\omega), c_{k+1}(i\omega), \dots, c_n(i\omega)$. For the purpose of the next theorem we let Z be a non-empty subset of $\mathcal{L}_{n \times n}^\infty$. Define

$$\mathcal{V}_k(\omega) = \bigcap_{Z \in Z} \mathcal{V}_k(Z(i\omega)), \quad (2.6)$$

i.e. $\mathcal{V}_k(\omega)$ is the largest subspace contained in every $\mathcal{V}_k(Z(i\omega))$.

Theorem 2.2. Let Z be a non-empty subset of $\mathcal{L}_{n \times n}^\infty$. Suppose that $Z \in Z$ and $\Phi \in \mathcal{L}_{n \times n}^1$ with $\Phi(i\omega) \geq 0$ and $\Phi(i\omega) \neq 0$ for each $\omega \in \mathbb{R}$. If for some $k \leq n$

$$\text{Range } \Phi(i\omega) \subseteq \mathcal{V}_k(\omega) \quad \text{for each } \omega \in \mathbb{R}, \quad (2.7)$$

and

$$\left| \int_{-\infty}^{\infty} \text{trace}(Z\Phi) d\omega \right| \leq \left| \int_{-\infty}^{\infty} \text{trace}(Z'\Phi) d\omega \right| \quad \forall Z' \in Z, \quad (2.8)$$

and

$$\left| \int_{-\infty}^{\infty} \text{trace}(Z\Phi) d\omega \right| = \sigma_k(Z) \int_{-\infty}^{\infty} \text{trace}(\Phi) d\omega, \quad (2.9)$$

then

$$\sigma_k(Z) \leq \sigma_k(Z') \quad \forall Z' \in Z, \quad (2.10)$$

and for each $Z' \in Z$ with $\sigma_k(Z') = \sigma_k(Z)$

$$\left| \int_{-\infty}^{\infty} \text{trace}(Z'\Phi) d\omega \right| = \left| \int_{-\infty}^{\infty} \text{trace}(Z\Phi) d\omega \right| \quad (2.11)$$

□

The proof of this theorem is given in the Appendix.

Remark. When $k = 1$, condition (2.7) of Theorem 2.2 is of course always satisfied. □

It may be of some interest to see what is required of a matrix valued function Z to satisfy condition (2.9) of Theorem 2.2. Define $\theta(\omega) = \arg \text{trace}(Z\Phi(i\omega))$.

Theorem 2.3. Let Z be an arbitrary function in $\mathcal{L}_{n \times n}^\infty$ and $\Phi \in \mathcal{L}_{n \times n}^1$ with $\Phi(i\omega) \geq 0$ and $\Phi(i\omega) \neq 0$ for each $\omega \in \mathbb{R}$. Suppose that $\text{Range } \Phi(i\omega) \subseteq \mathcal{V}_k(\omega)$ for every ω . Then

$$\left| \int_{-\infty}^{\infty} \text{trace}(Z\Phi) d\omega \right| = \sigma_k(Z) \int_{-\infty}^{\infty} \text{trace}(\Phi) d\omega \quad (2.12)$$

if and only if

- (i) for every $j = 1, 2, \dots, n-k+1$ we have $s_{k+j-1}(Z(i\omega)) = \sigma_k(Z)$ for almost all ω such that $s_j(\Phi(i\omega)) > 0$;
- (ii) θ is constant a.e.;
- (iii) $e^{-i\theta} \cdot Z(i\omega)\Phi(i\omega) \geq 0$ a.e. □

For the proof, see the Appendix.

Observe that in Theorem 2.3 statement (i) actually means that $s_k(Z(i\omega)), s_{k+1}(Z(i\omega)), \dots, s_{k+j-1}(Z(i\omega))$ are all equal to $\sigma_k(Z)$ for almost all ω such that $s_j(\Phi(i\omega)) > 0$. In our attempt to successively minimize the $\sigma_k(Z)$ -values, we should like Z to satisfy the conditions of Theorem 2.2. In view of the above observation we may expect to find promising candidates among those Z for which all singular values are constant, i.e., independent of ω , and $s_k(Z) = s_{k+1}(Z) = \dots = s_n(Z)$.

If $Z(i\omega)$ in Theorem 2.2 is nonnegative definite, as far as the range of Φ is concerned the matrix Z acts as if it would be of the equalizing type $\lambda^2 I$ used in ³. This is the content of the following corollary.

