Mode competition in a system of two parametrically driven pendulums; the Hamiltonian case*

E.J. Banning, J.P. van der Weele

Center for Theoretical Physics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

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Abstract

We study the mode competition in a Hamiltonian system of two parametrically driven pendulums, linearly coupled by a torsion spring. First we make a classification of all the periodic motions in four main types: the trivial motion, two ‘normal modes’, and a mixed motion. Next we determine the stability regions of these motions, i.e., we calculate for which choices of the driving parameters (angular frequency Ω and amplitude A) the respective types of motion are stable. To this end we take the (relatively simple) uncoupled case as our starting point and treat the coupling K as a control parameter. Thus we are able to predict the behaviour of the pendulums for small coupling, and find that increasing the coupling does not qualitatively change the situation anymore. One interesting result is that we find stable (and also Hopf bifurcated) mixed motions *outside* the stability regions of the other motions. Another remarkable feature is that there are regions in the (A, Ω) -plane where all four motion types are stable, as well as regions where all four are *unstable*. As a third result we mention the fact that the coupling (i.e. the torsion spring) tends to destabilize the normal mode in which the pendulums swing in parallel fashion. The effects of the torsion spring on the stability region of this mode is, surprisingly enough, not unlike the effect of dissipation.

1. Introduction

1.1. The system

The world is full of coupled oscillators. They turn up in every area of science, and it is therefore no wonder that they are the subject of much research. One of the problems of interest is *mode competition*, i.e. a competition between two or more ‘normal’ modes of the system, resulting in a mixed motion. If the system is nonlinear such mixed

*This paper is dedicated to the memory of Drs. Jan Banning, father of the first author.

motions often lead to chaotic behaviour. For example, the chaotic signal of the pulsating star R Scuti is probably the result of two competing vibrational modes, as has recently been pointed out by Buchler et al. [1]. Mode competition has also been observed in more down to earth systems, for instance in the fluid experiments of Ciliberto and Gollub [2,3] and Lichter et al. [4–6]. These experiments have provided us with some of the finest illustrations of mode competition, and yet a systematic theoretical study in these cases is hampered by the fact that the fluid has an infinite number of degrees of freedom, whereas eventually only two turn out to be essential. Thus the analysis requires one or more, usually approximate, reductions, at the cost of rigor and clarity. This problem does not exist if one takes a simple mechanical system of only two degrees of freedom, for which the equations of motion can be written down immediately. Skeldon et al. [7,8] investigated a parametrically driven double pendulum and indeed found mode competition, including the transition to chaotic behaviour. Skeldon's pendulum, despite its elegance, is still not the ideal model to study mode competition. As any experimental pendulum it is damped, while a theorist would prefer to start with a Hamiltonian system, and furthermore the double pendulum is such that the coupling is nonlinear and cannot be turned off. To remedy these shortcomings we have constructed our own system, consisting of two *linearly* coupled parametrically driven pendulums, which can also be *decoupled*. In this paper we study the Hamiltonian case (without damping) and in a forthcoming paper we shall treat the dissipative (damped) case.

Our system, depicted in Fig. 1, consists of two equal pendulums (mass m , length l) suspended from a bar and coupled by a linear torsion spring. The bar is being moved up and down harmonically, with amplitude A and angular frequency Ω , and each pendulum has one degree of freedom: its angle ϑ with the vertical. Disregarding any dissipative effects (for the moment) the equations of motion of this system are readily found to be

$$\ddot{\vartheta}_1 + f(t) \sin \vartheta_1 + K(\vartheta_1 - \vartheta_2) = 0, \quad (1.1a)$$

$$\ddot{\vartheta}_2 + f(t) \sin \vartheta_2 - K(\vartheta_1 - \vartheta_2) = 0. \quad (1.1b)$$

Here the coupling parameter K is given by

$$K \equiv \frac{C}{ml^2}, \quad (1.2)$$

where C is the torsion-spring constant and the function $f(t)$ contains the driving force:

$$f(t) = \frac{1}{l}(g + A\Omega^2 \cos \Omega t). \quad (1.3)$$

In the numerical calculations we will always take the length of the pendulums $l = 1$ m.

The type of forcing in this system is called *parametric*, because it appears in the equations of motion as a modulation of the gravitational acceleration g , which is one

Fig. 1. The system, consisting of two parametrically driven pendulums, coupled by a linear torsion spring.

of the main parameters of the system. Thus, our system consists of two coupled, parametrically driven pendulums. In order to keep things as simple as possible the coupling is kept linear, so the nonlinearity of the system is entirely due to the pendulums themselves.

For small amplitudes it is convenient to use so-called normal coordinates and normal modes. For large amplitudes one (strictly) cannot speak of ‘normal modes’ because this would imply that any solution of the equations of motion could be written as a linear superposition of these modes, and this is not true. Still, in the existing literature, it is customary to speak of modes (and mode competition). We will therefore also use normal coordinates, defined by:

$$\phi_1 \equiv \vartheta_1 + \vartheta_2, \quad (1.4a)$$

$$\phi_2 \equiv \vartheta_1 - \vartheta_2. \quad (1.4b)$$

The equations of motion then become

$$\ddot{\phi}_1 + 2f(t) \sin \frac{1}{2} \phi_1 \cos \frac{1}{2} \phi_2 = 0, \quad (1.5a)$$

$$\ddot{\phi}_2 + 2f(t) \sin \frac{1}{2} \phi_2 \cos \frac{1}{2} \phi_1 + 2K\phi_2 = 0, \quad (1.5b)$$

with $f(t)$ and K as before. Note that these equations are coupled, even for $K = 0$. That is, the normal modes are coupled even if the pendulums are *not*. We will see later that, as far as mode competition is concerned, this makes a description in terms of ϑ_1, ϑ_2 more appropriate than in terms of ϕ_1, ϕ_2 . In this sense it would be more natural to speak of *pendulum* competition rather than *mode* competition.

The equations (1.5a)–(1.5b) can also be written in the form of four coupled first-order differential equations:

$$\frac{d}{dt} \begin{pmatrix} \phi_1 \\ \dot{\phi}_1 \\ \phi_2 \\ \dot{\phi}_2 \end{pmatrix} = \begin{pmatrix} \dot{\phi}_1 \\ -2f(t) \sin \frac{1}{2} \phi_1 \cos \frac{1}{2} \phi_2 \\ \dot{\phi}_2 \\ -2f(t) \sin \frac{1}{2} \phi_2 \cos \frac{1}{2} \phi_1 - 2K\phi_2 \end{pmatrix}. \quad (1.6)$$

This is the form we will often use when dealing with the (stroboscopic) phase space of the system in normal coordinates $\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2$. The stroboscope will be synchronized to the forcing in such a way that the velocities and positions of the pendulums are measured at the moment the bar of suspension passes through its lowest position. The stroboscopic map is computed using a fourth order Runge–Kutta algorithm.

We will study the dynamics of our (Hamiltonian) system by means of an investigation of the periodic orbits, which are in tune with the driving force. In stroboscopic phase space these orbits appear as (series of) points. Each orbit has four eigenvalues, which reflect the linear stability properties of the orbit, and these eigenvalues play an important role in our analysis. In particular we focus on elliptic orbits (also called *stable* orbits, with all four eigenvalues on the unit circle in the complex plane) because these will change into attractors when dissipation is added, and are therefore the most relevant from a practical point of view.

We will ignore the homogeneous solutions to the equations of motion, which have their own periodicity, and present themselves as quasiperiodic motions around the elliptic points. Even though these motions do not die away in the Hamiltonian case, they become transient when dissipation is added. So, also in this sense, our discussion anticipates the dissipative case.

The present paper is organized as follows. In Section 2 we describe the case of zero coupling ($K = 0$). This case consists simply of two independent pendulums, and serves as a useful preparation for the coupled system ($K > 0$), which is considered in Section 3. This third section contains our main results. In Section 4 we summarize and draw a few conclusions.

1.2. Four types of motion

As stated above, we consider only periodic motions of the pendulums, which are in tune with the driving force. Throughout the paper we will use the classification shown in Fig. 2, putting these motions into four categories.

First of all we have the motion in which both pendulums move only in the vertical direction (following the bar), with $\vartheta_1(t) = \vartheta_2(t) = 0$ for all time. This is called the *downward equilibrium motion*, or 0-motion. In (stroboscopic) phase space the 0-motion is just the origin of the coordinate system, i.e., $(\vartheta_1, \dot{\vartheta}_1, \vartheta_2, \dot{\vartheta}_2) = (0, 0, 0, 0)$, or, if we work in normal coordinates $(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) = (0, 0, 0, 0)$.

Fig. 2. The four types of motion.

The second motion is the one in which both pendulums move in phase with each other, i.e., $\vartheta_1(t) = \vartheta_2(t)$. Because this is essentially a one-pendulum motion, we denote it as the 1-motion. In phase space the orbit of such a motion lies in the plane for which $\phi_2 = 0$, $\dot{\phi}_2 = 0$. We call this the 1-plane. From Eq. (1.6), or any other form of the equations of motion, it follows that the 1-plane is invariant: once an orbit lies in the 1-plane it will stay in it forever.

In the third type of motion both pendulums move exactly 180 degrees out of phase with each other, i.e., $\vartheta_1(t) = -\vartheta_2(t)$. We call this the 2-motion. In phase space the orbit of this motion lies in the 2-plane, defined by $\phi_1 = 0$, $\dot{\phi}_1 = 0$. Again, once an orbit lies in the 2-plane it will stay in it.

Finally we have the M-motion, or *mixed motion*. These are the periodic motions that cannot be characterized as either a 0-, 1- or 2-motion. In general, the orbit in phase space of an M-motion is not restricted to a two-dimensional plane but uses all four dimensions.

We are interested in the regions of stability and the bifurcation curves of the motions in the (A, Ω) -plane, for various values of K . A period-doubling bifurcation curve (where two eigenvalues are at -1) will be labeled with a bold lowercase character (**a**, **b**, **c**, ...) whereas an equal-period bifurcation curve (two eigenvalues at $+1$) will be denoted by a bold uppercase character (**A**, **B**, **C**, ...). It should be pointed out that this convention is not completely unambiguous; when different motions are considered it is often possible to denote the same line in the (A, Ω) -plane by both an upper case and a lowercase character.

2. Zero coupling

2.1. Stability of the 0-motion (for $K = 0$)

When we confine ourselves to small amplitudes of both pendulums, the equations of motion (1.5) reduce to, keeping only terms linear in ϕ_1 and ϕ_2 :

$$\ddot{\phi}_1 + f(t)\phi_1 = 0, \quad (2.1a)$$

$$\ddot{\phi}_2 + f(t)\phi_2 = 0, \quad (2.1b)$$

which are just two identical Mathieu equations, describing two identical parametrically driven one-pendulum systems. The literature on the Mathieu equation is extensive and the regions in the (A, Ω) -plane for which the downward equilibrium of the pendulum (the 0-motion) is unstable are well known: see Fig. 3a. The regions of instability appear as a series of white tongues, originating from the horizontal axis at

$$\Omega = \frac{2}{n} \sqrt{\frac{g}{l}}, \quad \text{with } n = 1, 2, 3, \dots \quad (2.2)$$

For values of (A, Ω) inside these tongues the pendulum is said to be in parametric resonance.

