A new computer method for temperature measurement based on an optimal control problem

N. Damean*, Z. Houkes, P.P.L. Regtien

University of Twente, Dept. of Electrical Engineering (BSC/MI), P.O. Box 217, 7500 AE, Enschede, The Netherlands

Received 19 September 1995; revised 30 October 1995

Abstract

A new computer method to measure extreme temperatures is presented. The method reduces the measurement of the unknown temperature to the solving of an optimal control problem, using a numerical computer. Based on this method, a new device for temperature measurement is built. It consists of a hardware part that includes some standard temperature sensors and it also has a software section.

The problem of temperature measurement, according to this new method, is mathematically modelled by means of the one-dimensional heat equation, with boundary and initial conditions, describing the heat transfer through the device.

The principal hardware component of the new device is a rod. The variation of the temperature which is produced near one end of the rod is determined using some temperature measurements in the other end of the rod and the new computer method which is described in this work.

This device works as an attenuator of high temperatures and as an amplifier of low temperatures. In fact, it realizes an extension of the standard working range of temperature sensors at very high and very low values.

The mathematical model of the device and the computer method are explained in detail and some possible practical implementations and a collection of simulations are also presented.

0. Nomenclature

C_1	Thermal exchange coefficient between the rod and the measurement medium [m^{-1}]
d	Diameter of the rod [mm]
$H^S(\Omega)$, ($S \in \mathbb{N}$)	Space of the functions $Y \in L^2(\Omega)$ which have all the derivatives up to the order S belonging to $L^2(\Omega)$
Inf	Greatest lower bound
L	Length of the rod [mm]
L_1	Length of the part of the rod on which the temperature sensors are not placed [mm]
$L^P(\Omega)$, ($P \in \mathbb{N}^*$)	Space of the functions P -integrable on Ω
m	Number of the division parts of T , $m \in \mathbb{N}^*$
$\mathbb{N}^*$	Set of the positive integers without zero
n	Number of the division parts of L , $n \in \mathbb{N}^*$
$\mathbb{R}_+^*$	Set of the positive real numbers without zero
$\mathbb{R}^n$	Linear space of n dimension on the real corp
T	Duration of contact between the hot end and the point where the temperature is measured [s]

* Corresponding author.

t	Time [s]
U	Unknown temperature to be measured [K]
$U = [M_1, M_2]$	Temperature range including the unknown temperature U [K]
v	Number of the temperature sensors, $v \in \mathbb{N}^*$
x	Distance along the rod relating to the hot end [mm]
$Y(t, x)$	Temperature of the rod at the moment t in the point x [K]
$Y^m(t, x)$	Temperature of the rod at the moment t of a point x which is measured by the temperature sensors [K]
α	Thermal conductivity coefficient of the rod [$\text{m}^2 \text{s}^{-1}$]
Δt	Time step [s]
Δx	Space step [m]

1. Introduction

It is known that for temperature measurement there are different kinds of transducers. They are divided into two main classes: modulating transducers and self-generating transducers. Thermistors, resistance-temperature detectors and semiconductor temperature sensors are in the first class. Thermocouples are important elements in the second class (see [1, 2]).

All these standard transducers have something in common; they can only measure standard temperatures, in the range 200–2000 K, with good accuracy.

Pyrometric methods are used for measurement of high temperatures out of this range. Unfortunately, some integration effects appear using these methods, so practical results are disturbed by a lot of errors.

Some cryogenic temperature sensors were developed in the last decade. However, the technical feasibility of these sensors and their ability to give good metrological performances may be adversely affected by a broad working range (see [3, 4]).

At the same time, some special precautionary measures must be taken so that the temperature measurement instruments are not destroyed in the two cases mentioned above.

Thus, it is necessary to develop new sensors and methods for extreme, i.e. very high and very low, temperature measurements.

With this aim, the starting point of this work was the need to find a new method for the measurement of the temperature of molten metal in the melting pot.

In this work the authors propose a new method to measure extreme temperatures using standard sensors in their usual range, where their behaviour is known with great accuracy. One possible way to obtain this extension is to use an adequate software tool.

Therefore, the authors consider the problem in which the unknown temperature is not determined directly using temperature sensors. It is determined through the formulation of the measurement problem as an optimal control problem. This last problem will be solved using a numerical algorithm.

The principal component of the device for which a new computer method is developed is a long cylindrical rod with a diameter as small as possible. This rod is made of a material chosen to satisfy the following requirements: the melting temperature is higher than the temperature to be measured; its variations of thermal conductivity coefficient and thermal exchange coefficient with temperature should be known and should be as small as possible; its thermal conductivity coefficient should be as big as possible so that the flow of heat passes rapidly through the rod; it should be perfectly homogeneous.

The rod is covered by a thermic insulating layer made of ceramic fibre. If necessary, a protective cover against corrosive action of the measurement medium may also be added.

One end of the rod, called the 'hot end' will be in contact with the point at which the temperature is measured. This end is covered with a detachable lid made of a thermic insulating material and a protective cover against corrosion.

The other end of the rod is named the 'cold end'. In its vicinity, a small distance along a generating line of the rod, a number v of standard temperature sensors, e.g. thermocouples, are placed equidistantly. They will be stuck directly on the rod under the thermic insulating layer. The temperature sensors are coupled with a computer.

Fig. 1. The scheme of the device.

The scheme of this device is shown in Fig. 1.

The operation of this device will be explained using the assumption that the unknown temperature U is higher than the temperature of the environment, and that it is constant during the time of measurement T .