Corollary 2.4. Let Z and Φ be as in Theorem 2.3, but suppose in addition that $Z(i\omega) \geq 0$ for each $\omega \in \mathbb{R}$. Then (2.12) of Theorem 2.3 is equivalent to

$$Z\Phi = \sigma_k(Z) \cdot \Phi \quad \text{a.e.} \quad (2.13)$$

In this case $\theta = 0$ a.e. □

For the proof, see the Appendix.

Application of Theorem 2.2 to class k optimization

We return to the real rational Z of our system $\mathcal{P}$. The matrix function Φ is in the sequel assumed to be non-trivial, real rational, para-Hermitian and

strictly proper with $\Phi(i\omega) \geq 0$ for all real ω . Since Z is para-Hermitian, $\text{trace}(Z\Phi)$ is always nonnegative and the absolute values of the integrals in Theorems 2.2 and 2.3 may be omitted.

In elaborating the condition (2.9) of Theorem 2.2 we use the following result proved in 3.

Lemma 2.4. A sufficient condition for the minimization of the auxiliary criterion

$$\int_{-\infty}^{\infty} \text{trace}(Z\Phi) d\omega \quad (2.14)$$

is that

$$DV\Phi V^* (S^* W_1^* W_1 N_r - T^* W_2^* W_2 D_r) \quad (2.15)$$

has all its poles in $\text{Re}(s) \leq 0$. Here $N_r D_r^{-1} = H$ is a right coprime polynomial factorization of the plant transfer matrix. $\square$

Suppose that for some family Z we have found a $Z \in \mathcal{Z}$ together with m matrix functions $\Phi_1, \dots, \Phi_m$, $m \leq n$, such that for each k , $1 \leq k \leq m$, we have $\Phi_k \in \mathcal{L}_{n \times n}^1$ with $\Phi_k(i\omega) \geq 0$ and $\Phi_k(i\omega) \neq 0$ for each ω . For the family Z define for each k the space $\mathcal{V}_k(i\omega)$ as in (2.6). Suppose furthermore that for each k , $1 \leq k \leq m$,

(i) $\text{Range } \Phi_k(i\omega) \subseteq \mathcal{V}_k(i\omega)$ for each $\omega \in \mathbb{R}$,

and

(ii) $\int_{-\infty}^{\infty} \text{trace}(Z\Phi_k) d\omega \leq \int_{-\infty}^{\infty} \text{trace}(Z'\Phi_k) d\omega \quad \forall Z' \in \mathcal{Z}$,

and

(iii) $\int_{-\infty}^{\infty} \text{trace}(Z\Phi_k) d\omega = \sigma_k(Z) \cdot \int_{-\infty}^{\infty} \text{trace}(\Phi_k) d\omega$.

Then by Theorem 2.2 we must have $\sigma(Z;m) \stackrel{L}{\leq} \sigma(Z';m)$, i.e., the matrix function Z is class m optimal. As an example of a system where such Φ_k functions easily may be found consider the diagonal case in Section 1, assuming that each of the subsystems $\mathcal{V}_k$ is one-dimensional. Using the results in Kwakernaak^{2,3} we may find for each subsystem $\mathcal{V}_k$, $1 \leq k \leq m$, an equalizing class 1 optimal Z_k together with a Φ_k satisfying Lemma 2.1. It is obvious that Z of the whole system for each $0 \leq k \leq m$ satisfies (i), (ii) and (iii) above. Thus Z is class m optimal. For $m = n$ we obtain super-optimality.

3. ALGORITHM FOR CLASS k AND SUPER-OPTIMIZATION

We intend to find optimal compensators G that make Z take the form

$$Z = \sum_{i=1}^n \lambda_i^2 a_i^* a_i, \quad (3.1)$$

where each λ_i is a real constant and $\{a_i\}_{i=1}^n$ is an orthonormal independent set of stable rational row vectors. By this we mean that for each ω we have the orthonormality condition

$$a_i(i\omega) a_j^*(i\omega) = \begin{cases} 1, & i = j, \\ 0 & \text{otherwise,} \end{cases} \quad (3.2)$$

and the stability condition

for each $i = 1, 2, \dots, n$ the row vector a_i has all its poles in $\text{Re}(s) < 0$.

For each i the positive number λ_i^2 is the i -th singular value of $Z(i\omega)$ and a_i^* a corresponding right singular vector. We briefly outline a procedure for finding a class k optimal or even super-optimal compensator.