We will concentrate our attention on the main tongue ($n = 1$), of which the right and left borders are given by [9, 10]:

$$\Omega = 2 \sqrt{\frac{g}{l}} \left[1 \pm \left(\frac{A}{l} \right) + \frac{3}{2} \left(\frac{A}{l} \right)^2 \pm \frac{5}{2} \left(\frac{A}{l} \right)^3 + \dots \right]. \quad (2.3)$$

In Fig. 3a we have indicated these borders by **a** and **b** respectively. Of all other tongues only those for $n = 2, 3$ and 4 are shown in the figure. Their width (for $A/l < 1$) decreases rapidly with n and therefore, for increasing n , they become less and less important.

The downward equilibrium (the 0-motion) is the origin of the stroboscopic phase space. It is a fixed point with its four eigenvalues organized in a degenerate pair. In

Fig. 3. (a) Stability diagram of the 0-motion for $K = 0$. (b) Eigenvalues and stroboscopic phase portraits at the points marked in Fig. 3a. The four dimensionality of the phase space has been represented schematically by two planes, intersecting each other in a single point.

Fig. 3b we show the eigenvalues together with the corresponding stroboscopic phase space portraits at the three points marked in Fig. 3a. In situation 1, to the right of the tongue of instability, the origin is an elliptic fixed point (eigenvalues on the unit-circle in the complex plane); that is, the downward equilibrium is stable. Upon entering the tongue, at line **a**, the origin undergoes a period-doubling bifurcation (eigenvalues leave the circle at -1). We then get situation 2, where the origin has turned into a saddle point; that is, the downward equilibrium is now unstable (eigenvalues on negative real axis). Next, upon leaving the tongue at line **b** we have a *reverse* period-doubling bifurcation, which means that the saddle point turns into an elliptic

point once more; as a by-product we also get 2 saddle points, constituting an unstable cycle of period 2 of the stroboscopic map. This brings us to situation 3. Going further to the left in the (A, Ω) -plane, the origin will lose and regain stability in an analogous manner at all the higher order resonance tongues; for $n = 2$ this happens by means of equal-period bifurcations (eigenvalues at $+1$), for $n = 3$ by means of period-doubling bifurcations again (eigenvalues at -1), for $n = 4$ by means of equal-period bifurcations, and so on.

2.2. Stability of the 1- and 2-motion (for $K = 0$)

When we enter the instability-tongue of the 0-motion from the right, the origin of the stroboscopic phase space undergoes a period-doubling bifurcation, giving birth to stable motions of period $2T$, with $T = 2\pi/\Omega$ denoting the period of the forcing. Fig. 4 schematically depicts the motion of one of the pendulums during one forcing period. We see that the pendulums have zero angular velocity (just reaching their greatest angle) every time the bar of suspension passes its lowest point.

When both pendulums exhibit this $2T$ periodic motion (in tune with the driving force) we have either a 1- or a 2-motion, depending on whether the pendulums are in phase or precisely out of phase with each other¹. In stroboscopic phase space these $2T$ motions appear as two pairs of elliptic fixed points of period 2. One pair lies in the 1-plane and constitutes the 1-motion. The other pair lies in the 2-plane and constitutes the 2-motion.

Both motions are governed by the same equation of motion. For the 1-motion we have $\phi_2 = 0$, and Eq. (1.5a) reduces to:

$$\ddot{\phi}_1 + 2f(t) \sin \frac{1}{2} \phi_1 = 0, \quad (2.4a)$$

and for the 2-motion we have $\phi_1 = 0$, and Eq. (1.5b) becomes (with $K = 0 \text{ s}^{-2}$):

$$\ddot{\phi}_2 + 2f(t) \sin \frac{1}{2} \phi_2 = 0. \quad (2.4b)$$

Indeed, in the uncoupled case the 1-motion and the 2-motion are entirely equivalent, and so for the moment we may drop the suffixes and continue with the equation

$$\ddot{\phi} + 2f(t) \sin \frac{1}{2} \phi = 0. \quad (2.5)$$

The stroboscopic phase space of this equation is two dimensional and the stability of the orbits is governed by two eigenvalues. This is logical, since for zero coupling the four-dimensional system consists in fact of two separate, identical two-dimensional systems. The stability regions of the 1- and 2-motions are shown in Fig. 7a.

When the 1- and 2-motions are born (at the line A) they are mirror symmetrical with respect to both the lines $\phi = 0$ and $\dot{\phi} = 0$. This is depicted in Fig. 5.

¹ Quasiperiodic motions of the pendulums (not precisely in tune with the driving force) are also possible, but these are not considered in the present paper.

Fig. 4. Motion of a pendulum in parametric resonance. The periodic orbit itself (mirror-symmetrical with respect to both $\vartheta = 0$ and $\dot{\vartheta} = 0$) is represented by the dashed curve, and the motion depicted (solid curve) is in fact a quasiperiodic orbit close to it.

Fig. 5. Schematic representation of the symmetrical 1- and 2-motions. The former (a) is restricted entirely to the $(\phi_1, \dot{\phi}_1)$ -plane, and the latter (b) to the $(\phi_2, \dot{\phi}_2)$ -plane.

At line A the eigenvalues of the orbit are equal to $+1$. When we decrease Ω , keeping A/l constant, the amplitude of the symmetrical motion (of both pendulums) increases, and the eigenvalues travel over the unit circle from $+1$ to reach -1 at line c. One might now expect that the eigenvalues would leave the unit circle to go on along the negative real axis, or in other words, a period doubling bifurcation. However, due to the symmetry in the orbit this is not possible. This can be understood as follows.

Consider the symmetrical orbit of period 2, which has the form $\{\Phi, -\Phi, \Phi, -\Phi, \dots\}$, with $\Phi = (\phi, 0)$. The eigenvalues of this orbit are those of the second iterate of the stroboscopic map P . Let us take one of the two points of this orbit, say Φ ; the second iterate can then be written as

$$P^{(2)}(\Phi) \equiv P(P(\Phi)) = \Phi, \quad (2.6)$$

with linearization

$$L^{(2)}|_{\phi} = L|_{P(\phi)} L|_{\phi}. \quad (2.7)$$

We have

$$P(\Phi) = -\Phi, \quad (2.8)$$

and inserting this into Eq. (2.7), we get

$$L^{(2)}|_{\phi} = L|_{-\phi} L|_{\phi}. \quad (2.9)$$

In the equation of motion (2.5) the coordinate ϕ appears as the argument of a sine-function; hence, in the linearization (derivative) of the stroboscopic map ϕ appears as the argument of a cosine-function. Because the cosine is an even function it follows that

$$L|_{-\phi} = L|_{\phi}, \quad (2.10)$$

and Eq. (2.9) now becomes

$$L^{(2)}|_{\phi} = (L|_{\phi})^2. \quad (2.11)$$

Or equivalently, denoting the eigenvalues of $L^{(2)}|_{\phi}$ as $\lambda_{1,2}$ and those of $L|_{\phi}$ as $\mu_{1,2}$,

$$\lambda_{1,2} = \mu_{1,2}^2. \quad (2.12)$$

Now, for $\lambda_{1,2}$ to lie on the negative real axis (excluding -1), $\mu_{1,2}$ must lie on the imaginary axis (excluding $+i$ and $-i$), but this is *not* possible in a two-dimensional Hamiltonian system. The eigenvalues $\lambda_{1,2}$ can therefore not leave the unit circle at -1 .

So there is no period-doubling bifurcation at line **c** (which, for this reason, is only dashed in Fig. 7a and not solid) and the eigenvalues of the symmetrical motion have to go on along the unit circle. That is, they go right through -1 and travel on to reach $+1$ again at line **D**. Here the eigenvalues finally leave the unit circle and move onto the positive real axis. At this point the symmetrical motion becomes unstable by means of an equal-period bifurcation, giving birth to two non-symmetrical motions of period $2T$. Fig. 6 shows these non-symmetrical motions.

The eigenvalues of the non-symmetrical orbits originate at $+1$ and travel over the unit circle to reach -1 at line **e**. Now, unlike at line **c**, there is no symmetry anymore to prevent the eigenvalues from leaving the unit circle and they simply enter the negative real axis. This means that the non-symmetrical orbits become unstable by means of a period doubling bifurcation. Indeed, from this point on, the period-doubling sequence resumes its normal course. Thus, at line **e** two non-symmetrical orbits of period 4 are born, bifurcating at line **f** into orbits of period 8, and so on, until (soon afterwards) period 2^{∞} is reached, the so-called threshold of chaos. Fig. 7b shows the positions of the relevant eigenvalues at the six points marked in Fig. 7a. To complete the picture, Fig. 7c shows the stroboscopic phase space portraits for the corresponding orbits.

Fig. 6(a, b). Schematic representation of the non-symmetrical 1- and 2-motions. The difference in left and right amplitude is exaggerated in the figure, for sake of clarity. In reality, typical amplitudes (at $A/l = 0.16$) are 5.13 rad and 4.78 rad, respectively. The amplitude of the (unstable) symmetrical $2T$ motion lies between these two values, at 4.94 rad.

This completes our treatment of the 1- and 2-motions in the uncoupled case. We will come back to them in Section 3, but then for non-zero coupling. The observant reader may have noticed that our treatment of the 1- and the 2-motions is not really complete, in the sense that there exist also periodic 1- and 2-motions of another type than the ones we have been dealing with. In particular, there is a whole series of periodic motions in which the pendulums *rotate* around the bar of suspension, without ever coming to a halt. The most conspicuous of those has period T , but also many other periodicities (such as $\frac{2}{3}T$, $\frac{1}{2}T$ or $\frac{2}{5}T$) are possible. We shall not pursue these rotating motions. For one thing, the 2-motion of this type (i.e., with the two pendulums rotating in opposite directions) will not survive if the coupling is turned on. Secondly, these rotating motions are not very relevant for an understanding of nonlinear oscillators in general, since they are rather specific to pendulums.

2.3. Stability of the M -motions (for $K = 0$)

Although it may seem somewhat artificial to discuss the notion of an M - or mixed motion in a system of two uncoupled pendulums, it provides an instructive introduction for the M -motions in the coupled case.

- stable symmetrical 2T (for $K = 0$)
- stable non-symmetrical 2T (for $K = 0$)
- stable non-symmetrical 4T (for $K = 0$)

Fig. 7. (a) Stability diagram of the various (symmetrical and non-symmetrical) 1- and 2-motions, for $K = 0$. The line A is identical to line a of Fig. 3a, but since it is a + 1 line for the 1- and 2-motion, it is now denoted by an uppercase character. (b) Eigenvalues of the respective motions at the points marked 1 to 6 in (a). (c) Stroboscopic phase portraits at the points marked 1 to 5 in (a).

Fig. 8. Schematic representation of the four types of M-motion, denoted by A,B,C and D.