This device is introduced into the medium of measurement so that its hot end is in the measurement point and the lid is removed for a short time T . During this period, temperature versus time variations are measured with the v sensors near the cold end. In fact, the hot end is excited by a step function having value U as amplitude, and its influence over a vicinity of the cold end is observed.

To compute U , an optimal control problem is formulated.

The cost function of this is the quadratic mean error between the values which are calculated depending on U and those which are measured in a vicinity of the cold end, during the time of measurement T . The unknown temperature U is the admissible control that minimizes the cost function.

Thus, the value of U is computed from the measured temperatures by the v sensors as the solution of an optimal control problem.

It is clear from the above description that the temperature of the cold end will be much lower than the measured temperature U . Therefore, it can be considered that this device performs the function of a temperature attenuator.

The central idea of the measurement process is the calculation of the unknown temperature U using the temperature variations measured by the v sensors during the time of measurement T .

Obviously, the device works in a similar manner when the temperature to be measured is lower than the temperature of the environment. In this case it can be considered that it performs the function of a temperature amplifier.

2. Theoretical approach

2.1. Mathematical modelling

The phenomenon described above is mathematically modelled by a partial differential equation of parabolic type, called the heat equation,

$$Y_t(t, x) - \alpha Y_{xx}(t, x) = 0, \quad \text{for } (t, x) \in (0, T) \times (0, L), \quad (1)$$

with the boundary conditions, of the Newton type,

$$Y_x(t, 0) = C_1(Y(t, 0) - U), \quad \text{for } t \in [0, T], \quad (2)$$

$$Y_x(t, L) = 0, \quad \text{for } t \in [0, T], \quad (3)$$

and the initial condition,

$$Y(0, x) = Y_0(x), \quad \text{for } x \in [0, L], \quad (4)$$

$\alpha \in \mathbb{R}_+^*$, $C_1 \in \mathbb{R}_+^*$, $Y_0 \in H^1(0, L)$ (see [5]).

Eq. (1) and the conditions in Eqs. (2) and (3) show that there is no loss of heat through the exterior surface and the cold end of the rod. In fact, only the hot end comes into contact with the measurement medium.

The presence of the unknown temperature U is observed in the boundary condition in Eq. (2).

The condition in Eq. (4) shows that it is necessary to know the initial temperature distribution of the rod. In many practical situations, $Y_0(x)$ is independent of x ; in other words:

$$Y_0(x) = \text{constant} = \text{environment temperature.}$$

The existence and the regularity results for the problem in Eqs. (1)–(4) are well known from [6]. For instance, the unique solution of it, denoted by $Y(t, x)$, is in $H^1((0, T) \times (0, L))$.

The problem is to compute U , using the mathematical model and the set of measured temperatures in a vicinity of the cold end of the rod, during the time of measurement T .

At first sight, this problem is inverse parabolic in one space dimension with Newton-type boundary conditions.

At the same time, the least square approach to this problem shows that it could be considered in terms of an optimal control problem, denoted by (P), with the following typical elements, as they were defined in [7]:

- The equations of the state system are given by Eqs. (1)–(4).
- The class of the admissible controls is $U = \{u | u \in [M_1, M_2]\}$, where M_1 and M_2 are chosen from the beginning by physical limitations on the problem to be extreme values where the unknown temperature U can be situated.
- The cost function to be minimized is

$$\Phi(u) = \int_0^T \int_{L_1}^L [Y(t, x) - Y^m(t, x)]^2 dx dt \quad (5)$$

for $Y^m \in L^2((0, T) \times (L_1, L))$, $u \in U$. Quantity $Y(t, x)$ is the solution of the above problem in Eqs. (1)–(4) which corresponds to u , $Y^m(t, x)$ is the temperature of a point near the cold end which is measured by the standard temperature sensors. The strict convex function Φ displayed above has the sense of the quadratic mean deviation between the values which are calculated and those which are measured in the vicinity of the cold end.

The optimal control problem (P) can now be formulated as:

$$\text{find an admissible control } U^* \in [M_1, M_2] \text{ that minimizes the cost function } \Phi. \quad (\text{P})$$

There are different methods to treat this optimal control problem. One of these methods, based on a variational approach, will be developed in the following subsection. Another way that could be used is based on the abstract differential Riccati equation (see [8]).

We note the complexity of Eqs. (1)–(4) (control U is in a Newton-type boundary condition corresponding to a partial differential equation of parabolic type). Therefore, the variational approach seems to be more attractive from the theoretical and the practical points of view.

2.2. Variational solution

Concerning the enunciated optimal problem (P), it was proved in [9] that it admits only one solution.

Thus, there exists a pair $\{U^*, Y^*(t, x)\}$, so that $U^* \in U$ is the optimal control of the problem (P) and $Y^*(t, x)$ is the optimal state which corresponds to U^* . This is equivalent to:

$$\Phi(U^*) = \int_0^T \int_{L_1}^L [Y^*(t, x) - Y^m(t, x)]^2 dx dt = \inf_{u \in U} \Phi(u).$$

In the following, we take into account the development of a descent algorithm (see [10]). For that we need to establish the necessary conditions for optimisation.

Consider an admissible pair $\{u, Y(t, x)\}$ from the adequate equations (1)–(4) of the state system and the corresponding pair of variations $\{u + \epsilon z, Y(t, x) + \epsilon W(t, x)\}$ when control u is perturbed by a small quantity ϵz . In the previous pair, ϵ is a positive real number so that $u + \epsilon z$ is an admissible control.

