Extensive Coverings and Exact Core Bounds

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The topic of the presentation is the determination of exact bounds for the nonempty core of a cooperative game in characteristic function form. The exact lower and upper core bounds are determined in terms of the extremal excesses with respect to reduced upper and lower coverings. *Journal of Economic Literature* Classification Number: 026. © 1994 Academic Press, Inc.

1. INTRODUCTION

Let $\langle I, v \rangle$ be a cooperative game with a player set $I = \{1, 2, \dots, n\}$ and a characteristic function $v: 2^I \rightarrow \mathbb{R}$ satisfying $v(\emptyset) = 0$. A nonempty subset S of the player set I (notation, $S \subseteq I$) is called a coalition. The coalition S is said to be proper if it differs from the grand coalition I (notation, $S \subset I$). Throughout the paper, it is supposed that the cooperative game v is in the zero-reduced form, i.e., $v(\{i\}) = 0$ for all $i \in I$ and $0 \leq v(S) \leq v(I)$ for all $S \subset I$. To exclude the zero game, suppose also $v(I) > 0$. For any $T \subseteq I$, let the subgame $\langle T, v_T \rangle$ be defined by means of the restriction of the function v to 2^T ; i.e., $v_T(S) := v(S)$ for all $S \subseteq T$. The number of elements of an arbitrarily finite set B is denoted by $|B|$.

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The core is a well-known solution concept for cooperative games with side payments. The core $C(v)$ of the n -person game v is defined as the set of payoff vectors $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ satisfying $\mathbf{x} \cdot \chi_I = v(I)$ and $\mathbf{x} \cdot \chi_S \geq v(S)$ for all $S \subset I$. Here the incidence vector $\chi_S = (\chi_S(1), \chi_S(2), \dots, \chi_S(n)) \in \mathbb{R}^n$ of a coalition $S \subseteq I$ is given by $\chi_S(i) = 1$ for all $i \in S$ and $\chi_S(i) = 0$ for all $i \in I - S$.

The main purpose of the paper is to determine exact upper and lower bounds for the nonempty core by means of the notions of upper and lower coverings. The organization of the paper is as follows. The treatment of the essential notions of reduced upper and lower coverings is started in Section 2. In Section 3 we introduce the idea of excess with respect to reduced upper and lower coverings and it is proved that the upper and lower extremal excesses are exact lower and upper bounds for the nonempty core (see Theorem 3.2). We also describe a class of games for which these exact upper core bounds are equal to the marginal contributions of the players with respect to the formation of the grand coalition (see Theorem 3.3).

2. REDUCED UPPER AND LOWER COVERINGS

The notions of a covering and an (reduced) extensive covering, respectively, were introduced in Bondareva (1963, 1983). In the present paper we pay attention to an (reduced) upper and a lower covering which can be regarded as two partial cases of an (reduced) extensive covering. The notions of an (reduced) upper and a lower covering were already introduced in Bondareva (1988) and, as such, the content of that earlier paper is partly related to the content of the present paper. The relevant definitions are as follows.

DEFINITION 2.1. Let $i \in I$ and let $\mathbf{0}$ be the zero vector.

(i) A vector $\boldsymbol{\lambda} = (\{\lambda_S\}_{S \subseteq I})$ is called an *extensive covering* if

$$\sum_{S \subseteq I} \lambda_S \chi_S = \mathbf{0}. \quad (2.1)$$

(ii) A nonnegative vector $\bar{\boldsymbol{\lambda}}^i = (\{\bar{\lambda}_S^i\}_{S \subseteq I}, \gamma)$ is called an *upper i -covering* if either

$$\bar{\lambda}_{\{i\}}^i = 1, \bar{\lambda}_S^i = 0 \text{ for all } S \subset I, S \neq \{i\}, \text{ and } \gamma = 0,$$

or

$$\bar{\lambda}_{\{i\}}^i = 0 \quad \text{and} \quad \sum_{S \subset I} \bar{\lambda}_S^i \chi_S - \gamma \chi_I = \chi_{\{i\}}. \quad (2.2)$$