Step 1 (class 1 solution). Calculate a class 1 optimal solution P_1, Q_1, R_1 together with a real rational matrix function Φ_1 as described in section 2. Let μ_1 be the corresponding minimal value of $\sigma_1(Z)$. Φ_1 is constructed to satisfy the sufficient condition of Lemma 2.4. It may be chosen to have the form

$$\Phi_1 = \Delta_1^* \Delta_1,$$

where Δ_1 is a rational row vector, i.e. Φ_1 is of rank 1. (For details see 3).

Step 2 (class 2 solution). Choose an orthonormal set $\{a_{2i}\}_{i=1}^n$ of stable real rational row vectors with $a_{2i} = \Delta_1/\phi$, where ϕ is a real rational function satisfying $\Delta_1 \Delta_1^* = \phi^* \phi$. The first subscript in a_{2i} is the step number of the algorithm. Since the row vector a_{2i} is used in all the remaining steps, we also denote it by a_i instead of a_{2i} . Let μ_1 be the minimal value of $\sigma_1(Z)$ obtained in step 1. We now restrict the family $\mathcal{Z}$ of possible Z to the set of all Z of the form

$$Z = \mu_1^2 a_1^* a_1 + \lambda^2 \sum_{i=2}^n a_{2i}^* a_{2i}, \quad \lambda \in \mathbb{R}, \quad (3.3)$$

for which the corresponding compensator G stabilizes

the system. Using coprime factorization write

$$\begin{bmatrix} \mu_1 a_1 \\ \mu_1 a_{22} \\ \vdots \\ \mu_1 a_{2n} \end{bmatrix} = \Gamma_2 L_2^{-1}, \quad \Gamma_2 = \begin{bmatrix} \gamma_{21} \\ \gamma_{22} \\ \vdots \\ \gamma_{2n} \end{bmatrix}, \quad (3.4)$$

with $\gamma_{21}, \gamma_{22}, \dots, \gamma_{2n}$ polynomial row vectors. Since the a_{21} 's are stable, Γ_2 has all its roots in $\text{Re}(s) < 0$. Let $K_2 = \text{diag}(1, \lambda/\mu_1, \dots, \lambda/\mu_1) \cdot \Gamma_2$ and $J_2 = K_2 L_2^{-1}$. Then

$$J_2^* J_2 = \mu_1^2 a_1^* a_1 + \lambda^2 \sum_{i=2}^n a_{2i}^* a_{2i} = Z.$$

Consider the equations

$$M L_2 R = D B_1 P + N B_2 Q, \quad (3.5a)$$

$$R^* (\gamma_{21}^* \gamma_{21} + \frac{\lambda^2}{\mu_1} \sum_{i=2}^n \gamma_{2i}^* \gamma_{2i}) R = P^* A_1^* A_1 P + Q^* A_2^* A_2 Q. \quad (3.5b)$$

Suppose that (3.5) has a solution P, Q, R such that $\det(R)$ has all its roots in $\text{Re}(s) < 0$. Then $F = M L_2 R$ has all roots in $\text{Re}(s) < 0$. Let the compensator G be defined by (2.2). Substitution of (3.5) into (2.1) shows that the corresponding Z has the form (3.3). This suggests that we try to solve (3.5) to obtain compensators giving Z the desired form.

For $\lambda = \mu_1$ we may use the class 1 optimal solution P_1, Q_1, R_1 to obtain a starting solution of (3.5) as follows. By matrix fraction conversion we may find L_2, R_s such that $R_s L_2 = L_2 R_s$. Let $P_s = P_1 L_2, Q_s = Q_1 L_2$. Then P_s, Q_s, R_s is a solution.

By solving (3.5) while $|\lambda|$ is decreased from $|\mu_1|$ we reach a value $\lambda = \mu_2$ where one or several of the roots of $\det(R)$ crosses from the left- to the right-half plane (possibly via infinity). At this point the solution R, P, Q again contains a common factor. Cancelling this we obtain a solution of reduced degree, i.e. a solution such that $\deg(\det(R)) < \deg(\deg(R_s))$. For this reduced solution we construct a Φ_2 such that in Theorem 2.2 the minimality of $\sigma_2(Z)$ can be proved, i.e., $\int \text{trace}(Z \Phi_2) d\omega$ is minimized over Z and $\text{Range}(\Phi_2(i\omega)) \subset \text{span}(a_{22}^*(i\omega), a_{23}^*(i\omega), \dots, a_{2n}^*(i\omega))$. We indicate briefly how such a Φ_2 may be found.