Subtracting adequate equations (1)–(4) from the corresponding equations when the pair of variations $\{u + \epsilon z, Y(t, x) + \epsilon W(t, x)\}$ occur, the following parabolic system is obtained:

$$W_t(t, x) - \alpha W_{xx}(t, x) = 0, \quad \text{for } (t, x) \in (0, T) \times (0, L), \quad (6)$$

$$W_x(t, 0) = C_1(W(t, 0) - z), \quad \text{for } t \in [0, T], \quad (7)$$

$$W_x(t, L) = 0, \quad \text{for } t \in [0, T], \quad (8)$$

$$W(0, x) = 0, \quad \text{for } x \in [0, L]. \quad (9)$$

Define the cost function:

$$J(g) = \int_0^T \int_{L_1}^L [Y(t, x) - Y^m(t, x)]^2 dx dt + I_U(g), \quad \text{for } g \in U, \quad (10)$$

where I_U is the indicator function of the set U .

Consider U^* to be the optimal control of the problem (P).

Hence:

$$\frac{J(U^* + \epsilon z) - J(U^*)}{\epsilon} \geq 0, \quad \text{for } \epsilon > 0.$$

Developing this formula according to Eq. (10) and letting $\epsilon \rightarrow 0$ we obtain:

$$2z \int_0^T \int_{L_1}^L [Y^*(t, x) - Y^m(t, x)] dx dt + I'_U(U^*, z) \geq 0, \quad \text{for } z \in U. \quad (11)$$

In the relation in Eq. (11), I'_U denotes the derivative of I_U .

The adjoint state system is now introduced:

$$A_t(t, x) + \alpha A_{xx}(t, x) = Y^*(t, x) - Y^m(t, x), \quad \text{for } (t, x) \in (0, T) \times [L_1, L], \quad (12)$$

$$A_t(t, x) + \alpha A_{xx}(t, x) = 0, \quad \text{for } (t, x) \in (0, T) \times (0, L_1),$$

$$A_x(t, 0) = C_1 A(t, 0), \quad \text{for } t \in [0, T], \quad (13)$$

$$A_x(t, L) = 0, \quad \text{for } t \in [0, T], \quad (14)$$

$$A(T, x) = 0, \quad \text{for } x \in [0, L]. \quad (15)$$

The existence and the uniqueness of the solution $A(t, x)$ corresponding to the adjoint state system are proved in [11].

We now multiply Eq. (6) by $A(t, x)$ and we integrate on $(0, T) \times (0, L)$. Using the integration by parts formula and Eq. (8), the following relation is carried out:

$$\begin{aligned} \int_0^T \int_0^L W_t(t, x) A(t, x) dx dt - \alpha \int_0^T \int_0^L A_{xx}(t, x) W(t, x) dx dt \\ + \alpha \int_0^T [W_x(t, 0) A(t, 0) - W(t, 0) A_x(t, 0)] dt = 0. \end{aligned} \quad (16)$$

A simple calculation shows also that

$$\int_0^L W_t(t, x) A(t, x) dx = \frac{d}{dt} \left[\int_0^L W(t, x) A(t, x) dx \right] - \int_0^L W(t, x) A_t(t, x) dx. \quad (17)$$

Now, using the formulae in Eqs. (9), (12), (15) and (17), the relation in Eq. (16) becomes:

$$- \int_0^T \int_{L_1}^L W(t, x) [Y^*(t, x) - Y^m(t, x)] dx dt + \alpha \int_0^T [W_x(t, 0) A(t, 0) - W(t, 0) A_x(t, 0)] dt = 0. \quad (18)$$

Introducing Eqs. (7) and (13) into Eq. (18), results in

$$- \int_0^T \int_{L_1}^L W(t, x) [Y^*(t, x) - Y^m(t, x)] dx dt - \alpha C_1 z \int_0^T A(t, x) dt = 0. \quad (19)$$

Joining Eq. (11) with Eq. (19) produces

$$- 2\alpha C_1 z \int_0^T A(t, x) dt + I'_U(U^*, z) = 0, \quad \text{for } z \in U. \quad (20)$$

Hence, from Eq. (20):

$$2\alpha C_1 A(t, 0) \in \partial I'_U(U^*),$$

where ∂ denotes the subdifferential and, finally, the optimal control U^* is the argument which minimizes the function Φ on U_0 .

The closed interval denoted by U_0 is determined of admissible control u considered earlier and u^* is defined as

$$u^* = \begin{cases} M_2, & \text{if } \int_0^T A(t, 0) dt > 0 \\ M_1, & \text{if } \int_0^T A(t, 0) dt < 0, \end{cases} \quad (21)$$

where $A(t, 0)$ is computed with adequate equations (12)–(15) when the admissible pair $\{u, Y(t, x)\}$ is used.

The relation in Eq. (21) with Eqs. (12)–(15) make the necessary conditions for optimisation.

Therefore, the results presented in this subsection show that the entire problem has only a solution in accordance with physical reality.

At the same time these suggest the possibility for numerical implementation which will be developed in the following section.

3. Numerical approach

For the numerical calculation of the temperature U , which is an optimal control problem, as seen before, one type of descent algorithm will be used (see [10, 12]).

Using Eq. (21) as the main equation, the following iterative algorithm is developed (k being the iteration number):

Step 0: Choose $u^{(0)} \in U$; set $k := 0$.

The algorithm is initialized with an admissible control and a counter of the iterative algorithm is set to zero.