We are looking for stable periodic motions that do not classify as 0-, 1- or 2-motions. That is, we want periodic motions in which the two pendulums perform different types of motion. For instance, an M-motion of period T can be constructed with one pendulum in its downward equilibrium and the other performing a rotation with the period of the forcing. However, it is evident that this motion will not survive a non-zero coupling, because the torsion spring would continue to wind and eventually break. For this reason we will not discuss this particular M-motion. More generally, we dismiss any M-motion with a rotating pendulum.

An M-motion of period $2T$ (discarding rotations) can be constructed in four ways. We call them types A to D, see Fig. 8. These motions may be expected to survive (more or less distorted of course) if we turn on the coupling. Types A and B have one pendulum in the downward equilibrium position, while the other pendulum performs a symmetrical respectively a non-symmetrical motion of period $2T$. In types C and D both pendulums perform a non-symmetrical motion of period $2T$. In type C they move in phase with each other and in type D they are 180 degrees out of phase with each other.

The regions of stability of the four M-motions are shown in Fig. 9. For types A and B (Fig. 9a, b) these are just combinations of the stability region of the downward equilibrium (cf. Fig. 3a) and that of the respective motions of period $2T$ (cf. Fig. 7a).

Fig. 9(a, b, c). Stability diagram of the various M-motions, for $K = 0$. Some of the lines depicted in these and the following pictures do not belong to the described motions but are nevertheless useful as a guide to the eye; they are not solid but dotted, and are labeled with open-faced characters. (d) Eigenvalues of the M-motion type A, at the points marked in (a) (still at $K = 0$). Type A will turn out to be the most important M-motion for $K > 0$.

The region of stability of types C and D (Fig. 9c) is of course identical to that of the non-symmetrical motions of period $2T$ (cf. Fig. 7a).

Fig. 9 continued

From these pictures it is immediately clear that type A will be the most important of the four M-motions. Its eigenvalues, at the points marked in Fig. 9a, are depicted in Fig. 9d. Note that the eigenvalues of the downward-equilibrium (gray) are *squared* in this figure, because we are dealing with the twice iterated stroboscopic map; the line where the eigenvalues of the downward equilibrium are at $\pm i$ is therefore now a -1 line (hence the i^2 symbol in Fig. 9a, b).

Going to periods higher than 2 one may construct many more M-motions. For instance, one can make combinations of the 2^k -orbits (with $k > 1$) bifurcated from the one-pendulum motion of period $2T$, but these orbits are only stable in a very small region of the (A, Ω) -plane, which becomes even smaller when the coupling is turned on. The same holds for the M-motions constructed from them, and they may safely be put aside.

3. Non-zero coupling

3.1. A few words about the notation

We are now ready to go on to the real problem, i.e., the full system of two coupled parametrically driven pendulums. We use the same classification for the types of motion as in the previous section (0-, 1-, 2-, or M-motion), and determine the region of stability in the (A, Ω) -plane for each type. For the 0-, 1-, and 2-motion this amounts to a straightforward continuation of the uncoupled case; for the M-motion one has to be more careful, because the coupling will change the characteristics of this motion. In the stability diagrams the bifurcation lines will be labeled with the same letters as in the previous section, to indicate from which $K = 0$ line they have originated. Lines where two of the four eigenvalues are at $+1$ (-1) will be labeled with bold uppercase (lowercase) characters; now and again we also need an index to distinguish between several lines originating from the same $K = 0$ line. For small K the lines will lie very

close to the $K = 0$ lines, but for increasing K the lines can deviate quite substantially. The value $K = 1 \text{ s}^{-2}$, which is used in most of the pictures, can already be regarded as quite large.

As before, the stability of an orbit is determined by its four eigenvalues. For the 0-motion two of them can be associated with stability properties within the 1-plane, and the other two with the 2-plane. For the 1- and the 2-motion two of the eigenvalues can be associated with stability properties within their respective planes, while the other two then govern the stability in directions perpendicular to these planes. The eigenvalues belonging to the 1-direction will be depicted as solid black circles, and the eigenvalues belonging to the 2-direction by open circles. For the M-motions such a classification is not possible, and we will use solid gray and solid black circles for the eigenvalue pairs.

3.2. Stability of the 0-motion

The region of stability of the downward equilibrium (i.e., the origin in stroboscopic phase space) can be found from the equations of motion for small amplitude:

$$\ddot{\phi}_1 + f(t) \phi_1 = 0, \quad (3.1a)$$

$$\ddot{\phi}_2 + [f(t) + 2K] \phi_2 = 0. \quad (3.1b)$$

Just as in the uncoupled case these are two separate Mathieu equations. They are however no longer identical: in the second equation $f(t)$ has been replaced by $f(t) + 2K$, or in other words, the factor g/l has been replaced by $g/l + 2K$. As a result of this the main tongue of instability (cf. Fig. 3a) unfolds into two tongues, originating from the Ω -axis at:

$$\Omega = 2 \sqrt{\frac{g}{l}}, \quad (3.2a)$$

and

$$\Omega = 2 \sqrt{\frac{g}{l} + 2K}. \quad (3.2b)$$

The tongues are shown in Fig. 10a, for $K = 1 \text{ s}^{-2}$. In the white region the origin of the stroboscopic phase space is unstable. In the left part (cf. 3.2a) it is unstable with respect to the 1-direction, and in the right part (cf. 3.2b) with respect to the 2-direction. Where the two parts overlap, the origin is unstable in all directions.

The higher order tongues are being split in an analogous manner. At the values of A/l considered here they are still apart, but at sufficiently (unrealistically) large values they overlap, just as the tongues of the $n = 1$ resonance. Of all these higher order tongues we have depicted only those for $n = 2$, originating from the horizontal axis at $\Omega = \sqrt{g/l}$ and $\Omega = \sqrt{g/l + 2K}$. They are of much less importance than the main tongue.

Fig. 10. (a) Stability diagram of the 0-motion, for $K = 1 \text{ s}^{-2}$. (b) Eigenvalues and stroboscopic phase portraits at the points marked in (a).

Fig. 10b shows the positions of the eigenvalues of the origin, and a sketch of the stroboscopic phase space portraits at the five points marked in Fig. 10a. Traveling from right to left in the (A, Ω) -plane the origin will first, at line a_1 , become unstable by means of a period-doubling bifurcation in the 2-direction, giving birth to a 2-motion of period $2T$. Crossing line a , stability in the 1-direction will also be lost and

a 1-motion of period $2T$ will be born. Decreasing Ω further will cause the origin to become stable again, first in the 2-direction at line $\mathbf{b}_1$ and then in the 1-direction at line $\mathbf{b}$. In an analogous manner stability will be lost and regained at the resonance tongues of higher order; for $n = 2$ by means of equal period bifurcations, for $n = 3$ by means of period doubling bifurcations, for $n = 4$ by means of equal period bifurcations, and so on.

3.3. Stability of the 1-motion

For $K = 0$ we found that the 1- and 2-motions are both governed by the same two-dimensional Eq. (2.5). For $K > 0$ this is no longer the case, and we therefore have to deal with each of them separately. In this section we consider the 1-motion.

We start out with the equations of motion (1.6), which can also be written in the more compact form

$$\dot{\Phi} = \Gamma(\Phi, t), \quad (3.3)$$

where Φ is shorthand for $\Phi(t) = (\phi_1(t), \dot{\phi}_1(t), \phi_2(t), \dot{\phi}_2(t))$. If $\Phi(t)$ does not differ much from a $2T$ periodic 1-motion $\Phi_0(t)$, say $\Phi(t) = \Phi_0(t) + \Delta\Phi(t)$, we can write

$$\Gamma(\Phi, t) = \Gamma(\Phi_0, t) + L_D(\Phi_0, t)\Delta\Phi + O((\Delta\Phi)^2), \quad (3.4)$$

where $L_D(\Phi_0, t)$ is the derivative-matrix

$$L_D(\Phi_0, t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -f(t)\cos\frac{1}{2}\phi_1\cos\frac{1}{2}\phi_2 & 0 & f(t)\sin\frac{1}{2}\phi_1\sin\frac{1}{2}\phi_2 & 0 \\ 0 & 0 & 0 & 1 \\ f(t)\sin\frac{1}{2}\phi_1\sin\frac{1}{2}\phi_2 & 0 & -f(t)\cos\frac{1}{2}\phi_1\cos\frac{1}{2}\phi_2 - 2K & 0 \end{pmatrix}. \quad (3.5)$$

The 1-motion is confined to the 1-plane (for which $\phi_2 = 0$). The derivative-matrix will therefore be in block-diagonal form, and the same can be said for the derivative-matrix of the stroboscopic map (see also Appendix A). The stability with respect to the 1-plane is determined by the 1-eigenvalues and the stability with respect to directions perpendicular to it by the 2-eigenvalues. The 1-eigenvalues correspond to the upper left block and the 2-eigenvalues to the lower right block of $L^{(2)}$ (i.e. the derivative of the second iterate of the stroboscopic map). Because the upper left block is not affected by K the corresponding eigenvalues are also unaffected, i.e., the 1-eigenvalues are the same as in the case $K = 0$. The 2-eigenvalues, however, *do* depend on K .

In Section 2.2 we found that there are two kinds of 1-motion, namely the symmetrical and the non-symmetrical variety. Let us first discuss the symmetrical 1-motion.

3.3.1. The symmetrical 1-motion

The region of stability of the symmetrical 1-motion for $K = 0.01 \text{ s}^{-2}$ (a relatively small value) is presented in Fig. 11.

Fig. 11. Stability diagram of the symmetrical 1-motion, for $K = 0.01 \text{ s}^{-2}$.

For $K = 0$ the region of stability was just the region between the lines **A** and **D**; at these lines all four eigenvalues of the symmetrical 1-motion were equal to $+1$. For $K > 0$ the lines **A** and **D** still play an important role, since the two 1-eigenvalues are not altered by the coupling and remain at $+1$ along these lines. The $+1$ lines for the 2-eigenvalues, on the other hand, are altered by the coupling; in the figure they are denoted by **A_{II}**, **D_I** and **R**. So now the region of stability is bounded by the lines **A**, **A_{II}**, **D** and **R**. Only within the shaded region(s) all four eigenvalues lie on the unit circle.

The situation for larger K is qualitatively the same. The lines **A_{II}**, **D_I** and **R** are just shifted. Fig. 12a shows the (A, Ω) -plane for $K = 1 \text{ s}^{-2}$. At this value line **D_I** has been shifted so much that it is no longer in the picture. Fig. 12b shows the eigenvalues and sketches of the stroboscopic phase space at the points marked in Fig. 12a.