Let $N_r D_r^{-1} = H$ be a right coprime factorization of the plant transfer matrix. Then there exists a polynomial

matrix B such that $B_1 P_\infty = N_r B$ and $B_2 Q_\infty = -D_r B$. $R_\infty, P_\infty, Q_\infty$ is the starting solution in step 1. From the proof of Theorem 2.3(b) in [3] we know that if $X = B_1 P, Y = B_2 Q$

$$S^* W_1^* W_1 N_r - T^* W_2^* W_2 D_r = D^* F^{-*} E^* B^{-1},$$

where $E = P_\infty^* A_1^* A_1 P + Q_\infty^* A_2^* A_2 Q$. Let Φ_2 have the form

$$\Phi_2 = L_2 R E^{-1} C C^* E^{-*} R^* L_2^*$$

where C is some proper real rational matrix, $E = E_+ E_-$ and E_+ has all its roots in $\text{Re}(s) > 0$ and E_- has all its roots in $\text{Re}(s) < 0$. It is possible to choose C such that for the reduced R, Φ_2 is strictly proper of rank 1 and $\text{Range } \Phi_2 \subset \text{span}(a_{22}^*, a_{23}^*, \dots, a_{2n}^*)$. Since Φ_2 is of rank 1 we may write

$$\Phi_2 = \Delta_2^* \Delta_2$$

for some strictly proper rational row vector Δ_2 . Note that $\Delta_2 \Delta_1^* = 0$.

Using $F = M L_2 R$ we obtain

$$D V \Phi_2 V^* (S^* W_1^* W_1 N_r - T^* W_2^* W_2 D_r) = M L_2 R E^{-1} C B^{-1}$$

which has all its poles in $\text{Re}(s) \leq 0$ as desired. From Lemma 2.4. we see that Φ_2 minimizes $\int \text{trace}(Z \Phi_2) d\omega$. Thus by Theorem 2.2 the Z corresponding to the reduced solution P, Q, R must have minimal $\sigma_2(Z)$.

Since $\text{Range}(\Phi_1) = \text{span}(a_1^*)$ and $\text{Range}(\Phi_2) \subset \text{span}(a_{22}^*, \dots, a_{2n}^*)$, it is easily seen that Theorem 2.2 is satisfied, not only for Φ_2 , but for Φ_1 as well. Thus by Theorem 2.2 the reduced solution is class 2 optimal, and for the corresponding Z we have $\sigma_1(Z) = \mu_1^2$ and $\sigma_2(Z) = \mu_2^2$. Let P_2, Q_2, R_2 denote this class 2 optimal solution corresponding to $\lambda = \mu_2$.

Step 3 (class 3 solution).

Choose an orthonormal set $(a_{3i})_{i=1}^n$ of stable real rational row vectors such that $a_{31} = a_1$ and $a_{32} = \Delta_2 / \phi$, where ϕ is a real rational function satisfying $\Delta_2 \Delta_2^* = \phi^* \phi$. Since the row vector a_{32} is used in all the remaining steps, we also denote it by a_2 . We now restrict the family Z to be the set of all Z of the form

$$Z = \mu_1^2 a_1^* a_1 + \mu_2^2 a_2^* a_2 + \lambda^2 \sum_{i=3}^n a_{3i}^* a_{3i}, \quad \lambda \in \mathbb{R}, \quad (3.6)$$

for which the corresponding compensator G stabilizes the system. Using coprime factorization write

$$\begin{bmatrix} \mu_1 a_1 \\ \mu_2 a_2 \\ \mu_2 a_{33} \\ \vdots \\ \mu_2 a_{3n} \end{bmatrix} = \Gamma_3 L_3^{-1}, \quad \Gamma_3 = \begin{bmatrix} \gamma_{31} \\ \gamma_{32} \\ \gamma_{33} \\ \vdots \\ \gamma_{3n} \end{bmatrix},$$

with $\gamma_{31}, \gamma_{32}, \dots, \gamma_{3n}$ polynomial row vectors. Since the a_{31} 's are stable, L_3 has all its roots in $\text{Re}(s) < 0$. Let $K_3 = \text{diag}(1, 1, \lambda/\mu_2, \dots, \lambda/\mu_2)\Gamma_3$ and $J_3 = K_3 L_3^{-1}$. Then

$$J_3^* J_3 = \mu_1^2 a_1^* a_1 + \mu_2^2 a_2^* a_2 + \lambda^2 \sum_{i=3}^n a_{3i}^* a_{3i} = Z.$$