Step 1: Compute $Y^{(k)}(t, x)$, i.e. the state system described by Eqs. (1)–(4):

$$Y_t^{(k)}(t, x) - \alpha Y_{xx}^{(k)}(t, x) = 0, \quad \text{for } (t, x) \in (0, T) \times (0, L), \quad (22)$$

$$Y_x^{(k)}(t, 0) = C_1(Y^{(k)}(t, 0) - u^{(k)}), \quad \text{for } t \in [0, T], \quad (23)$$

$$Y_x^{(k)}(t, L) = 0, \quad \text{for } t \in [0, T], \quad (24)$$

$$Y^{(k)}(0, x) = Y_0(x), \quad \text{for } x \in [0, L]. \quad (25)$$

Step 2: Compute $A^{(k)}(t, x)$, i.e. the adjoint state system described by Eqs. (12)–(15):

$$A_t^{(k)}(t, x) + \alpha A_{xx}^{(k)}(t, x) = Y^{(k)}(t, x) - Y^m(t, x), \quad \text{for } (t, x) \in (0, T) \times [L_1, L],$$

$$A_t^{(k)}(t, x) + \alpha A_{xx}^{(k)}(t, x) = 0, \quad \text{for } (t, x) \in (0, T) \times (0, L_1),$$

$$A_x^{(k)}(t, 0) = C_1 A^{(k)}(t, 0), \quad \text{for } t \in [0, T],$$

$$A_x^{(k)}(t, L) = 0, \quad \text{for } t \in [0, T],$$

$$A^{(k)}(T, x) = 0, \quad \text{for } x \in [0, L].$$

Step 3: Use the relation in Eq. (21) to compute the optimal control at iteration k , denoted as $u^{*(k)}$, from:

$$u^{*(k)} = \begin{cases} M_2, & \text{if } \int_0^T A^{(k)}(t, 0) dt > 0 \\ M_1, & \text{if } \int_0^T A^{(k)}(t, 0) dt \leq 0. \end{cases}$$

An approximation of the optimal control problem (P) is derived.

Step 4: Compute $\delta^{(k)} \in [0, 1]$ as a solution of the minimization process:

$$\min \{ \Phi(\delta u^{(k)} + (1 - \delta)u^{*(k)}); \quad \delta \in [0, 1] \},$$

where Φ is the cost function of the problem in Eqs. (1)–(4) given by the relation in Eq. (5).

It is verified if the optimal control computed in Step 3 minimizes the cost function.

Step 5: Set:

$$u^{(k+1)} := \delta^{(k)} u^{(k)} + (1 - \delta^{(k)}) u^{*(k)}.$$

A new admissible control $u^{(k+1)}$ placed between $u^{(k)}$ and $u^{*(k)}$ is computed.

Step 6: ('the stopping criterion'):

```

IF  $|u^{(k+1)} - u^{(k)}| \leq \epsilon_s$ 
  THEN  $U^* = u^{(k+1)}$ ;
  STOP.
ELSE  $k := k + 1$ ;
  GO TO Step 1.

```

The 'stopping criterion' in Step 6 can also be
Step 6':

```

IF  $|\Phi(u^{(k+1)}) - \Phi(u^{(k)})| \leq \epsilon'_s$ 
  THEN  $U^* = u^{(k+1)}$ ;
  STOP.
ELSE  $k := k + 1$ ;
  GO TO Step 1.

```

Symbols ϵ_s and ϵ'_s from Steps 6 and Step 6' denote two small positive real values chosen in accordance with practical requirements.

As observed from the practical implementation of this algorithm, there are difficulties in trying to find a good 'guess' of $u^{(0)}$ in Step 0.

To avoid this, a local variation procedure is used in Step 0, namely the azimuth mark method (AMM) (see [13]).

As explained further, constant controls are used. Their form is $u_0 = M_1 + Q$, $Q \in \mathbb{R}_+^*$, so that $u_0 \in U$.

To implement this procedure, Step 0 is divided into two parts presented in the following.

Step 0.1: The class of admissible controls $[M_1, M_2]$ is divided into 8 equal parts. The values of the cost function in the 9 division ends are computed using the formula in Eq. (5). These values form the vector φ .

```

 $\Delta U := (M_2 - M_1)/8$ ;
DO - FOR  $l = 0, 1, \dots, 8$ 
   $Q := l \Delta U$ 
   $u_0 := M_1 + Q$ 
  Compute  $\Phi(u_0)$  using (5)
   $\varphi(l) = \Phi(u_0)$ 
END - DO.

```

Clearly, the number of parts must be in accordance with the physical implementation of the algorithm.

Step 0.2: Locate the best two values of vector φ (where the cost function has the smallest values) and denote the corresponding values of admissible control by u_1 and u_2 . The centre c of an azimuth mark (AM) and its radius r are computed.

Find l_1 and l_2 from:

```

 $\varphi(l_1) := \min\{\varphi(l) | l = 0, 1, \dots, 8\}$ 
 $\varphi(l_2) := \min\{\{\varphi(l) | l = 0, 1, \dots, 8\} - \{\varphi(l_1)\}\}$ 
 $u_1 := M_1 + l_1 \Delta U$ ;
 $u_2 := M_2 + l_2 \Delta U$ ;

```


$$c := \frac{(u_1 + u_2)}{2};$$

$$r := \frac{|u_1 - u_2|}{2};$$

Start the AMM.

Some considerations about AMM are now presented.