So, let us try to explain the position of **A_{II}**, **D_I** and **R**. For line **A_{II}** this is relatively easy. Looking at Fig. 12a we see that line **A** (at which the symmetrical 1-motion is born) partly lies inside the instability region of the 0-motion. Because the symmetrical 1-motion is born from the 0-motion by a period doubling bifurcation, the eigenvalues of the former start their life as the squares of those of the latter. For the 1-eigenvalues of the symmetrical 1-motion this simply means that they are at $+1$ on line **A** (as we have already seen). The 2-eigenvalues, however, are not. Along the upper part of line **A**, inside the right-hand instability tongue for the 0-motion, the 2-eigenvalues are on the positive real axis; as we go to the left in the (A, Ω) -plane they move towards $+1$ and onto the unit circle. Line **A_{II}** lies therefore to the left of line **A**. For the lower part of line **A**, outside the right-hand instability tongue of the 0-motion, the 2-eigenvalues start their life not on the real axis but on the unit circle. As a consequence, line **A_{II}** here marks the moment when the 2-eigenvalues leave the unit circle to enter the positive real axis.

Fig. 12. (a) Stability diagram of the symmetrical 1-motion, for $K = 1 \text{ s}^{-2}$. (b) Eigenvalues and stroboscopic phase portraits at the points marked in (a). The 2'-planes owe their name to the fact that they are parallel to the 2-plane.

The shapes of lines **D**₁ and **R** are more peculiar. We see that they come close together in a kind of snout (see Fig. 11), separated by line **c**₁ (where the 2-eigenvalues are at -1). In traversing the snout, the 2-eigenvalues make a *complete* journey around the unit circle. Apparently, the effect of K on the 2-eigenvalues is highly non-trivial here. This is brought out in a clear way by the behaviour of the 2-eigenvalues along line **D**. For small Ω (on line **D**) they are found to lie on the positive real axis, far away from $+1$; for high Ω they lie on the unit circle, close to $+1$. At some intermediate value of Ω they are equal to $+1$. This effect of K can be explained by considering the lower right block of $L^{(2)}$, which provides the 2-eigenvalues (see also Appendix A). Instead of the eigenvalues themselves we can also consider the *trace* of the lower right block, which is just the sum of the eigenvalues. (Since their product is always equal to 1, the determinant of the block, the sum completely fixes the eigenvalues.) We therefore evaluate

$$\text{Tr } L_{\text{LR}}^{(2)} \quad (3.6)$$

along line **D**, where the subscript LR stands for lower right. If $\text{Tr } L_{\text{LR}}^{(2)}$ is *smaller* than the corresponding quantity $\text{Tr } L_{\text{UL}}^{(2)}$ for the upper left block (i.e., the sum of the 1-eigenvalues, which is equal to 2 on line **D**), the 2-eigenvalues will lie on the unit circle (since they are squares they cannot lie on the negative real axis). Vice versa, if $\text{Tr } L_{\text{LR}}^{(2)}$ is larger than $\text{Tr } L_{\text{UL}}^{(2)}$ the 2-eigenvalues will lie on the positive real axis.

In appendix A we derive the following approximate expression:

$$\begin{aligned} \text{Tr } L_{\text{LR}}^{(2)} = \text{Tr } L_{\text{UL}}^{(2)} &+ (-2K_{\Omega}) \left(4 + \frac{4}{3} \int_0^2 \frac{-4\pi^2}{\Omega^2 l} (g + A\Omega^2 \cos 2\pi\tau) \cos \frac{1}{2} \phi_1(\tau) d\tau + \text{h.o.t.} \right) \\ &+ (-2K_{\Omega})^2 \left(\frac{4}{3} + \frac{4}{15} \int_0^2 \frac{-4\pi^2}{\Omega^2 l} (g + A\Omega^2 \cos 2\pi\tau) \cos \frac{1}{2} \phi_1(\tau) d\tau + \text{h.o.t.} \right) \\ &+ \dots \\ &+ (-2K_{\Omega})^k \left(2 \frac{4^k}{(2k)!} + 2 \frac{4^k}{(2k+1)!} \right. \\ &\quad \times \left. \int_0^2 \frac{-4\pi^2}{\Omega^2 l} (g + A\Omega^2 \cos 2\pi\tau) \cos \frac{1}{2} \phi_1(\tau) d\tau + \text{h.o.t.} \right) \\ &+ \dots, \end{aligned} \quad (3.7)$$

where $\tau = \Omega t / 2\pi$ is the rescaled, dimensionless time (such that the period of the driving force is equal to 1) and $K_{\Omega} = 4\pi^2 K / \Omega^2$ the appropriately scaled, dimensionless coupling coefficient.

Fig. 13. Numerical data, relevant to the symmetrical 1-motion, for $\Omega = 1.9$ rad/s, $A/l = 0.001$. The maximum value of $\phi_1(\tau)$ is 6.19 rad, almost 2π , so $\cos \frac{1}{2}\phi_1(\tau)$ oscillates between the values $+1$ and $\cos(3.095) = -0.999$.

Let us first consider the effect of the coupling for small Ω along line **D**. For these values of Ω the pendulums spend most of their time near the upward equilibrium position, where $\cos \frac{1}{2}\phi_1(\tau)$ equals -1 , as can be seen from Fig. 13. In the limit for small Ω or, equivalently, small A/l (along line **D**), the peaks in Fig. 13 become narrower and narrower. At the same time the factor $(-4\pi^2/\Omega^2 l)(g + A\Omega^2 \cos 2\pi\tau)$ becomes very large and negative, being entirely dominated by its first term. As a consequence, the integrals in Eq. (3.7) will be *large* and *positive*. Furthermore, in this limit the dominant term of Eq. (3.7) is the one with the highest power in K_Ω , having the lowest power in Ω . (To be precise, the higher order terms belonging to lower powers of K_Ω also include terms with the same power in Ω , but these turn out to be less important.) Since we are dealing with a symmetrical motion of period $2T$ we know the highest power of K_Ω to be even (see the example in Appendix A), so the dominant term has positive sign, and therefore

$$\text{Tr } L_{\text{LR}}^{(2)} \gg \text{Tr } L_{\text{UL}}^{(2)}. \quad (3.8)$$

This implies that the 2-eigenvalues lie far away from $+1$, on the positive real axis, in agreement with the numerical observations.

For large values of Ω the dominant term of Eq. (3.7) is the one with the lowest power in K_Ω . We know the amplitude of the 1-motion to decrease with increasing Ω , and for

Fig. 14. Numerical data, relevant to the symmetrical 1-motion, for $\Omega = 100$ rad/s, $A/l = 0.55$. The maximum value of $\phi_1(\tau)$ is 2.2151 rad, so $\cos \frac{1}{2} \phi_1(\tau)$ oscillates between the values $+1$ and $\cos(1.1076) = 0.447$.

sufficiently large Ω the function $\cos \frac{1}{2} \phi_1(\tau)$ will not stray far from $+1$ anymore, as can be seen from Fig. 14. At the same time the factor $(-4\pi^2/\Omega^2 l)(g + A\Omega^2 \cos 2\pi\tau)$ becomes sinusoidal, being dominated by its second term. This results in a (relatively small) *positive* value for the integrals of Eq. (3.7). We thus find

$$\text{Tr } L_{\text{LR}}^{(2)} < \text{Tr } L_{\text{UL}}^{(2)}, \quad (3.9)$$

implying that the 2-eigenvalues lie on the unit circle, again in agreement with our numerical observations.

The stability region of the symmetrical 1-motion is, all things considered, quite remarkable. For one thing, it actually consists of two regions which are only connected in one point. Another interesting feature is line **R**. Along this line the stable symmetrical 1-motion recombines (hence the symbol **R**) with two unstable non-symmetrical mixed motions, resulting in an *unstable* symmetrical 1-motion. The pictures suggest that line **R** is an uplifted version of the Ω -axis and indeed, line **R** approaches the Ω -axis for $K \downarrow 0$.

This is noteworthy since it means that (except for a very small region of frequencies close to $\Omega = 2\sqrt{g/l}$, between line **A_{II}** and **A**) in order to get a stable 1-motion the driving amplitude A has to exceed a certain threshold value, which grows with K . Such a threshold effect is usually associated with *dissipation*, where the driving has to overcome the resistive force of the damping, but in the present it is the *coupling* (i.e. the torsion spring) which apparently obstructs the 1-motion.

This concludes our discussion of the symmetrical 1-motion. Along line **D** it gives way to a non-symmetrical 1-motion, along **D_I** to a mixed motion of type C (we will come to the mixed motions in Section 3.5), and along **R** the symmetrical 1-motion loses stability without yielding a new motion. In the remainder of this section we will concentrate on the non-symmetrical 1-motion, which is born along line **D**.

3.3.2. The non-symmetrical 1-motion

The non-symmetrical 1-motion comes into existence by means of an equal-period bifurcation at line **D**. In the uncoupled case this motion becomes unstable at line **e** (cf. Fig. 7a) by means of a period-doubling bifurcation, its eigenvalues leaving the unit circle at -1 . For non-zero coupling the situation is more complicated. This is again due to the 2-eigenvalues, which are affected by the coupling (whereas the 1-eigenvalues are not).

Fig. 15 depicts the stability diagram for the non-symmetrical 1-motion for $K = 1 \text{ s}^{-2}$. In this figure there are two lines we have not seen before: line $\mathbf{e}_1$ (where the 2-eigenvalues are at -1) and the so-called *cutting line* $\mathbf{L}_c$, which cuts off the region of stability on the left side. On this line the 2-eigenvalues are at $+1$. For the sake of clarity, Fig. 16a depicts an enlargement of the relevant part of the stability diagram, and Fig. 16b shows the corresponding eigenvalues and stroboscopic phase space portraits.

Let us first consider the line $\mathbf{e}_1$. Of course this line can never get beyond line **D** (because the non-symmetrical 1-motion simply does not exist there). In fact, the only possible point where line $\mathbf{e}_1$ can (and indeed does) touch line **D** is the intersection point of lines $\mathbf{e}_1$ and **D**, because this is the only point on line **D** where the 2-eigenvalues of the mother orbit (the symmetrical 1-motion) are at -1 . Furthermore, following the same arguments which led to Eq. (3.9), it can be shown that the 2-eigenvalues of the non-symmetrical 1-motion are on the negative real axis along the upper part of line **e**, implying that $\mathbf{e}_1$ lies to the right of **e**. Thus, along its upper part, line $\mathbf{e}_1$ lies between **e** and **D**.

Analogously, along the lower part of **e**, we can infer that the 2-eigenvalues lie very far away from -1 , implying that $\mathbf{e}_1$ lies far from **e** for low values of Ω . This large shift

Fig. 15. Stability diagram of the non-symmetrical 1-motion, for $K = 1 \text{ s}^{-2}$.

Fig. 16. (a) Enlargement of the relevant part of Fig. 15. (b) Eigenvalues and stroboscopic phase portraits at the points marked in (a).

cannot be to the right, because it is blocked by line **D**, so line e_1 lies to the left of the line e , as in Fig. 16a. Somewhere between the upper and the lower part of line e the effect of K will be zero. This is where lines e_1 and e intersect.

The line L_C has no $K = 0$ equivalent. From simple considerations, however, it is clear that this line must originate from the point where **D** and **R** intersect, and also that it should lie to the left of the line e_1 ; this is corroborated by the numerical results.

3.4. Stability of the 2-motion

We now come to the 2-motion. Just as for the 1-motion one has a symmetrical and a non-symmetrical variety. Let us first discuss the symmetrical 2-motion.