Similarly to (3.5) we consider the equations

$$M L_3 R = D B_1 P + N B_2 Q, \quad (3.7a)$$

$$\begin{aligned} R^* (\gamma_{31}^* \gamma_{31} + \gamma_{32}^* \gamma_{32} + \frac{\lambda^2}{\mu_2^2} \sum_{i=3}^n \gamma_{3i}^* \gamma_{3i}) R = \\ = P^* A_1^* A_1 P + Q^* A_2^* A_2 Q. \end{aligned} \quad (3.7b)$$

Again, if P, Q, R is a solution the compensator $G = B_2 Q (B_1 P)^{-1}$ lets Z have the form (3.6). Thus we proceed by trying to solve (3.7).

For $\lambda = \mu_2$ we may use the class 2 optimal solution P_2, Q_2, R_2 to obtain a starting solution of (3.7) as follows. By matrix fraction conversion we may find $\hat{L}_3, \hat{R}_3$ such that $L_2 R_2 \hat{L}_3 = L_3 \hat{R}_3$. Let $P_3 = P_2 \hat{L}_3, Q_3 = Q_2 \hat{L}_3$. Then P_3, Q_3, R_3 is a solution.

By solving (3.7) while $|\lambda|$ is decreased from $|\mu_2|$ we reach a value $\lambda = \mu_3$ where some of the roots of $\det(R)$ crosses from the left- to the right-half plane (possibly via infinity). At this point the solution R, P, Q contains a common factor. Cancelling this we obtain a solution of reduced degree, i.e. one with $\deg(\det(R)) < \deg(\deg(R_3))$. For this solution we construct a strictly proper real rational Φ_3 such that in Theorem 2.2 the minimality of $\sigma_3(Z)$ may be proved, i.e., such that $\int \text{trace}(Z \Phi_3) d\omega$ is minimized over Z and $\text{Range } \Phi_3 \subseteq \text{span}(a_{33}^*, a_{34}^*, \dots, a_{3n}^*)$. This Φ_3 may be chosen to have rank 1. Thus for some strictly proper real rational row vector Δ_3 , we may write

$$\Phi_3 = \Delta_3^* \Delta_3.$$

Note that $\Delta_3 \Delta_1^* = \Delta_3 \Delta_2^* = 0$.

Since $\text{Range}(\Phi_1) = \text{span}(a_1^*)$ and $\text{Range}(\Phi_2) \subseteq \text{span}(a_2^*)$ and $\text{Range}(\Phi_3) \subseteq \text{span}(a_{33}^*, \dots, a_{3n}^*)$, Theorem 2.2 is

satisfied, not only for Φ_3 , but for Φ_1 and Φ_2 as well. Thus by Theorem 2.2 the reduced solution is class 3 optimal and for the corresponding Z we have $\sigma_1(Z) = \mu_1^2, \sigma_2(Z) = \mu_2^2$ and $\sigma_3(Z) = \mu_3^2$. Let P_3, Q_3, R_3 denote this class 3 optimal solution corresponding to $\lambda = \mu_3$.

Remaining steps. It is clear from step 3 how to continue this process. Use the Δ_k of Φ_k from step k to define the orthonormal set $\{a_{ki}\}$ such that the first $k-2$ vectors remain the same as in step k and $a_{k(k-1)}$ is directed along Δ_k . Define Γ_{k+1} and L_{k+1} analogously to Γ_3 and L_3 and consider the corresponding linear and quadratic equations similar to (3.7). Use P_k, Q_k and R_k to construct a starting solution of step $k+1$ at $\lambda = \mu_k$. Solve the linear and quadratic equations while decreasing λ until $\det(R)$ loses degree. Obtain the reduced, class $k+1$ optimal solution $P_{k+1}, Q_{k+1}, R_{k+1}$ by cancelling common factors. Use this solution to construct Δ_{k+1} and Φ_{k+1} . Define $\{a_{(k+1)i}\}$ so that the first $k-1$ vectors remain the same as in step k , and with $a_{(k+1)k}$ directed along Δ_{k+1} . Use $P_{k+1}, Q_{k+1}, R_{k+1}$ to obtain a starting solution of step $k+1$, and so on. Continue the process until a class n optimal solution P_n, Q_n, R_n is obtained. For this solution $\sigma_k(Z) = \mu_k^2, k = 1, 2, \dots, n$. Consequently P_n, Q_n, R_n is a super-optimal solution for the system.