We denote the AM having centre c and radius r by $M_{c,r} = \{c-r, c, c+r\}$. We denote the set of ends of the AM by $E(M_{c,r}) = \{c-r, c+r\}$. We use $c(i)$ to denote the value of c at iteration i of this local variation procedure and we use $r(i)$ to denote the value of r at the same iteration.

AM $M_{c(i),r(i)}$ is of type 1, and it is denoted by $M_{c(i),r(i)}^1$ if there is a $w \in E(M_{c(i),r(i)})$ so that $\Phi(w) < \Phi(c(i))$. (It means that the argument which minimizes the cost function on $M_{c(i),r(i)}$ is contained in the subset $E(M_{c(i),r(i)})$.) The function Φ is given by Eq. (5) as the cost function of the problem represented in Eqs. (1)–(4), but in a general case it can be a strict convex function.

AM $M_{c(i),r(i)}$ is of type 2, and is denoted as $M_{c(i),r(i)}^2$ if $\Phi(w) \geq \Phi(c(i))$, for all $w \in E(M_{c(i),r(i)})$. (It means that the argument which minimizes the cost function on $M_{c(i),r(i)}$ is $c(i)$.)

Denote initial AM as $M_{c(0),r(0)}$. Define recursively the string $M_{c(n),r(n)}$ in the following manner:

- (1) if $M_{c(i),r(i)}^1$, $c(i+1)$ will be set as one of the points of $E(M_{c(i),r(i)})$ where the minimum of Φ is achieved, and $r(i+1) = r(i)$;
- (2) if $M_{c(i),r(i)}^2$, $\Gamma_{(i)}$ is considered to be a new AM having $c(i)$ as an end, the other end being one of the points $w \in E(M_{c(i),r(i)})$ where a minimum of Φ on $E(M_{c(i),r(i)})$ is achieved. The radius of $\Gamma_{(i)}$ is $r = r(i)/2$.

Using previous notations to mark the type of AM, there are 2 cases:

(2.1) if $\Gamma_{(i)}^1$, it is set $c(i+1) = (c(i) + w)/2$ and $r(i+1) = r(i)/2$. These define $M_{c(i+1),r(i+1)}$;

(2.2) if $\Gamma_{(i)}^2$, it is set $c(i+1) = c(i)$ and $r(i+1) = r(i)/2$. These define $M_{c(i+1),r(i+1)}$.

Therefore, this technique shows that if $M_{c(i),r(i)}^1$ when the method is applied, and a $M_{c(j),r(j)}^2$ is obtained after a finite number of iterations, the last AM is left in 2 stages, and a new AM is obtained that has a reduced radius.

Other specifications, as the convergence of this technique, are displayed in [13].

Two stop criterions which could be used for this procedure are:

IF $|r(i+1) - r(i)| \leq \epsilon_m$

THEN STOP AMM.

ELSE $i := i + 1$;

CONTINUE AMM;

and

IF $i = i_0$

THEN STOP AMM.

ELSE $i := i + 1$;

CONTINUE AMM;

where ϵ_m and i_0 are real and natural numbers, respectively. They are specified at the beginning of the algorithm.

Using the first stop criterion of AMM, it is noted that $u^{(0)}$ is a real number in the interval $(c(i) - r(i), c(i) + r(i))$. A common choice used to initialize this algorithm is $u^{(0)} = c(i)$.

Steps 1 and 2 are solved using some implicit finite difference approximations, namely the O'Brian et al. formula and the Crank–Nicolson formula (see [14]). Boundary conditions impose the need to use a false boundary for practical implementation of these formulae.

Step 3 does not cause any serious problems from an applicative point of view.

Concerning Step 4, at the beginning, we used a carefully supervised loop with the form $(n\delta \in \mathbb{N}^*)$:

```

 $\Delta\delta := 1/n\delta;$ 
DO – FOR  $l = 0, 1, \dots, n\delta$ 
   $\delta := l \Delta\delta$ 
  Compute  $\Phi[\delta u^{(k)} + (1 - \delta)u^{*(k)}]$  using (5)
END – DO.

```

This procedure, with constant length steps to solve a minimization problem, could be improved (from a computational time point of view) using variable length steps.

So, later, we used a carefully supervised loop of the form ($m\delta \in \mathbb{N}^*$):

```

DO – FOR  $l = 0, 1, \dots, m\delta$ 
   $\delta := \sum_{k=1}^l (1/2)^k$ 
  Compute  $\Phi[\delta u^{(k)} + (1 - \delta)u^{*(k)}]$  using (5)
END – DO.

```

Between $n\delta$ and $m\delta$ there is the following relation:

$$2^{m\delta} = n\delta \quad (26)$$

obtained from the condition to have the same smallest length step in the two loops.

This means that the second loop has a smaller number of steps than the first. Generally, the second procedure is faster but less accurate than the first procedure.

The double integral from Step 4 as well as Steps 5 and 6, (Step 6') do not lead to potential problems from a numerical point of view.

4. Implementation of the algorithm

In the following, we will treat Steps 1 and 2 only, because the others could be implemented without difficulties as was seen in the previous section.

Steps 1 and 2 are solved using some implicit finite difference approximations; the grids in effect with equidistant knots are:

$$\begin{aligned}
 0 &= t_1 < t_2 < \dots < t_{m+1} = T, \\
 x_0 < 0 &= x_1 < x_2 < \dots < x_{n+1} = L.
 \end{aligned}$$

The existence of a 'false' boundary denoted by x_0 is observed in the previous relation.

In the following, we will present a detailed algorithm using the O'Brian et al. formula.

The implementation of the algorithm and the results obtained when the Crank–Nicolson formula is used are similar (see [14]).