(iii) A nonnegative vector $\underline{\lambda}^i = (\{\underline{\lambda}_S^i\}_{S \subset I}, \delta)$ is called a *lower i -covering* if

$$\delta \chi_I - \sum_{S \subset I} \underline{\lambda}_S^i \chi_S = \chi_{\{i\}}. \quad (2.3)$$

An upper i -covering $\bar{\lambda}^i = (\{\bar{\lambda}_S^i\}_{S \subset I}, \gamma)$ is said to be *reduced* if the collection $\{\{\chi_S: \bar{\lambda}_S^i > 0\}, \chi_I\}$ of vectors is linearly independent. Similarly, a lower i -covering $\underline{\lambda}^i = (\{\underline{\lambda}_S^i\}_{S \subset I}, \delta)$ is said to be *reduced* if the collection $\{\{\chi_S: \underline{\lambda}_S^i > 0\}, \chi_I\}$ of vectors is linearly independent. Without proofs we list some straightforward, but interesting, relationships between (reduced) upper, lower, and extensive coverings.

Remark 2.2. Let $i \in I$.

(i) An upper i -covering $\bar{\lambda}^i = (\{\bar{\lambda}_S^i\}_{S \subset I}, \gamma)$ induces the extensive covering $\lambda = (\{\lambda_S\}_{S \subset I})$ where $\lambda_I := -\gamma$, $\lambda_{\{i\}} := \bar{\lambda}_{\{i\}}^i - 1$, $\lambda_S := \bar{\lambda}_S^i$ for all $S \subset I$, $S \neq \{i\}$.

(ii) A lower i -covering $\underline{\lambda}^i = (\{\underline{\lambda}_S^i\}_{S \subset I}, \delta)$ induces the extensive covering $\lambda = (\{\lambda_S\}_{S \subset I})$ where $\lambda_I := -\delta$, $\lambda_{\{i\}} := \underline{\lambda}_{\{i\}}^i + 1$, $\lambda_S := \underline{\lambda}_S^i$ for all $S \subset I$, $S \neq \{i\}$.

(iii) A (reduced) upper i -covering $\bar{\lambda}^i = (\{\bar{\lambda}_S^i\}_{S \subset I}, \gamma)$ induces the (reduced) lower i -covering $(\{\bar{\lambda}_{I-S}^i\}_{S \subset I}, \sum_{S \subset I} \bar{\lambda}_S^i - \gamma)$.

(iv) A (reduced) lower i -covering $\underline{\lambda}^i = (\{\underline{\lambda}_S^i\}_{S \subset I}, \delta)$ induces the (reduced) upper i -covering $(\{\underline{\lambda}_{I-S}^i\}_{S \subset I}, \sum_{S \subset I} \underline{\lambda}_S^i - \delta)$.

The sets of upper, reduced upper, lower, and reduced lower i -coverings, respectively, are denoted by $\bar{\Lambda}_i$, $\bar{\Lambda}_i^*$, $\underline{\Lambda}_i$, and $\underline{\Lambda}_i^*$. As it was for coverings (cf. Bondareva, 1963), the set $\bar{\Lambda}_i^*$ ($\underline{\Lambda}_i^*$, respectively) is the finite set of extreme points of $\bar{\Lambda}_i$ ($\underline{\Lambda}_i$). Due to Remark 2.2 (iii–iv), we obtain that $|\bar{\Lambda}_i^*| = |\underline{\Lambda}_i^*|$ for all $i \in I$.

A pair $(\bar{\lambda}^i, \underline{\lambda}^i)$ of a reduced upper and lower i -covering is called *complementary* if the reduced lower i -covering $\underline{\lambda}^i$ can be obtained from the reduced upper i -covering $\bar{\lambda}^i$ (and vice versa) as indicated in Remark 2.2 (iii–iv). Without going into detail, we present the table of all reduced upper and lower 1-coverings over the four-player set $I = \{1, 2, 3, 4\}$ (Table I). For notational reasons, we usually delete the braces and write χ_{12} instead of $\chi_{\{1,2\}}$, etc.

3. EXACT BOUNDS FOR THE CORE

In this section we present the two main theorems of the paper. The first main theorem expresses that the so-called upper and lower extremal excesses are exact lower and upper bounds for the nonempty core.