CONCLUSION

We have presented an extension of the polynomial methods for minimax frequency domain optimization developed in Kwakernaak³ to super-optimization. The super-optimal compensator is expressed in terms of a reduced degree solution to a linear and a quadratic polynomial matrix equation. These equations are similar to those used in the class 1 problem. Since equations of this type have been successfully solved in³, we expect that essentially the same computational methods may be used, although further work is needed to develop a numerical algorithm. The paper only deals with the mixed sensitivity problem. Many of the ideas apply to the standard H_∞ problem as well. It is therefore not unlikely that super-optimization in the standard H_∞ may be treated by methods similar to those given here.

APPENDIX

The space $\mathbb{C}^{n \times n}$ equipped with the norm $|\cdot|_p$ is denoted M_p . We make use of the following result about the spaces M_p , which may be found in ¹, pp 130 - 132.

Lemma A1. The general form of a bounded linear functional ϕ on M_p ($1 < p \leq \infty$) is given by

$$\phi(X) = \text{trace}(AX),$$

where A is some matrix in M_q . Here $p^{-1} + q^{-1} = 1$ if $1 < p < \infty$ and $q = 1$ if $p = \infty$. Moreover,

$$|\phi| := \sup_{X \in M_p} \frac{|\text{trace}(AX)|}{|X|_p} = |A|_q.$$

Thus M_p^* is isometrically isomorphic to M_q . M_p is reflexive for $1 < p < \infty$. $\square$

We need two preliminary results.

Lemma A2. Let $Z \in \mathcal{L}_{n \times n}^\infty$ and $\Phi \in \mathcal{L}_{n \times n}^1$ with $\Phi(i\omega) \geq 0$ and $\Phi(i\omega) \neq 0$ for each $\omega \in \mathbb{R}$. Then

$$\left| \int_{-\infty}^{\infty} \text{trace}(Z\Phi) d\omega \right| \leq \|Z\|_\infty \cdot \int_{-\infty}^{\infty} \text{trace}(\Phi) d\omega. \quad \square$$

Proof: Take $p = 1$ in the second statement of Lemma A1 to obtain $\text{trace}(Z\Phi(i\omega)) \leq |Z(i\omega)|_\infty \cdot |\Phi(i\omega)|_1$. The result follows immediately from the fact that $\text{trace}(\Phi(i\omega)) = |\Phi(i\omega)|_1$ for $\Phi(i\omega) \geq 0$.

Lemma A3. Let Z be a non-empty subset of $\mathcal{L}_{n \times n}^\infty$ and let $\Phi \in \mathcal{L}_{n \times n}^1$. Suppose that $\Phi(\omega) \geq 0$ and $\Phi(i\omega) \neq 0$ for each $\omega \in \mathbb{R}$. If for some $Z_0 \in Z$

$$\left| \int_{-\infty}^{\infty} \text{trace}(Z_0\Phi) d\omega \right| \leq \left| \int_{-\infty}^{\infty} \text{trace}(Z\Phi) d\omega \right| \quad (A.1)$$

for each $Z \in Z$, and

$$\left| \int_{-\infty}^{\infty} \text{trace}(Z_0\Phi) d\omega \right| = \|Z_0\|_\infty \cdot \int_{-\infty}^{\infty} \text{trace}(\Phi) d\omega, \quad (A.2)$$

then

$$(i) \quad \|Z_0\|_\infty \leq \|Z\|_\infty \text{ for all } Z \in Z,$$

$$(ii) \text{ for each } Z \in Z \text{ with } \|Z\|_\infty = \|Z_0\|_\infty$$

$$\left| \int_{-\infty}^{\infty} \text{trace}(Z_0\Phi) d\omega \right| = \left| \int_{-\infty}^{\infty} \text{trace}(Z\Phi) d\omega \right|. \quad \square$$

Proof: From (A.2), (A.1) and Lemma A2 we obtain for any $Z \in \mathcal{L}_{n \times n}^\infty$

$$\begin{aligned} \|Z_0\|_\infty \cdot \int_{-\infty}^{\infty} \text{trace}(\Phi) d\omega &= \left| \int_{-\infty}^{\infty} \text{trace}(Z_0\Phi) d\omega \right| \leq \\ &\leq \left| \int_{-\infty}^{\infty} \text{trace}(Z\Phi) d\omega \right| \leq \|Z\|_\infty \cdot \int_{-\infty}^{\infty} \text{trace}(\Phi) d\omega. \end{aligned}$$

This gives (i). By taking $\|Z\|_\infty = \|Z_0\|_\infty$ in the above inequalities we obtain (ii).