Denote:

$$\begin{aligned}
 y(j, i) &= Y^{(k)}(t_j, x_i), \quad \text{for } j = 1, 2, \dots, m+1, \quad i = 0, 1, \dots, n+1, \\
 u &= u^{(k)}
 \end{aligned} \quad (27)$$

The following usual discretizations are performed:

$$Y_t^{(k)}(t, x)|_{j,i} \approx \frac{y(j, i) - y(j-1, i)}{\Delta t}, \quad \text{for } j = 1, 2, \dots, m+1, \quad i = 0, 1, \dots, n+1, \quad (28)$$

$$Y_{xx}^{(k)}(t, x)|_{j,i} \approx \frac{y(j, i-1) - 2y(j, i) + y(j, i+1)}{\Delta x^2}, \quad \text{for } j = 1, 2, \dots, m+1, \quad i = 1, 2, \dots, n, \quad (29)$$

$$Y_x^{(k)}(t, 0)|_{j,1} \approx \frac{y(j, 1) - y(j, 0)}{\Delta x}, \quad \text{for } j = 1, 2, \dots, m+1, \quad (30)$$

$$Y_x^{(k)}(t, L)|_{j,n+1} \approx \frac{y(j, n+1) - y(j, n)}{\Delta x}, \quad \text{for } j = 1, 2, \dots, m+1, \quad (31)$$

where

$$\Delta t = \frac{T}{m}, \quad \Delta x = \frac{L}{n}, \quad C = \frac{\Delta t}{\Delta x^2}. \quad (32)$$

Using these discretizations, the value of $Y^{(k)}(t, x)$ is obtained in Step 1.

Some preparations are necessary for this aim.

The relations in Eqs. (22), (27)–(29) and (32) imply:

$$-\alpha C y(j, i-1) + (1 + 2\alpha C) y(j, i) - \alpha C y(j, i+1) = y(j-1, i), \\ \text{for } j = 2, 3, \dots, m+1, \quad i = 1, 2, \dots, n. \quad (33)$$

Using Eqs. (23), (27) and (30), we get

$$y(j, 0) = (1 - C_1 \Delta x) y(j, 1) + C_1 \Delta x u, \quad \text{for } j = 1, 2, \dots, m+1. \quad (34)$$

The relations in Eqs. (24) and (31) make

$$y(j, n+1) = y(j, n), \quad \text{for } j = 1, 2, \dots, m+1. \quad (35)$$

Using Eq. (27), the following discretization is obtained for the relation in Eq. (25):

$$y(1, i) = Y_0(x_i), \quad \text{for } i = 1, 2, \dots, n+1. \quad (36)$$

Using Eq. (34) and the relation in Eq. (33), particularized for $i = 1$, the following result is obtained:

$$(\alpha C C_1 \Delta x + \alpha C + 1) y(j, 1) - \alpha C y(j, 2) = y(j-1, 1) + \alpha C C_1 \Delta x u, \quad \text{for } j = 2, 3, \dots, m+1. \quad (37)$$

Using Eq. (35) and the relation in Eq. (33), particularized for $i = n$, the following result is obtained:

$$-\alpha C y(j, n-1) + (1 + \alpha C) y(j, n) = y(j-1, n), \quad \text{for } j = 2, 3, \dots, m+1. \quad (38)$$

Thus, the value of $Y^{(k)}(t, x)$ is carried out in Step 1 as follows:

- from Eqs. (33), (37) and (38), the quantities $y(j, i)$, for $j = 2, 3, \dots, m+1$, $i = 1, 2, \dots, n$, are computed solving the following system of linear equations:

$$\begin{bmatrix} \alpha C C_1 \Delta x + \alpha C + 1 & -\alpha C & 0 & 0 & \dots & 0 & 0 & 0 \\ -\alpha C & 1 + 2\alpha C & -\alpha C & 0 & \dots & 0 & 0 & 0 \\ 0 & -\alpha C & 1 + 2\alpha C & -\alpha C & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -\alpha C & 1 + 2\alpha C & -\alpha C \\ 0 & 0 & 0 & 0 & \dots & 0 & -\alpha C & 1 + 2\alpha C \end{bmatrix} \begin{bmatrix} y(j, 1) \\ y(j, 2) \\ y(j, 3) \\ \vdots \\ y(j, n-1) \\ y(j, n) \end{bmatrix} = \begin{bmatrix} y(j-1, 1) + \alpha C C_1 \Delta x u \\ y(j-1, 2) \\ y(j-1, 3) \\ \vdots \\ y(j-1, n-1) \\ y(j-1, n) \end{bmatrix};$$

- $y(j, n + 1)$ is computed from the relation in Eq. (35), for $j = 2, 3, \dots, m + 1$;
- $y(1, i)$ is computed from the relation in Eq. (36), for $i = 1, 2, \dots, n + 1$.

In a similar manner, as in Step 1, we will develop Step 2.