TABLE I
ALL NINE REDUCED UPPER AND LOWER I-COVERINGS OVER
{1, 2, 3, 4}

Reduced upper I-covering over I	Complementary reduced lower I-covering over I
$X_1 = X_1$	$X_1 = X_I - X_{234}$
$X_1 = X_{12} + X_{134} - X_I$	$X_1 = X_I - X_2 - X_{34}$
$X_1 = X_{13} + X_{124} - X_I$	$X_1 = X_I - X_3 - X_{24}$
$X_1 = X_{14} + X_{123} - X_I$	$X_1 = X_I - X_4 - X_{23}$
$X_1 = X_{12} + X_{13} + X_4 - X_I$	$X_1 = 2X_I - X_{123} - X_{24} - X_{34}$
$X_1 = X_{12} + X_3 + X_{14} - X_I$	$X_1 = 2X_I - X_{124} - X_{23} - X_{34}$
$X_1 = X_2 + X_{13} + X_{14} - X_I$	$X_1 = 2X_I - X_{134} - X_{23} - X_{24}$
$X_1 = \frac{1}{2}X_{12} + \frac{1}{2}X_{13} + \frac{1}{2}X_{14} - \frac{1}{2}X_I$	$X_1 = X_I - \frac{1}{2}X_{23} - \frac{1}{2}X_{24} - \frac{1}{2}X_{34}$
$X_1 = X_{123} + X_{124} + X_{134} - 2X_I$	$X_1 = X_I - X_2 - X_3 - X_4$

DEFINITION 3.1. Let $i \in I$.

(i) The *excess* of an upper i -covering $\bar{\lambda}^i = (\{\bar{\lambda}_S^i\}_{S \subseteq I}, \gamma)$ and a lower i -covering $\underline{\lambda}^i = (\{\underline{\lambda}_S^i\}_{S \subseteq I}, \delta)$, respectively, in the game v are given by

$$e^v(\bar{\lambda}^i) := \sum_{S \subseteq I} \bar{\lambda}_S^i V(S) - \gamma v(I),$$

$$e^v(\underline{\lambda}^i) := \delta v(I) - \sum_{S \subseteq I} \underline{\lambda}_S^i v(S).$$

(ii) The *upper* and *lower extremal excess*, respectively, for player i in the game v are given by

$$\begin{aligned} \bar{e}_i^v &:= \max_{\bar{\lambda}^i \in \bar{\Lambda}_i} e^v(\bar{\lambda}^i) = \max_{\bar{\lambda}^i \in \bar{\Lambda}_i^*} e^v(\bar{\lambda}^i), \\ \underline{e}_i^v &:= \min_{\underline{\lambda}^i \in \underline{\Lambda}_i} e^v(\underline{\lambda}^i) = \min_{\underline{\lambda}^i \in \underline{\Lambda}_i^*} e^v(\underline{\lambda}^i). \end{aligned}$$

THEOREM 3.2. If $\langle I, v \rangle$ is a game with $C(v) \neq \emptyset$, then

$$\min_{x \in C(v)} x_i = \bar{e}_i^v \quad \text{and} \quad \max_{x \in C(v)} x_i = \underline{e}_i^v \quad \text{for all } i \in I.$$

Proof of Theorem 3.2. Suppose that $\langle I, v \rangle$ is a game with $C(v) \neq \emptyset$ and let $i \in I$. Consider the following linear programming problem LP and its dual problem DLP.

$$\begin{aligned}
& \text{(LP) Minimize } x \cdot \chi_{\{i\}} && \text{subject to } x \in \mathbb{R}^n, x_k \geq 0 \text{ for} \\
& && \text{all } k \in I, x \cdot \chi_S \geq v(S) \text{ for all} \\
& && S \subset I, \text{ and } -x \cdot \chi_I \geq -v(I). \\
& \text{(DLP) Maximize } \sum_{S \subset I} \lambda_S v(S) - \gamma v(I) && \text{subject to } (\{\lambda_S\}_{S \subset I}, \gamma) \geq \mathbf{0} \\
& && \text{and } \sum_{S \subset I} \lambda_S \chi_S - \gamma \chi_I \leq \chi_{\{i\}}. \tag{3.1}
\end{aligned}$$