Proof of Theorem 2.2.

To simplify notation we occasionally suppress the argument $i\omega$ in $Z(i\omega)$, $c_j(i\omega)$, $d_j(i\omega)$, etc. This should not cause any serious confusion.

Using Schmidt expansion we write

$$Z = \sum_{j=1}^n s_j(Z) (\cdot, c_j) d_j, \quad (A.3)$$

where $\{c_j(i\omega)\}$ and $\{d_j(i\omega)\}$ are two systems of orthonormal vectors in $\mathbb{C}^n$. For $0 \leq k \leq n$ associate

$$Z_k = \sum_{j=k}^n s_j(Z) (\cdot, c_j) d_j \quad (A.4)$$

with Z . Similarly define for Z' the matrix function Z'_k . Since $\Phi \geq 0$ and $\text{Range}(\Phi) \subseteq V_k$, the Schmidt expansion of Φ may be written as

$$\Phi = \sum_{r=1}^p s_r(\Phi) (\cdot, e_r) e_r, \quad (A.5)$$

where $p = n-k+1$ and $\{e_r\}$ is a suitable orthonormal system. Since $\text{Range} \Phi(i\omega) \subseteq \text{span} \{c_k, \dots, c_n\}$, we must have $c_j \perp e_r$ for each $j < k$ and each r . But this means that $Z\Phi = Z_k\Phi$. We also note from (A.4) that $\sigma_k(Z) = \sigma_1(Z_k)$. Similarly $Z'\Phi = Z'_k\Phi$ and $\sigma_k(Z') = \sigma_1(Z'_k)$. This allows us to apply Lemma A.3 to Z_k and Z'_k instead of Z and Z' . Observe that the conclusions of the Lemma are precisely those we want.

Proof of Theorem 2.3:

(only if): Again we suppress the argument $i\omega$. Assume that

$$\left| \int_{-\infty}^{\infty} \text{trace}(Z\Phi) d\omega \right| = \sigma_k(Z) \cdot \int_{-\infty}^{\infty} \text{trace}(\Phi) d\omega.$$

In the proof of Theorem 2.2 we noticed that $Z\Phi = Z_k\Phi$. From (A.4) we see that $s_j(Z_k) = s_{j+k-1}(Z)$. Our assumption may therefore be written

$$\left| \int_{-\infty}^{\infty} \text{trace}(Z_k\Phi) d\omega \right| = \sigma_1(Z_k) \cdot \int_{-\infty}^{\infty} \text{trace}(\Phi) d\omega. \quad (A.6)$$

Let $p = n-k+1$. From Theorem 8.5 on p. 104 and Corollary 4.1 on page 49 of ¹ we obtain

$$\begin{aligned}
|\text{trace}(Z_k \Phi)| &\leq \sum_{j=1}^p s_j(Z_k \Phi) \leq \sum_{j=1}^p s_j(Z_k) s_j(\Phi) \leq \\
&\leq \sigma_1(Z_k) \cdot \sum_{j=1}^p s_j(\Phi) = \sigma_1(Z_k) \cdot \sum_{j=1}^p s_j(\Phi) \leq \\
&\leq \sigma_1(Z_k) \cdot \text{trace}(\Phi).
\end{aligned}$$

Integrating we obtain

$$\begin{aligned}
|\int \text{trace}(Z_k \Phi) d\omega| &\leq \int |\text{trace}(Z_k \Phi)| d\omega \leq \\
\int \sum_{j=1}^p s_j(Z_k \Phi) d\omega &\leq \int \sum_{j=1}^p s_j(Z_k) s_j(\Phi) d\omega \leq \\
\int \sigma_1(Z_k) \cdot \sum_{j=1}^p s_j(\Phi) d\omega &= \sigma_1(Z_k) \cdot \int \text{trace}(\Phi) d\omega \quad (\text{A.7})
\end{aligned}$$