Denote:

$$a(j, i) = A^{(k)}(t_j, x_i), \quad \text{for } j = 1, 2, \dots, m + 1, \quad i = 0, 1, \dots, n + 1.$$

The value of $A^{(k)}(t, x)$ is obtained as follows:

- the quantities $a(j, i)$, for $j = 1, 2, \dots, m, i = 2, 3, \dots, n + 1$, are computed by solving the following system of linear equations:

$$\begin{bmatrix} \frac{-\alpha C}{1 + C_1 \Delta x} + 1 + 2\alpha C & -\alpha C & 0 & 0 & \dots & 0 & 0 & 0 \\ -\alpha C & 1 + 2\alpha C & -\alpha C & 0 & \dots & 0 & 0 & 0 \\ 0 & -\alpha C & 1 + 2\alpha C & -\alpha C & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -\alpha C & 1 + 2\alpha C & -\alpha C \\ 0 & 0 & 0 & 0 & \dots & 0 & -\alpha C & 1 + 2\alpha C \end{bmatrix} \begin{bmatrix} a(j, 2) \\ a(j, 3) \\ a(j, 4) \\ \vdots \\ a(j, n) \\ a(j, n + 1) \end{bmatrix} = \begin{bmatrix} a(j + 1, 2) - \Delta t w(j, 2) \\ a(j + 1, 3) - \Delta t w(j, 3) \\ a(j + 1, 4) - \Delta t w(j, 4) \\ \vdots \\ a(j + 1, n) - \Delta t w(j, n) \\ a(j + 1, n + 1) - \Delta t w(j, n + 1) \end{bmatrix},$$

where

$$w(j, i) = \begin{cases} Y^{(k)}(t_j, x_i) - Y^m(t_j, x_i), & \text{for } (t_j, x_i) \in (0, T) \times [L_1, L) \\ 0 & \text{for } (t_j, x_i) \in (0, T) \times (0, L_1); \end{cases}$$

- $a(j, 1) = a(j, 2)/(1 + C_1 \Delta x)$, for $j = 1, 2, \dots, m + 1$;
- $a(m + 1, i) = 0$, for $i = 2, 3, \dots, n + 1$.

The matrix of both systems is a band matrix (the bandwidth is 3), therefore both systems are solved by a special Gaussian routine.

5. Results

For numerical tests, it is considered that the rod is made from tungsten, having the following characteristics: $C_1 = 2.2413 \cdot 10^3 \text{ m}^{-1}$, $d = 3 \text{ mm}$, $L = 30 \text{ mm}$, $L_1 = 20 \text{ mm}$, $\alpha = 63.3 \cdot 10^{-6} \text{ m}^2 \text{ s}^{-1}$.

We also consider that $n = m = 144$.

The initial distribution of the temperature in the rod is:

$$Y_0(x) = 300 \text{ K}, \quad \text{for } x \in [0, L].$$

The following simulations are achieved.

5.1. Computation of temperature variation of rod versus time

The first 2 steps of the above algorithm are used, with some minor modifications.

The results are displayed in Fig. 2 when the temperature of the hot end is $U = 3650 \text{ K}$.

The curve **c1** corresponds to the hot end, curve **c4** corresponds to the cold end and curves **c2** and **c3** correspond to points placed equidistantly on the rod, between the ends.

It is observed that the temperature in the vicinity of the cold end is lower than the temperature in the

Fig. 2. The temperature variations in the rod.

vicinity of the hot end, as was expected from the physical point of view. At the same time, the temperature of the hot end can be considered in a stationary state after 8 s, but the other temperatures are in a transitory state a long time.

This simulation could also be used to select a value for time T . The value of T must be chosen so that it is not too large because the temperature U must be constant during the measurement process and the temperature of the cold end must not be too high. At the same time, T must not be too small because some important temperature variations must be felt by the temperature sensors in the cold end.

5.2. Numerical simulations of the algorithm

Some numerical results will be analyzed in the following; they concern the entire algorithm presented above.

For this purpose, it is necessary to specify a value for the number of sensors, denoted by v .

At the beginning, this algorithm is tested using a number of sensors corresponding to some working conditions which are nearly ideal. Therefore, the case when the temperature is known at every discretization point near the cold end is analyzed. This case is named 'theoretical running'.

After that, in 'real running', consideration is given to a realistic number of temperature sensors, and the numerical results are compared with the previous results.

The value of other variables used in the algorithm are: $M_1 = 0$ K, $M_2 = 4000$ K and $T = 1$ s.

Note: Therefore, the class of admissible controls is chosen so that $U = [0, 4000]$. It covers all industrial applications. The values of M_1 and M_2 can be adjusted for a proper application, according to the real working conditions of this device.

The stopping criterion shown in Step 6 is used.

5.2.1. Theoretical running

In fact, the target of this simulation is to choose the best variant from those presented above. The problems are to decide between using constant or variable length steps in Step 4 and between using or not using AMM for initialization. Thus, we need to choose the best from 4 variants of this algorithm.

To compare these simulations from some points of view, a number of steps must be used in Step 4. The smallest length step of every variant is the same and this value is less than a half from the constant ϵ_s used in stopping criterion Step 6.

In these conditions, it is expected that the simulation error defined as

$$e_s(\%) = \frac{|U - U^*|}{U} \cdot 100$$

will be uniform in all 4 variants.

In the previous relation, U^* means the final value of U computed with the algorithm Steps 0–6, as seen above.

Using Eq. (26), this condition is displayed as

$$\frac{M_2 - M_1}{NC} = \frac{M_2 - M_1}{2^{NV}} = \frac{2\epsilon_m}{NCM} = \frac{2\epsilon_m}{2^{NVM}} < \frac{\epsilon_s}{2}. \quad (39)$$

where

NC = the number of constant length steps when AMM is not used;

NV = the number of variable length steps when AMM is not used;

NCM = the number of constant length steps when AMM is used;

NVM = the number of variable length steps when AMM is used.

To compare these simulations, it is necessary to use an ideal number of temperature sensors, as was mentioned above. This number in this present simulation is $v=49$.