Note that the nonempty core $C(v)$ is included in the feasible region of the problem LP because of $v(\{k\}) = 0$ for all $k \in I$. In fact, the minimization of the i th coordinate in the problem LP is not influenced by the restriction of the feasible region to the core. Namely, each feasible vector x for the problem LP induces the core element $y = x + [v(I) - x \cdot \chi_I] \chi_{\{i\}}$ where $j \in I - \{i\}$ is arbitrary. Consequently, the duality theorem for linear programs yields the equality

$$\min_{x \in C(v)} x_i = \max \left[\sum_{S \subset I} \lambda_S v(S) - \gamma v(I) : (\{\lambda_S\}_{S \subset I}, \gamma) \geq \mathbf{0} \text{ and subject to (3.1)} \right].$$

In the remainder of the proof, we establish that the maximum in the right side is attained on the set $\bar{\Lambda}_i^*$ of reduced upper i -coverings. Let the maximum be attained in $\lambda = (\{\lambda_S\}_{S \subset I}, \gamma) \geq \mathbf{0}$ satisfying (3.1). Define the set T by

$$T := \left\{ k \in I : \chi_{\{i\}}(k) > \sum_{S \subset I} \lambda_S \chi_S(k) - \gamma \right\}.$$

Evidently, $T = \emptyset$ implies that λ is an upper i -covering. In case $T \neq \emptyset$, we define the positive real number ρ by

$$\rho := \min_{k \in T} \left[\chi_{\{i\}}(k) - \sum_{S \subset I} \lambda_S \chi_S(k) + \gamma \right].$$

We assert $T \neq I$. Assume $T = I$. Then we derive from the definition of ρ that $\rho \leq \gamma$ as well as $\sum_{S \subset I} \lambda_S \chi_S - (\gamma - \rho) \chi_I \leq \chi_{\{i\}}$. So, $(\{\lambda_S\}_{S \subset I}, \gamma - \rho) \geq \mathbf{0}$ satisfies the constraint (3.1) and, moreover, the maximum of the problem DLP is increased by the positive amount $\rho v(I)$. Since the latter observation is impossible, we conclude that $T \neq I$. Define the vector $\lambda' = (\{\lambda'_S\}_{S \subset I}, \gamma)$ by

$$\lambda'_T := \lambda_T + \rho \text{ and } \lambda'_S := \lambda_S \text{ for all } S \subset I, S \neq T.$$

The vector $\lambda' \geq \mathbf{0}$ satisfies the constraint (3.1) because

$$\sum_{S \subset I} \lambda'_S \chi_S - \gamma \chi_I = \sum_{S \subset I} \lambda_S \chi_S - \gamma \chi_I + \rho \chi_T \leq \chi_{\{i\}},$$

where the last inequality is an equality for all $k \in I - T$ and at least one player $k \in T$. Further, the maximum of the problem DLP is also attained in the feasible vector λ' because

$$\begin{aligned} \sum_{S \subset I} \lambda'_S v(S) - \gamma v(I) &= \sum_{S \subset I} \lambda_S v(S) - \gamma v(I) + \rho v(T) \\ &\geq \sum_{S \subset I} \lambda_S v(S) - \gamma v(I). \end{aligned}$$

By continuing the above procedure, we conclude in the first step that the maximum of the problem DLP is attained on the set $\bar{\Lambda}_i$ of upper i -coverings and, in the second step, the maximum is attained on the set $\bar{\Lambda}_i^*$ of reduced upper i -coverings. This completes the proof of the equality $\min_{x \in C(v)} x_i = \bar{e}_i^v$. The proof of the equality $\max_{x \in C(v)} x_i = \underline{e}_i^v$ is similar and is left to the reader. ■