3.4.1. The symmetrical 2-motion

In Fig. 17a we depict the stability region for the symmetrical 2-motion, and in Fig. 17b we present the eigenvalues and sketches of the stroboscopic phase space for the five points marked in Fig. 17a. This 2-motion comes into existence at line A_1 , by means of a period-doubling bifurcation of the 0-motion, with all of its four eigenvalues on the unit circle. Going to the left in the (A, Ω) -plane the eigenvalues travel over the unit-circle towards -1 . Due to the symmetry of the orbit they are not allowed to enter the negative real axis (see the argument presented in Section 2.2) so line c_{II} , where the 1-eigenvalues reach -1 , and line c_{III} , where the 2-eigenvalues reach -1 , have no direct physical significance and the eigenvalues just stay on the unit circle. The 1-eigenvalues reach $+1$ before the 2-eigenvalues do. At this moment the symmetrical 2-motion becomes unstable by means of an equal-period bifurcation in the 1-direction at line D_{II} , where the 1-eigenvalues enter the positive real axis. It gives way to an M-motion of type D, which will be discussed in Section 3.5.

3.4.2. The non-symmetrical 2-motion

The non-symmetrical 2-motion of period $2T$ is born, by an equal-period bifurcation, when the 2-eigenvalues of the symmetrical 2-motion enter the positive real axis, at line D_{III} . The non-symmetrical 2-motion thus starts its life *unstable*, because its 1-eigenvalues are on the positive real axis; they will enter the unit-circle for lower values of Ω . However, for realistic values of A/l the 2-eigenvalues will, if the coupling is as large as $K = 1 \text{ s}^{-2}$, already have entered the negative real axis at this time. In other words: for these parameter values the non-symmetrical 2-motion will never be stable. Only if K is sufficiently small will we observe stable non-symmetrical 2-motions. For this reason we present in Fig. 18 the stability diagram for $K = 0.01 \text{ s}^{-2}$; for this value of the coupling parameter the non-symmetrical 2-motion still has a small stable band.

And that is all. The 2-motions and their regions of stability turn out to be remarkably simple in comparison with the 1-motion.

Fig. 17. (a) Stability diagram of the symmetrical 2-motion, for $K = 1 \text{ s}^{-2}$. (b) Eigenvalues and stroboscopic phase portraits at the points marked in (a). The $1'$ -planes owe their name to the fact that they are parallel to the 1-plane.

3.5. The stability of the M-motions

3.5.1. Preliminaries: a few words on Krein theory

It might be worth nothing that it is not until now, with the treatment of the M-motions in the coupled system ($K > 0$), that we encounter motions with a truly four-dimensional character. Due to the coupling we cannot separate the four dimensions of the stroboscopic phase space into 2×2 dimensions, and the eigenvalues can no longer be characterized as belonging to either the 1- or the 2-direction. The four dimensionality of the M-motions allows a new type of eigenvalue-configuration:

Fig. 18. Stability diagram of the non-symmetrical 2-motion, for $K = 0.01 \text{ s}^{-2}$.

Fig. 19. Quadruplet configuration of eigenvalues in a Hamiltonian system. The product of the four eigenvalues equals 1.

besides pairs on the unit circle ($\lambda, \bar{\lambda}$ with $|\lambda| = 1$) and pairs on the real axis (λ, λ^{-1}), the eigenvalues can now also appear in the form of a quadruplet ($\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}$). In Fig. 19 we show a possible quadruplet configuration. An interesting question is of course: when do we get a quadruplet? The answer is given by Krein theory.

According to Krein theory, as described by Howard and MacKay [11], one can assign a so-called Krein signature $Kr(\lambda)$, i.e. a plus or a minus, to each non-degenerate pair of eigenvalues on the unit circle. This signature will not change as long as the pair remains non-degenerate and away from the real axis.

Howard and MacKay describe what happens when, under a continuous perturbation, two eigenvalue pairs move towards each other on the unit circle. Typically, two pairs with *equal* signature will not collide but repel each other. This is called Krein repulsion, see Fig. 20a. And even if they collide, which is non-typical, Krein's theorem [11] states that they cannot leave the unit circle to form a quadruplet (they just go through each other). On the other hand, when two pairs with *opposite* signatures meet, they typically *do* collide and form a quadruplet. This is called Krein collision, see Fig. 20b. Two eigenvalues then move *outside* the unit circle (while the other two move *inside* the unit circle), which means that the orbit becomes unstable by means of a Hamiltonian Hopf bifurcation. Upon a further change of parameters the quadruplet will follow a trajectory through the complex plane and may eventually reenter the unit circle or come down upon the real axis, to be split into two pairs again.

Another important thing concerning Krein signatures is the squaring rule [12] which states that, if λ is an eigenvalue of the matrix M , then

$$\text{Kr}(\lambda^2) \text{ of } M^2 = \text{sign}(\lambda + \bar{\lambda}) \text{Kr}(\lambda) \text{ of } M. \quad (3.10)$$

In our system this squaring rule is particularly useful since, due to the symmetry, the dynamical behaviour is often conveyed not by the eigenvalues themselves, but by their square roots (see also Section 2.2). Furthermore, this rule is used in appendix B to show that in case of a period doubling bifurcation, the eigenvalue pair of the daughter orbit will have *minus* the signature of the corresponding pair of the mother orbit. In contrast, in case of an equal period bifurcation the Krein signature of the daughter-pair will be the same as that of the mother-pair.

Fig. 20. Krein repulsion (a) and Krein collision (b).

In the remainder of this section we will discuss the effect of non-zero coupling on the stability regions of the various M-motions, the most important of which is type A.

3.5.2. M-motion type A

When a non-zero coupling is introduced in the M-motions of type A and B, the pendulum which was initially in the downward equilibrium will try to follow the motion of the other one which, in turn, will be somewhat hampered in its motion. The amplitude of the former will thus increase from zero to some finite value, while the amplitude of the latter is decreased. This results in the modified type A and B motions depicted in Fig. 21. The symmetry is *not* affected by the coupling: type A is still symmetrical and type B is still non-symmetrical.

The orbit in phase space of M-motion type A for $K = 0$ was characterized by $|\phi_1(t)| = |\phi_2(t)|$. In stroboscopic phase space it was positioned on the line for which $|\phi_1| = |\phi_2|$ and $\dot{\phi}_1 = \dot{\phi}_2 = 0$. For $K > 0$ this is no longer the case. From Fig. 21 we see that, for growing K , the type A tends to resemble a 1-motion (with both pendulums moving in phase with each other), which means that the orbit in stroboscopic phase space moves closer to the 1-plane; so we now have $|\phi_1| > |\phi_2|$, with $\dot{\phi}_1 = \dot{\phi}_2 = 0$ still holding. Actually, this connection with the 1-motion is only natural, since the type A motion bifurcates from the 1-motion by means of an equal period bifurcation at line A_{II} (Fig. 12a).

The stability diagram, for $K = 1 \text{ s}^{-2}$, is shown in Fig. 22a. The structure of the $K = 0$ case is still clearly visible (cf. Fig. 9a). New features are the region where the motion is Hopf bifurcated, and the small bubble of stability near the point marked 8. The former is a result of Krein collision and the latter is due to Krein repulsion.

In order to understand what is happening we recall the uncoupled case. For $K = 0$ the eigenvalues which govern the type A motion are the eigenvalues of the individual pendulums. That is to say, one of the eigenvalue pairs corresponds to the (one-pendulum) downward equilibrium and the other to a symmetrical (one-pendulum) motion of period $2T$. The periodicity of the type A motion as a whole is $2T$, i.e. twice the driving period, and the eigenvalues we are dealing with are those of the twice iterated stroboscopic map. As far as the downward equilibrium (period T) is concerned, this means that we are dealing with the *squares* of the eigenvalues. To the right of line **a** (see Fig. 3a) the downward equilibrium has negative Krein signature. Traversing the instability tongue will change this signature, so just to the left of line **b** it is positive. The *squared* eigenvalue pair will then have negative signature, by virtue of the squaring rule (3.10), since $\text{sign}(\lambda + \bar{\lambda})$ is negative. We also know the signature of the symmetrical motion of period $2T$, since it has been born (by means of a period doubling bifurcation) from the downward equilibrium at line **A** (see Fig. 7a). Between lines **A** and **c** it will therefore have positive signature. This means that the two eigenvalue pairs of the type A mixed motion will have *opposite* Krein signatures just to the left of line **B** and below line **c** (around the point marked 2). For zero coupling this

Fig. 21. Schematic representation of the M-motions types A and B, both for $K = 0$ and $K > 0$.

has little meaning, because both pendulums are totally independent and the eigenvalue pairs can therefore never interact. For $K > 0$ however, they *do* interact, and more specifically, they collide and yield a Hopf bifurcation. This happens along the dotted line **H** in Fig. 22a. For growing K this line unfolds in both directions to become the boundary of the Hopf region, i.e., a region where the four eigenvalues form a quadruplet. For small K this region is still very thin, hardly distinguishable from line **H** itself. For $K = 1 \text{ s}^{-2}$ (Fig. 22a) the region has grown considerably, with a boundary **H**₁.

Let us consider the situation for $K = 1 \text{ s}^{-2}$ in some more detail. Just to the left of the shifted line **B**_{II} (and below line **c**_{IV}) the eigenvalue pairs of the type A mixed motion have opposite Krein signatures. Both pairs will travel along the unit circle and pass through -1 (one after the other), where they change signature. This change of signature is due to the fact that all eigenvalues of the type A motion are squares (see also the symmetry argument in Section 2.2):

$$\lambda_{1,2,3,4} = \mu_{1,2,3,4}^2, \quad (3.11)$$

so when λ goes through -1 its square root μ crosses the imaginary axis, changing the sign of $\mu + \bar{\mu}$. The squaring rule then tells us that the Krein signature of the λ -pair changes. The first pair goes through -1 between points 2 and 2*, at line **c**_{IV}. In Fig. 22b we have sketched the evolution of the eigenvalues λ of the twice iterated stroboscopic map, as well as their (relevant) square roots μ . The latter are actually the more important ones. For instance, naively one might expect the two λ -pairs to repel each other somewhere near point 2*, but because their (square root) μ -pairs are *not* so close together there is no reason why they should interact. The interaction has to wait

Fig. 22. Stability diagram of the (symmetrical) M-motion type A, for $K = 1 \text{ s}^{-2}$. (b) Eigenvalues (and their relevant square roots) at the points marked 1 to 5 in (a). (c) Eigenvalues (and their relevant square roots) at the points marked 2** and 6 in (a). (d) Eigenvalues (and their relevant square roots) at the points marked 3*, 7, 7*, 5*, 8 and 5** in (a).

until both λ -pairs have passed through -1 . The μ -pairs and hence also the λ -pairs, collide somewhere between points 2* and 3. Situation 3 corresponds with a Hopf bifurcated type A motion.