But because of (A.6) equality must hold throughout in (A.7). The fourth of the above inequalities then implies

$$\sum_{j=1}^p s_j(Z_k) s_j(\Phi) = \sigma_1(Z_k) \cdot \sum_{j=1}^p s_j(\Phi) \text{ a.e.}$$

Thus $s_{k+j-1}(Z) = s_j(Z_k(i\omega))$ must be equal to $\sigma_1(Z_k)$ a.e. where $s_j(\Phi(i\omega))$ differs from zero. This gives (i). Since the first inequality of (A.7) is an equality, we may find a real number α such that $\text{trace}(Z_k \Phi) = e^{i\alpha} |\text{trace}(Z_k \Phi)|$ a.e. (see [4] Theorem 1.39). Thus $\theta = \arg \text{trace}(Z\Phi) = \arg \text{trace}(Z_k \Phi) = \alpha$ a.e., which proves (ii). Using (A.7) we also see that

$$\begin{aligned}
|\text{trace}(Z\Phi(i\omega))| &= |\text{trace}(Z_k \Phi(i\omega))| = \\
\sum_{j=1}^p s_j(Z_k \Phi(i\omega)) &= \sum_{j=1}^p s_j(Z\Phi(i\omega)) \text{ a.e.}
\end{aligned}$$

According to Theorem 8.5 on p. 104 of [1] this can be true if and only if the matrix $e^{-i\theta} \cdot Z\Phi(i\omega)$ is nonnegative-definite a.e.

(if): Suppose that (i), (ii) and (iii) of the theorem hold. Without any loss of generality we may assume that $\theta = 0$. Thus $Z\Phi \geq 0$. Using polar representation of Z (see [1] p. 6-7), we may write $Z = UH$, where $H = [Z]$ and U is partially isometric, i.e. it maps $\mathcal{C}^n \ominus \ker(U)$ isometrically onto $\text{Range}(U)$. Using Schmidt expansion of $[Z]$ and Φ and recalling that Z and $[Z]$ have the same singular values, we find that $[Z]\Phi$ may be written $\sigma_k(Z)V\Phi$, where V is a partial isometry. Thus we get $Z\Phi = \sigma_k(Z)UV\Phi$. But since $Z\Phi \geq 0$ this implies that $\text{trace}(Z\Phi) = \sum s_1(Z\Phi) = \sigma_k(Z) \sum s_1(UV\Phi) = \sigma_k(Z) \sum s_1(\Phi) = \sigma_k(Z) \cdot \text{trace}(\Phi)$. Integrating we obtain the desired result.

Proof of Corollary 2.4.

Suppose that (2.12) of Theorem 2.3 holds. Since $Z(i\omega) \geq 0$, we may write

$$Z_k = \sum_{j=k}^n s_j(Z)(\cdot, c_j)c_j.$$

Using the Schmidt decomposition of Φ and (i) of Theorem 2.4 we see that for almost all ω the following equalities hold:

$$\begin{aligned}
Z\Phi - Z_k\Phi &= \Phi - \sum_{r=1}^p \sum_{j=k}^n s_r(\Phi) s_j(Z)(\cdot, e_r)(e_r, c_j)c_j = \\
&= \sum_{r=1}^p \sum_{j=k}^n s_r(\Phi) \sigma_k(Z)(\cdot, e_r)(e_r, c_j)c_j = \\
&= \sigma_k(Z) \cdot \sum_{r=1}^p s_r(\Phi)(\cdot, e_r) \sum_{j=k}^n (e_r, c_j)c_j = \\
&= \sigma_k(Z) \cdot \sum_{r=1}^p s_r(\Phi)(\cdot, e_r) \sum_{j=1}^n (e_r, c_j)c_j = \\
&= \sigma_k(Z) \cdot \sum_{r=1}^p (\cdot, e_r)e_r = \sigma_k(Z)\Phi.
\end{aligned}$$

On the other hand, if $Z\Phi = \sigma_k(Z)\Phi$ a.e., clearly the condition

$$\left| \int \text{trace}(Z\Phi) d\omega \right| = \sigma_k(Z) \cdot \int \text{trace}(\Phi) d\omega$$

holds. Since $\text{trace}(Z\Phi) = \sigma_k(Z) \cdot \text{trace}(\Phi(i\omega))$ a.e. we must have $\theta = 0$ a.e.

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