Consider the lower i -covering $\underline{\lambda}^i = (\{\underline{\lambda}_S^i\}_{S \subset I}, \delta)$ which is associated with the equality $\chi_I - \chi_{I-\{i\}} = \chi_{\{i\}}$; i.e., $\delta = 1$, $\underline{\lambda}_{I-\{i\}}^i = 1$, and $\underline{\lambda}_S^i = 0$ for all $S \subset I$, $S \neq I - \{i\}$. The excess of this particular lower i -covering is given by $e^v(\underline{\lambda}^i) = v(I) - v(I - \{i\})$ and, consequently, the inequality $\underline{e}_i^v \leq v(I) - v(I - \{i\})$ always holds. It is evident that the marginal contribution $v(I) - v(I - \{i\})$ of any player i in the game v can be interpreted as an upper bound for the core because of $x_i = x \cdot \chi_I - x \cdot \chi_{I-\{i\}} \leq v(I) - v(I - \{i\})$ for all $x \in C(v)$. Generally speaking, the marginal contribution $v(I) - v(I - \{i\})$ is not an exact upper bound for the core. The second main theorem deals with a sufficient condition on the game v for the marginal contributions $v(I) - v(I - \{i\})$, $i \in I$, to be exact upper bounds for the nonempty core.

THEOREM 3.3. *Let $i \in I$ and let $\langle I, v \rangle$ be a balanced game such that*

- (i) *the subgame $\langle I - \{i\}, v_{I-\{i\}} \rangle$ is also balanced, and*
- (ii) *$v(S \cup \{i\}) - v(S) \leq v(I) - v(I - \{i\})$ for all $S \subset I - \{i\}$.*

(3.2)

Then $\underline{e}_i^v = v(I) - v(I - \{i\})$.

Proof of Theorem 3.3. (a) Consider any lower i -covering $\underline{\lambda}^i = (\{\underline{\lambda}_S^i\}_{S \subset I}, \delta)$ satisfying $\underline{\lambda}_S^i = 0$ for all $S \subset I$ with $i \in S$. From (2.3) we derive that $\delta = 1$ and therefore, (2.3) reduces to $\sum_{S \subset I - \{i\}} \underline{\lambda}_S^i \chi_S = \chi_I - \chi_{\{i\}} = \chi_{I-\{i\}}$. In other words, the collection $\{S: S \subset I - \{i\}, \underline{\lambda}_S^i > 0\}$ of coalitions is balanced over $I - \{i\}$ (cf. Shapley, 1967). Together with the balancedness of the subgame $v_{I-\{i\}}$, this yields $\sum_{S \subset I - \{i\}} \underline{\lambda}_S^i v(S) \leq v(I - \{i\})$. Now we obtain that

$$\begin{aligned}
e^v(\underline{\lambda}^i) &= \delta v(I) - \sum_{S \subset I} \underline{\lambda}_S^i v(S) = v(I) - \sum_{S \subset I - \{i\}} \underline{\lambda}_S^i v(S) \\
&\geq v(I) - v(I - \{i\}).
\end{aligned}$$

So far, we conclude that

$$e^v(\underline{\lambda}^i) \geq v(I) - v(I - \{i\})$$

for all $\underline{\lambda}^i \in \underline{\Lambda}_i$ satisfying $\underline{\lambda}_S^i = 0$ for all $S \subset I$ with $i \in S$.

(b) As already noted, the inequality $e_j^v \leq v(I) - v(I - \{i\})$ always holds. Assume $e_j^v < v(I) - v(I - \{i\})$. Let $\lambda = (\{\lambda_S\}_{S \subset I}, \delta)$ be a reduced lower i -covering that attains the lower extremal excess, i.e., $e^v(\lambda) = e_j^v$. Because $e^v(\lambda) < v(I) - v(I - \{i\})$ with $\lambda \in \underline{\Lambda}_i$, it follows from the conclusion of part (a) that there exists a coalition $T \subset I$ with $i \in T$ and $\lambda_T > 0$.

Note that $T \neq \{i\}$ because the lower i -covering λ is reduced. Put $\mu := (1 + \lambda_T)^{-1}$ and define the vector $\lambda' = (\{\lambda'_S\}_{S \subset I}, \delta')$ by

$$\lambda'_T := 0, \quad \lambda'_{T-\{i\}} := \mu(\lambda_{T-\{i\}} + \lambda_T), \quad \delta' := \mu\delta,$$

and

$$\lambda'_S := \mu\lambda_S \quad \text{for all } S \subset I, S \neq T, S \neq T - \{i\}.$$