Just before point 4 the quadruplet comes down onto the positive real axis and is split into two eigenvalue pairs, one of which reenters the unit circle shortly afterwards. At point 5 we then get the situation with one pair on the unit circle and the other one on the positive real axis.

Not for all values of A/l will the chain of events be as described above. For *higher* values (e.g. for $A/l = 0.2$) situation 2, or rather 2**, is followed by situation 6, in which

Fig. 22. continued.

an equal period bifurcation has taken place before the μ -pairs have had the opportunity to collide, see Fig. 22c. In other words, instead of a Hopf bifurcated type A motion we now get a (non-symmetrical) mixed motion of type B. In contrast, for low values of A/l (e.g. for $A/l = 0.05$) the Krein collision comes so soon that the quadruplet, before it reaches the positive real axis, manages to return to the unit circle, forming two complex pairs again. Situation 3 is then followed by situation 7, see Fig. 22d.

One of the complex pairs goes towards $+1$ and enters the positive real axis, and as a result we get situation 5^* . This is a remnant of the $n = 2$ resonance tongue which we had in the case of $K = 0$. In the next step the μ -pair on the positive real axis turns back and reenters the unit circle; in the stability diagram this means that we enter the bubble of stability (situation 8). For $K = 0$ this μ -pair would travel all the way along the unit circle to enter the negative real axis at the $n = 3$ tongue. In the coupled case however, this is prohibited by the other μ -pair, which has the same Krein signature, and causes a repulsion. As a result the first μ -pair turns back and soon reenters the

Fig. 23. Stability diagram of the (non-symmetrical) M-motion type B, for $K = 1 \text{ s}^{-2}$.

positive real axis. We end up with situation 5**. Above some critical value of A/l (depending on K), at the top of the bubble, the return to the unit circle becomes altogether impossible (due to Krein repulsion) so situation 8 is then skipped and the orbit does not regain its stability.

3.5.3. M-motion type B

Let us now turn to the (non-symmetrical) type B motion. We have already remarked that this motion originates from the type A motion along line D_{IV} , in going from situation 2** to 6. The stability region of type B is shown in Fig. 23 for $K = 1 \text{ s}^{-2}$, and Fig. 24a is a magnification of the relevant part. In Fig. 24b we have indicated the eigenvalues at the marked points.

There are three main lines in the diagram. First of all we have the birthline D_{IV} , where the eigenvalue pair originally associated with the pendulum performing a symmetrical motion of period $2T$ is at $+1$. Secondly we have the line B_{III} , where the eigenvalue pair originally associated with the pendulum in its downward equilibrium is at $+1$. This line B_{III} originates from the intersection point of B_{II} and D_{IV} . Thirdly, there is a point on line D_{IV} where the eigenvalue pair originally associated with the downward equilibrium has the value -1 ; from this point two lines w_R and w_L (along which one eigenvalue pair is at -1) originate and form a wedge in the stability region. This wedge is thus an unfolding of the intersection point of lines C_{IV} and D_{IV} .

Traversing the wedge changes the Krein signature of one of the pairs, as can be deduced from the position of line C_{IV} and from the fact that the type B motion bifurcates by means of an equal-period bifurcation from the type A motion (see also

Fig. 24. (a) Enlargement of the relevant part of Fig. 23. (b) Eigenvalues at the points marked points in (a).

Appendix B). To the right side both pairs have the same signature, which means that they repel each other. To the left side of the wedge the eigenvalue pairs have opposite signature, resulting in a Krein collision and a Hamiltonian Hopf bifurcation of the orbit. The Hopf bifurcated region is shaded dark in Fig. 24a. In this region we have a quadruplet of eigenvalues. At the upper boundary of the region the quadruplet collapses onto the *negative* real axis (falling apart into two real pairs). On the left border of the region the quadruplet hits upon the *positive* real axis, again falling apart into two real pairs. Subsequently one of the pairs travels away from $+1$, while the other goes towards $+1$ and reenters the unit circle. This is a so-called symmetry restoring bifurcation; the type B motion ceases to exist and gives way to a type

A motion again. The bifurcation has no physical significance, however, since the motions (the daughter as well as the mother) are already unstable at this time.

3.5.4. *M-motion type C*

From Section 2.3 we know that, for $K = 0$, mixed motions of type C and D are constructed by letting both pendulums perform a non-symmetrical motion of period $2T$. These composite motions undergo a period doubling bifurcation at line e (for $K = 0$), which is simply the bifurcation line for the single pendulum motions. For $K > 0$ one might therefore expect the mixed motions of type C and type D to undergo a period doubling at some line e_{III} , respectively line e_{IV} , in the (A, Ω) -plane. This is however *not* the case. The reason for this lies in the symmetry of the orbits.

Let us consider Fig. 8, in which types C and D are depicted. If we denote the positions of the pendulums at time $t = 0$ by

$$\begin{aligned} \vartheta_1(0) &= a, \\ \vartheta_2(0) &= b, \end{aligned} \quad (3.12)$$

or, equivalently, in terms of the normal coordinates,

$$\begin{aligned} \phi_1(0) &= a + b, \\ \phi_2(0) &= a - b, \end{aligned} \quad (3.13)$$

then we get at $t = T$:

$$\begin{aligned} \vartheta_1(T) &= -b, \\ \vartheta_2(T) &= -a, \end{aligned} \quad (3.14)$$

or

$$\begin{aligned} \phi_1(T) &= -b - a = -\phi_1(0), \\ \phi_2(T) &= -b + a = \phi_2(0). \end{aligned} \quad (3.15)$$

The symmetry is not broken by turning on the coupling. This means that the linearization of the twice iterated stroboscopic map can be written as (see also Appendix A):

$$\begin{aligned} L^{(2)} &= \begin{bmatrix} L_{1,1} & L_{1,2} \\ L_{2,1} & L_{2,2} \end{bmatrix} \cdot \begin{bmatrix} L_{1,1} & -L_{1,2} \\ -L_{2,1} & L_{2,2} \end{bmatrix} \\ &= \begin{bmatrix} L_{1,1} & L_{1,2} \\ L_{2,1} & L_{2,2} \end{bmatrix} \cdot \begin{bmatrix} I_2 & \emptyset \\ \emptyset & -I_2 \end{bmatrix} \cdot \begin{bmatrix} L_{1,1} & L_{1,2} \\ L_{2,1} & L_{2,2} \end{bmatrix} \cdot \begin{bmatrix} I_2 & \emptyset \\ \emptyset & -I_2 \end{bmatrix} \end{aligned}$$

$$= \left(\begin{bmatrix} L_{1,1} & L_{1,2} \\ L_{2,1} & L_{2,2} \end{bmatrix} \cdot \begin{bmatrix} I_2 & \emptyset \\ \emptyset & -I_2 \end{bmatrix} \right)^2 \quad (3.16)$$

where the $L_{i,j}$ and I_2 are two-dimensional matrices, the latter representing the unit-matrix. So we see that the linearized map $L^{(2)}$ is actually the *square* of a (symplectic) matrix, and consequently its eigenvalues can be written as squares:

$$\lambda_{1,2,3,4} = \mu_{1,2,3,4}^2, \quad (3.17)$$

where the $\mu_{1,2,3,4}$ again can form complex conjugate pairs, real pairs, or a quadruplet. We have seen a situation like this before in the case of the symmetrical 1- and 2-motions.

The fact that the eigenvalues can be written as squares prohibits them from entering the negative real axis *unless* they are degenerate, as is the case for $K = 0$ (see Fig. 25). The degeneracy is lifted by the coupling, rendering a period doubling bifurcation impossible for both types C and D. But then, instead of a period doubling, what *do* we get?

Let us first turn our attention to the mixed motion of type C. From Section 3.3 we know that it is born from the symmetrical 1-motion by means of a symmetry breaking bifurcation at line D_1 . For $K = 1 \text{ s}^{-2}$ this line is shifted out of the picture (see Fig. 12a); the type C mixed motion is therefore of no practical interest for large coupling. For this reason we take $K = 0.01 \text{ s}^{-2}$. In Fig. 26 we present the stability region of type C at this value of K .

When the type C mixed motion is born along the right part of line D_1 , it is unstable since one of its eigenvalue pairs lies on the positive real axis. This pair enters the unit circle for smaller Ω , and as a byproduct two unstable daughter orbits are born. At this moment the type C motion becomes stable. Both its eigenvalue pairs now have the same Krein signature, and therefore repel each other. One of them passes through -1 , changes signature, and collides with the other pair, rendering the orbit unstable

Fig. 25. λ and μ configuration in case of zero coupling ($K = 0$). The four λ -eigenvalues are grouped in two degenerate pairs.

Fig. 26. The stability diagram of the M-motion type C, for $K = 0.01 \text{ s}^{-2}$.

my means of a Hamiltonian Hopf bifurcation. We now enter the wide Hopf region in Fig. 26. Traversing this region, the eigenvalues travel as a quadruplet through the complex plane to collapse (at the left border of the Hopf region) on the positive real axis, being split into two real pairs. The eigenvalues of the first pair go towards infinity and zero respectively, while those of the second pair reenter the unit circle; at this point the two daughters (the byproduct mentioned above) are united with their mother. The eigenvalue pair then travels the whole unit circle to reach $+1$ again at the left part of line D_1 , where the type C motion is recombined with the (unstable) symmetrical 1-motion and thus ceases to exist.

3.5.5. *M-motion type D*

The mixed motion of type D is born at line D_{II} (by an equal period bifurcation) from the symmetrical 2-motion, when the 1-eigenvalues of the latter leave the unit circle at $+1$. At this line the symmetrical 2-motion is still stable with respect to the 2-direction, and because of this the type D mixed motion starts its life with all of its four eigenvalues on the unit circle. That is, the type D motion is (initially) stable. The stability diagram for $K = 1 \text{ s}^{-2}$ is presented in Fig. 27.

Between lines c_{II} and c_{III} the eigenvalues of the mother orbit (the symmetrical 2-motion) have opposite Krein signatures. An equal period bifurcation preserves Krein signature (see Appendix B), so *below* the intersection point of lines D_{II} and c_{III} the eigenvalue pairs of the type D motion will start out with opposite Krein signature. Both pairs travel towards each other and before long undergo a Krein collision, rendering the orbit unstable by means of a Hopf bifurcation. For small

Fig. 27. Stability diagram of the M-motion type D, for $K = 1 \text{ s}^{-2}$.

values of A/l , where the eigenvalue pairs start their life very close together, this Hopf bifurcation takes place almost immediately, and the stability region is extremely thin. For larger values of A/l the stability region widens somewhat.

Analogously, *above* the intersection point of lines D_{II} and c_{III} the eigenvalue pairs of the type D motion start out with equal signature. They repel each other and one of the pairs is pushed towards -1 . One might expect this pair to enter the negative real axis, but due to the symmetry of the type D motion this is not possible, as explained in subsection 3.5.4. Instead, the pair passes through -1 and changes its signature. The two pairs now have opposite signature, and at some smaller value of Ω the stability of the orbit is again lost by means of a Hopf bifurcation.