The vector $\lambda' \geq 0$ is also a lower i -covering because

$$\begin{aligned}
\delta' \chi_I - \sum_{S \subset I} \lambda'_S \chi_S &= \mu \delta \chi_I - \sum_{S \subset I} \mu \lambda_S \chi_S + \mu \lambda_T (\chi_T - \chi_{T-\{i\}}) \\
&= \mu \chi_{\{i\}} + \mu \lambda_T \chi_{\{i\}} = \chi_{\{i\}}.
\end{aligned}$$

As a matter of fact, the lower i -covering λ' is also reduced since the linear independence of the collection $\mathcal{C} = \{\{\chi_S: \lambda_S > 0\}, \chi_I\}$ yields the linear independence of the collection $(\mathcal{C} - \{\chi_T\}) \cup \{\chi_{T-\{i\}}\}$ corresponding with λ' . Further, the excess of the lower i -covering λ' is given by

$$\begin{aligned}
e^v(\lambda') &= \delta' v(I) - \sum_{S \subset I} \lambda'_S v(S) = \mu \delta v(I) - \sum_{S \subset I} \mu \lambda_S v(S) \\
&\quad + \mu \lambda_T [v(T) - v(T - \{i\})] = \mu e^v(\lambda) + \mu \lambda_T [v(T) - v(T - \{i\})].
\end{aligned} \tag{3.3}$$

Recall that $e^v(\lambda) < v(I) - v(I - \{i\})$, while $v(T) - v(T - \{i\}) \leq v(I) - v(I - \{i\})$ because of (3.2). Now it follows from (3.3) that

$$e^v(\lambda') < (\mu + \mu \lambda_T) [v(I) - v(I - \{i\})] = v(I) - v(I - \{i\}).$$

In summary, the reduced lower i -covering λ satisfying $e^v(\lambda) < v(I) - v(I - \{i\})$ induces another reduced lower i -covering λ' satisfying $e^v(\lambda') < v(I) - v(I - \{i\})$ and $\{S \subset I: i \in S, \lambda'_S > 0\} \subsetneq \{S \subset I: i \in S, \lambda_S > 0\}$. Here the above procedure continues until we have a reduced lower i -covering $\lambda^i = (\{\lambda^i_S\}_{S \subset I}, \delta)$ satisfying $e^v(\lambda^i) < v(I) - v(I - \{i\})$ and $\lambda^i_S = 0$ for all $S \subset I$ with $i \in S$. The obtained result is, however, in contradiction to the conclusion of Part (a). This completes the proof of the theorem. ■

Note that Theorem 3.3 can be applied to the subclass of convex games (cf. Shapley, 1971). For any convex game $\langle I, v \rangle$ one can even show that $\bar{e}^v_i = v(\{i\})$ for all $i \in I$. Finally we remark that Condition (3.2) is stronger than the so-called semiconvexity condition for the game v (cf. Driessen and Tijs, 1985; Driessen, 1988).

REFERENCES

- BONDAREVA, O. N. (1963). "Some Applications of Linear Programming Methods to the Theory of Cooperative Games," *Problemy Kibernetiki* **10**, 119–139 [in Russian]; (1968) English translation in *Selected Russian Papers on Game Theory 1959–1965*. Princeton: Princeton Univ. Press.
- BONDAREVA, O. N. (1983). "Generalized Coverings and Some Necessary Conditions for the Existence of Solutions of a Cooperative Game," *Vestnik Leningrad Univ. Math.* **19**, 5–19 [in Russian]; (1984) English translation in the same journal Vol. 16, pp. 201–209.
- BONDAREVA, O. N. (1988). "Extensive Coverings and Bounds for Core Imputations and the Shapley Value," *Optimization* **43**(60), 94–101 [in Russian].
- DRIESSEN, T. S. H. (1988). *Cooperative Games, Solutions, and Applications*. Dordrecht: Kluwer.
- DRIESSEN, T. S. H., AND TIJS, S. H. (1985). "The τ -Value, the Core, and Semiconvex Games," *Int. J. Game Theory* **14**, 229–247.
- SHAPLEY, L. S. (1967). "On Balanced Sets and Cores," *Naval Res. Logistics Quart.* **14**, 453–460.
- SHAPLEY, L. S. (1971). "Cores of Convex Games," *Int. J. Game Theory* **1**, 11–26.