Once the Hopf bifurcation has taken place, in any of the above two ways, the eigenvalues travel as a quadruplet through the complex plane and eventually collapse on the positive real axis. This marks the end of the Hopf region. Subsequently one of the pairs reenters the unit circle, while the other remains on the positive real axis; the orbit does not regain its stability anymore.

4. Conclusions and discussion

In this paper we studied a Hamiltonian system of two identical, parametrically driven pendulums coupled by a linear torsion spring. The three control parameters in this system are the driving frequency Ω , the driving amplitude A and the coupling coefficient K . As a first step we made a classification of all possible steady motions into four types: the pendulums can move in a downward equilibrium motion (0-motion), they can move in phase with each other with the same amplitude (1-motion), in

opposite phase with the same amplitude (2-motion), or they can move – in or out of phase – with different amplitudes (M-motion). We then proceeded to determine the regions in the (A, Ω) -plane for which these motions were stable.

Our study was set up from the relatively simple case of two uncoupled pendulums ($K = 0$)². The stability regions for this case were determined and subsequently we investigated what kind of changes were brought about by turning on a small coupling. For the 0-motion and the 2-motion these changes were found to be quite straight forward: the stability regions were just shifted slightly. For the 1-motion and the M-motions, on the other hand, things were different. In these cases the stability regions underwent significant changes, quantitatively as well as qualitatively, even for infinitesimally weak coupling.

The 1-motion was shown to be unstable for driving amplitudes A below a certain *threshold* value, a small region around $\Omega = 2\sqrt{g/l}$ excepted. This kind of threshold is very common in *dissipative* systems, but in our Hamiltonian situation (in the absence of any damping effects) it came as a bit of a surprise. Apparently the torsion spring has a tendency to play an active part, thereby destabilizing any motion in which it does not participate, i.e. the 1-motion. The driving must overcome this destabilizing effect; hence the threshold.

Our study of the 1-motion furthermore revealed that the period doubling route to chaos becomes less and less important for growing K . The route only takes place within an increasingly narrow channel in the (A, Ω) -plane. This result is in agreement with the observation of previous authors [12–15] that period doubling is a rather special phenomenon in four (or higher) dimensional Hamiltonian systems, whereas it is ubiquitous in two dimensions. The pendulums, however, find themselves a new route to chaos via mixed motions.

The M-motions were found to undergo Hamiltonian Hopf bifurcations for $K > 0$. This is another clear departure from the uncoupled case, where Hopf bifurcations are not even possible (for $K = 0$ only equal-period and period doubling bifurcations are permitted). In the presence of dissipation (to be discussed in a future paper) Hopf bifurcations yield a limit-cycle, in which the pendulums (or modes) continually exchange energy, and which is often the precursor of a chaotic attractor. This is the kind of mode competition which is frequently reported by experimentalists.

In the case of the most important M-motion (type A) the Hopf bifurcation for small K takes place along a certain line H in the (A, Ω) -plane, the position of which could be accurately predicted with the aid of Krein theory. For growing K this line unfolds and encloses a rapidly growing Hopf region. The same kind of behaviour is also found for the M-motions of type B,C and D.

²The limit for very strong coupling ($K = \infty$) is also very simple. However, in this limit the 2- and M-motions do not exist, and therefore $K = \infty$ is a less convenient starting point than $K = 0$.

Fig. 28. The gray triangular region in the (A, Ω) -plane for $K = 1 \text{ s}^{-2}$ where the 0-motion, the symmetrical 1- and 2-motions, and the type A mixed motion are all stable. In the three black regions they are all *unstable*. The solid lines represent physically relevant bifurcations, while along the dotted lines bifurcations take place which have no direct physical meaning.

Once the coupling is turned on ($K > 0$) no new features emerge anymore upon increasing the value of K further. This is due, no doubt, to the fact that the coupling in our system is linear. In particular, the stability regions for the various motions remain qualitatively the same. Of course, quantitatively they do *not* remain the same: the regions shift and change in size upon increasing K . In this way the overlap of the various stability regions changes. For $K = 1 \text{ s}^{-2}$, for example, there are regions where *all* types of motion are stable as well as regions where *none* of the four motions are stable (Fig. 28). Furthermore, there exist regions in the (A, Ω) -plane where the *only* stable motion is an M-motion. This last observation may seem odd when the M-motion is interpreted as some kind of combination of the two ‘normal modes’. A more natural point of view (which removes the oddness of the stable M-motion) is simply to regard the M-motion as the result of a competition between the two *pendulums*, instead of between two *modes*. This conclusion was already anticipated in the introduction on the basis of the equations of motion, and here it is confirmed by our results.

Let us finally turn to the question of how our results compare with the experimental literature on mode competition. Strictly speaking we should postpone such a comparison until we have included dissipation in our model, since in experiments one always has to take dissipative effects into account. Nevertheless, some general features of our Hamiltonian system are present in dissipative systems as well.

Ciliberto and Gollub [2, 3] investigated the wave pattern in a vertically oscillating fluid layer. The excitation of the flat surface into a pure mode is governed by Mathieu’s equation, and at different driving frequencies one excites different modes.

When the natural frequencies of two modes are close, *both* modes can be excited (and hence compete) at one value of the driving frequency. This mode competition was observed to give rise to periodic as well as chaotic behaviour. A theoretical treatment of this problem can be found in the papers by Meron and Procaccia [16,17] and Crawford, Knobloch and Riecke [18].

Lichter et al. [4–6] studied parametrically forced cross-waves in a rectangular wave tank. The excitation of these waves can be described by two coupled nonlinear Schrödinger equations, yielding the same kind of instability tongues as the Mathieu equation. Also in this case mode competition was observed, with periodic and chaotic mixed motions.

Skeldon et al. [7,8] examined a parametrically driven double pendulum, a system of just two degrees of freedom. As far as the interaction of two modes is concerned, this system has the same linear structure as the problem of Ciliberto and Gollub. It is also in many ways similar to our own problem of two pendulums. For instance, in Skeldon's system one can also distinguish four types of motion: a trivial motion, two normal modes and a mixed mode. As usual, the normal modes are excited from the trivial motion within two tongues in the parameter plane, and they exhibit (amongst other things) symmetry breaking bifurcations. In the region enclosed by the symmetry breaking curves Skeldon found mixed modes. These mixed modes were observed to undergo Hopf bifurcations, torus doubling, torus gluing and eventually a transition to chaos.

So we see that the experiments exhibit a number of features which are also present in our Hamiltonian system: a trivial motion, normal modes, mixed motions (periodic as well as chaotic), Hopf bifurcations, period-doublings and even equal-period bifurcations. These similarities will become more apparent in our next paper, when we include dissipation into our model, and a detailed comparison has to wait until then. Even so, it is certainly advantageous to have treated the Hamiltonian case first, thus providing a basis for all forms of dissipation. After all there is a Hamiltonian system behind every dissipative system.

There is yet another property of our system which makes it better suited for a systematic treatment, and this concerns the coupling. In the first place, we can vary the coupling K and even turn it off, which is impossible in the above mentioned experiments. Secondly, our coupling is linear. In the experiments, where both the oscillators *and* the coupling are nonlinear, it is very hard (if not impossible) to pinpoint which of the two is responsible for the nonlinear effects, while in our system the nonlinear effects can be attributed completely to the *pendulums*. In a future paper we hope to treat nonlinearities in the coupling.

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Appendix A. Derivation of the trace formula

In this appendix we derive the approximate expression (3.7) for the sum of the 2-eigenvalues of the 1-motion:

$$\text{Tr } L_{\text{LR}}^{(2)}. \quad (\text{A.1})$$

We do this by applying Euler's integration scheme to the equations of motion. In the compact notation of Eq. (3.3) we then get:

$$\Phi_{n+1} \equiv \Phi(t_{n+1}) \approx M(\Phi_n, t_n) = \Phi_n + h\Gamma(\Phi_n, t) \quad \text{with } t_{n+1} = t_n + h. \quad (\text{A.2})$$

If we make the timestep h smaller and smaller this map becomes a better and better approximation of the differential equations. The twice iterated stroboscopic map is then simply the $(2N)$ th-iterate of the map $M(\Phi_n, t_n)$, where N represents the number of timesteps taken per period, i.e. $N = T/h$. The eigenvalues of the symmetrical motion of period $2T$ are just the eigenvalues of the derivative-matrix of the twice iterated stroboscopic map, denoted as $L^{(2)}$. This derivative-matrix equals the matrix product of the $2N$ derivative-matrices L_M of M , which have the form:

$$L_M(\Phi_n, t_n) = I + hL_D(\Phi_n, t_n), \quad (\text{A.3})$$

where I is the unity matrix and

$$L_D(\Phi_n, t_n) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -f(t_n) \cos \frac{1}{2} \phi_1(t_n) \cos \frac{1}{2} \phi_2(t_n) & 0 & f(t_n) \sin \frac{1}{2} \phi_1(t_n) \sin \frac{1}{2} \phi_2(t_n) & 0 \\ 0 & 0 & 0 & 1 \\ f(t_n) \sin \frac{1}{2} \phi_1(t_n) \sin \frac{1}{2} \phi_2(t_n) & 0 & -f(t_n) \cos \frac{1}{2} \phi_1(t_n) \cos \frac{1}{2} \phi_2(t_n) - 2K & 0 \end{pmatrix}. \quad (\text{A.4})$$

For a 1-motion (for which $\phi_2 = 0$) the derivative-matrix L_D , and therefore also $L^{(2)}$, will be in block-diagonal form. So the upper left block of $L^{(2)}$ equals the matrix product of $2N$ matrices of the form

$$\begin{bmatrix} 1 & h \\ X_n h & 1 \end{bmatrix}, \quad \text{with } X_n \equiv -f(t_n) \cos \frac{1}{2} \phi_1(t_n). \quad (\text{A.5})$$

For convenience we rescale time, such that the period of the driving force is equal to 1,

$$\tau = \frac{\Omega}{2\pi} t, \quad (\text{A.6})$$

and consequently $N = 1/h$. In rescaled form X_n reads

$$X_n = \frac{-4\pi^2}{\Omega^2 l} (g + A\Omega^2 \cos 2\pi\tau) \cos \frac{1}{2} \phi_1(\tau). \quad (\text{A.7})$$

Let us, as an example, take $N = 4$. The upper left block of the matrix $L^{(2)}$ then equals

$$L_{UL}^{(2)} = \begin{bmatrix} 1 & h \\ hX_7 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & h \\ hX_6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & h \\ hX_5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & h \\ hX_4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & h \\ hX_3 & 1 \end{bmatrix} \\ \times \begin{bmatrix} 1 & h \\ hX_2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & h \\ hX_1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & h \\ hX_0 & 1 \end{bmatrix} \quad (\text{A.8})$$

and its trace is (with $h = 1/4$):

$$\text{Tr } L_{UL}^{(2)} = 2 + h^2(7X_0 + 7X_1 + 7X_2 + 7X_3 + 7X_4 + 7X_5 + 7X_6 + 7X_7) \\ + h^4 \left(\begin{aligned} &5X_0X_2 + 8X_0X_3 + 9X_0X_4 + 8X_0X_5 + 5X_0X_6 + \\ &5X_1X_3 + 8X_1X_4 + 9X_1X_5 + 8X_1X_6 + 5X_1X_7 + \\ &5X_2X_4 + 8X_2X_5 + 9X_2X_6 + 8X_2X_7 + \\ &5X_3X_5 + 8X_3X_6 + 9X_3X_7 + \\ &5X_4X_6 + 8X_4X_7 + \\ &5X_5X_7 \end{aligned} \right) \\ + h^6 \left(\begin{aligned} &3X_0X_2X_4 + 4X_0X_2X_5 + 3X_0X_2X_6 + \\ &4X_0X_3X_5 + 4X_0X_3X_6 + 3X_0X_4X_6 + \\ &3X_1X_3X_5 + 4X_1X_3X_6 + 3X_1X_3X_7 + \\ &4X_1X_4X_6 + 4X_1X_4X_7 + 3X_1X_5X_7 + \\ &3X_2X_4X_6 + 4X_2X_4X_7 + 4X_2X_5X_7 + \\ &3X_3X_5X_7 \end{aligned} \right) \\ + h^8(X_0X_2X_4X_6 + X_1X_3X_5X_7). \quad (\text{A.9})$$

Incidentally, in the case of a symmetric orbit of period 2 (such as the 1-motion considered in the main text), many of the X_n are identical. In the present example we then have $X_0 = X_4$, $X_2 = X_6$ and $X_1 = X_3 = X_5 = X_7$. This gives a considerable simplification of the above expression.

The trace of the lower right block of $L^{(2)}$ has the same form as the upper left block provided we make the substitution:

$$X_n \rightarrow X_n - 2K_\Omega, \quad \text{with } K_\Omega = \frac{4\pi^2}{\Omega^2} K. \quad (\text{A.10})$$

The fact that K is replaced by K_Ω is a consequence of the time rescaling. The trace of the lower right block then equals (still in our example for $N = 4$, with $h = 1/4$):

$$\text{Tr } L_{LR}^{(2)} = \text{Tr } L_{UL}^{(2)} \\ + (-2K_\Omega) \left(56h^2 + 35h^4 \sum_{i=1}^8 X_i + \text{h.o.t.} \right)$$

Fig. 29. The curve $\cos \frac{1}{2} \phi_1(\tau)$ of the symmetrical 1-motion, for $\Omega = 2.55$ rad/s, $A/l = 0.020$, together with the curve $\cos 2\pi\tau$. The fourfold symmetry is evident: the picture is mirror symmetric with respect to the $\tau = 0 \bmod 1/2$.

$$\begin{aligned}
 &+ (-2K_\Omega)^2 \left(140h^4 + 21h^6 \sum_{i=1}^8 X_i + \text{h.o.t.} \right) \\
 &+ (-2K_\Omega)^3 \left(56h^6 + h^8 \sum_{i=1}^8 X_i \right) \\
 &+ (-2K_\Omega)^4 (2h^8). \tag{A.11}
 \end{aligned}$$

Note that the highest power of K_Ω is equal to N . If we are considering a symmetrical orbit of period 2, which has a fourfold symmetry (see Fig. 29), $2N$ will always be a multiple of four, and hence N will be even. The highest power of K_Ω will thus be even.

From expressions (A.8) and (A.11), for $N = 4$, we can easily infer the structure for general N . Recollecting that $h = 1/N$, we find that the coefficient of $-2K_\Omega$, which for $N = 4$ was equal to $56h^2 = 7/2$, in the limit $N \rightarrow \infty$ (or $h \rightarrow 0$) becomes

$$\lim_{N \rightarrow \infty} \left(\frac{1}{N} \right)^2 (2N - 1)2N = \lim_{N \rightarrow \infty} \left(\frac{1}{N^2} \right) (4N^2 - 2N) = 4. \tag{A.12}$$

In the same limit the coefficient of the terms $-2K_\Omega X_n$ (which had the value $35h^4 = 35/256$ for $N = 4$) can be shown to be equal to

$$\lim_{N \rightarrow \infty} \left(\frac{1}{N} \right)^4 \left[\frac{1}{6} (2N)^3 + O(N^2) \right] = \lim_{N \rightarrow \infty} \frac{4}{3} \frac{1}{N}. \tag{A.13}$$

So, writing the summation as an integral, with $h = 1/N = d\tau$ (in this limit h is indeed infinitesimally small), the second line of equation (A.11) becomes:

$$(-2K_\Omega) \left[4 + \frac{4}{3} \frac{-4\pi^2}{\Omega^2 l} \int_0^2 (g + A\Omega^2 \cos 2\pi\tau) \cos \frac{1}{2} \phi_1(\tau) d\tau \right]. \tag{A.14}$$

In the same way it is possible to obtain the values for the various coefficients for the terms $(-2K_\Omega)^k$, in the limit $N \rightarrow \infty$ ($h \rightarrow 0$). Substituting these coefficients in Eq. (A.11), we finally obtain:

$$\begin{aligned}
 \lim_{h \rightarrow 0} \text{Tr } L_{\text{LR}}^{(2)} &= \text{Tr } L_{\text{UL}}^{(2)} \\
 &+ (-2K_\Omega) \left(4 + \frac{4}{3} \frac{-4\pi^2}{\Omega^2 l} \int_0^2 (g + A\Omega^2 \cos 2\pi\tau) \cos \tfrac{1}{2} \phi_1(\tau) d\tau + \text{h.o.t.} \right) \\
 &+ (-2K_\Omega)^2 \left(\frac{4}{3} + \frac{4}{15} \frac{-4\pi^2}{\Omega^2 l} \int_0^2 (g + A\Omega^2 \cos 2\pi\tau) \cos \tfrac{1}{2} \phi_1(\tau) d\tau + \text{h.o.t.} \right) \\
 &+ \dots \\
 &+ (-2K_\Omega)^k \left(2 \frac{4^k}{(2k)!} + 2 \frac{4^k}{(2k+1)!} \frac{-4\pi^2}{\Omega^2 l} \right. \\
 &\quad \left. \times \int_0^2 (g + A\Omega^2 \cos 2\pi\tau) \cos \tfrac{1}{2} \phi_1(\tau) d\tau + \text{h.o.t.} \right) \\
 &+ \dots
 \end{aligned} \tag{A.15}$$

This is the trace formula (3.7) given in the main text.

Appendix B. Transfer of Krein signature at bifurcation

In our study of the M-motions, and in particular of the bifurcations of these motions, we often need to know the Krein signatures of the eigenvalues of the mother orbit as well as those of the daughter orbit(s). It would be very helpful, of course, if we had a relation between the Krein signatures of the mother and her daughters. In this appendix we establish such a relation, both for the case of a period-doubling bifurcation and an equal-period bifurcation. We proceed in three steps.

It may be noted that a similar relation for the (restricted) case of non-semisimple eigenvalues has been given in Ref. [12].

First step

We consider a Hamiltonian two dimensional system, periodically driven with period $2\pi/\Omega$ and driving amplitude A . The equation of motion is of the general form

$$\ddot{x} = f(x, A, \Omega t). \tag{B.1}$$

This system generates a two dimensional stroboscopic map. The linearized map is then a symplectic matrix L , of which the eigenvalues are either complex conjugates $(\lambda, \bar{\lambda})$ with modulus 1, or a real pair (λ, λ^{-1}) .

Consider a (stable) orbit of the system, the mother orbit, and its bifurcation line in the (A, Ω) -diagram, where it gives birth to one or two daughter orbits, depending on the bifurcation type. The eigenvalues of the mother orbit (i.e. of the linearized stroboscopic map L) have a definite Krein signature as long as they are complex. But on the bifurcation line the eigenvalues become real and their signature is therefore undefined. The eigenvalues of the daughter orbit also have a definite Krein signature as soon as they are complex.

Suppose the bifurcation to be a period doubling. A daughter orbit then has twice the period of the mother orbit, and *on* the bifurcation line its eigenvalues will be at $+1$, equal to the squares of those of the mother orbit. Fig. 30 depicts the bifurcation line in the (A, Ω) -diagram. We have marked it with a -1 , implying a period doubling bifurcation. The second line in the diagram represents some critical line for the daughter, i.e. a line where its eigenvalues are either at $+1$ or -1 (and where the eigenvalues *can* change Krein signature).

Second step

We now introduce a second system, which is identical to the first one, except for a coupling term K . The equations of motion are

$$\ddot{x} = f(x, A, \Omega t), \quad (\text{B.2a})$$

Fig. 30. Schematic representation of the bifurcation lines and eigenvalues of the system (B.1).

Fig. 31. Schematic representation of the bifurcation lines and eigenvalues of the system (B.2a)–(B.2b) for $K = 0$ and for $K > 0$.

$$\ddot{y} = f(y, A, \Omega t) + K\Omega^2 x. \quad (\text{B.2b})$$

The joint system is now four dimensional, with a four dimensional linearized stroboscopic map, and four eigenvalues.

We will initially take $K = 0$ and put the second system in precisely the same orbit as the first one. The linearized stroboscopic map is in block diagonal form, and its four eigenvalues coincide in one degenerate complex conjugate or real pair. In Fig. 31 we depict them as crosses and circles. Both eigenvalue pairs have the same Krein signature. At point X in Fig. 31 this signature is supposed to be *positive*.

Now let us perturb the system by increasing K slightly. The linearized stroboscopic map is now no longer in block diagonal form. The eigenvalues will shift slightly from their unperturbed position and the original bifurcation line will typically unfold into two bifurcation lines. A daughter orbit will also typically unfold into two orbits. We will consider the daughter orbit which is born along the line where the cross-eigenvalues are at -1 (see Fig. 31). We have chosen the sign of K and the position of point X in such a way that this line runs precisely through point X. The line where the circle-eigenvalues are at -1 lies above this point. Thus, due to continuity, the Krein signature of the circle-eigenvalues of the mother orbit is still *positive* (while the cross-eigenvalues do not have a definite signature). Since on the bifurcation line the eigenvalues of the daughter equal the squares of those of the mother, the squaring rule for signatures, i.e., Eq. (3.12), reveals that the circle-eigenvalues of the daughter have *negative* Krein signature.

Now move from point X to point Y in the diagram. Point Y was chosen to lie close to the birth line of the unperturbed daughter orbit and therefore, going from X to Y, nothing critical happens to the circle-eigenvalues of the daughter. They will therefore keep their *negative* Krein signature.

Third step

We finally decrease the coupling coefficient K back to zero (remaining at point Y). Due to the suitable choice of point Y the circle-eigenvalues will still keep their *negative* Krein signature and the cross-eigenvalues will again coincide with them for $K = 0$. The two (sub) systems are now completely identical again, and hence the cross-eigenvalues must have *negative* Krein signature too.

So we conclude that, in the case of a period-doubling bifurcation, the eigenvalues of the daughter orbit inherit the *opposite* Krein signature of the eigenvalues of the mother orbit.

In the same way one can show that for an equal period bifurcation (eigenvalues at $+1$) the Krein signatures of mother and daughter are *the same*.

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