A weak value for ϵ_s in Step 6 is used at this stage because a long calculus time is expected for some of these variants. Thus, $\epsilon_s=0.5$ is utilized.

According to Eq. (39), the other values are: $NC=16\,384$, $NV=14$, $NCM=128$, $NVM=7$, $\epsilon_m=15.625$.

These simulations are carried out for 14 values of unknown temperature U .

By analogy with the simulation error, the initial simulation error (when AMM is used) is defined as:

$$e_m(\%) = \frac{|U - U_m|}{U} \cdot 100.$$

In the previous relation, U_m means the value of $u^{(0)}$ computed in Step 0, using AMM.

The corresponding values of the criterion function Φ relating to U_m and U^* are denoted using Φ_m and Φ_s , respectively.

The results are presented in Table 1.

Following Table 1, it is observed that Φ_s and e_s are nearly similar in all variants for a certain temperature.

At the same time, the number of iterations of the principal algorithm can be greater when the variable length steps are used because the accuracy is lower in this case. This phenomenon appears especially when AMM is not used.

Sometimes some null values for Φ_s and e_s are obtained. These can be explained through the use of a 'favorable' length step.

These behaviours were forecasted from the beginning. According to expectation, the shortest computation time is obtained when AMM with variable length steps is used. In the same situation, the number of steps used in Step 4 is a minimum and the number of iterations of the principal algorithm is nearly a minimum.

Thus, only this variant of the algorithm, which is fast convergent and can work in real time, will be used in the following.

5.2.2. Real running

A realistic number of sensors, namely $v=3$, equidistantly placed in the vicinity of the cold end is considered in the following.

The authors suggested 2 variants for real running of the algorithm:

- only data obtained from the temperature sensors are used;
- data obtained from the temperature sensors and interpolation data for the points between the sensors are used; interpolation with spline functions is used.

Table 1
Computational results for theoretical running

U [K]	No. of iterations of AM	Φ_m	$e_m(\%)$	type step	No. iterations of principal algorithm	Φ_s	$e_s(\%)$
50	8	$0.28 \cdot 10^{-4}$	6.25	constant	2	0	$0.92 \cdot 10^{-4}$
				variable	5	$0.92 \cdot 10^{-8}$	0.11
				constant	2	$0.14 \cdot 10^{-8}$	$0.42 \cdot 10^{-1}$
				variable	2	$0.14 \cdot 10^{-8}$	$0.42 \cdot 10^{-1}$
350	7	$0.11 \cdot 10^{-3}$	1.79	constant	2	$0.37 \cdot 10^{-9}$	$0.26 \cdot 10^{-2}$
				variable	5	$0.81 \cdot 10^{-9}$	$0.46 \cdot 10^{-2}$
				constant	2	$0.63 \cdot 10^{-7}$	$0.42 \cdot 10^{-1}$
				variable	3	$0.10 \cdot 10^{-6}$	$0.53 \cdot 10^{-1}$
650	7	$0.11 \cdot 10^{-3}$	0.96	constant	2	$0.28 \cdot 10^{-7}$	$0.15 \cdot 10^{-1}$
				variable	6	$0.52 \cdot 10^{-7}$	$0.21 \cdot 10^{-1}$
				constant	2	$0.59 \cdot 10^{-7}$	$0.23 \cdot 10^{-1}$
				variable	3	$0.97 \cdot 10^{-7}$	$0.28 \cdot 10^{-1}$
950	8	$0.28 \cdot 10^{-4}$	0.33	constant	2	$0.67 \cdot 10^{-8}$	$0.51 \cdot 10^{-2}$
				variable	4	$0.77 \cdot 10^{-7}$	$0.17 \cdot 10^{-1}$
				constant	2	$0.10 \cdot 10^{-8}$	$0.22 \cdot 10^{-2}$
				variable	2	$0.10 \cdot 10^{-8}$	$0.22 \cdot 10^{-2}$
1250	7	0	0	constant	2	0	0
				variable	7	$0.65 \cdot 10^{-9}$	$0.13 \cdot 10^{-2}$
				constant	0	0	0
				variable	0	0	0
1550	8	$0.28 \cdot 10^{-4}$	0.20	constant	2	$0.64 \cdot 10^{-8}$	$0.30 \cdot 10^{-2}$
				variable	5	$0.14 \cdot 10^{-8}$	$0.14 \cdot 10^{-2}$
				constant	2	$0.14 \cdot 10^{-8}$	$0.14 \cdot 10^{-2}$
				variable	2	$0.14 \cdot 10^{-8}$	$0.14 \cdot 10^{-2}$
1850	7	$0.11 \cdot 10^{-3}$	0.34	constant	2	$0.16 \cdot 10^{-8}$	$0.13 \cdot 10^{-2}$
				variable	4	$0.49 \cdot 10^{-8}$	$0.23 \cdot 10^{-2}$
				constant	2	$0.61 \cdot 10^{-7}$	$0.79 \cdot 10^{-2}$
				variable	3	$0.96 \cdot 10^{-7}$	$0.10 \cdot 10^{-1}$
2150	8	$0.11 \cdot 10^{-3}$	0.29	constant	2	$0.14 \cdot 10^{-8}$	$0.11 \cdot 10^{-2}$
				variable	4	$0.48 \cdot 10^{-8}$	$0.20 \cdot 10^{-2}$
				constant	2	$0.61 \cdot 10^{-7}$	$0.68 \cdot 10^{-2}$
				variable	3	$0.96 \cdot 10^{-7}$	$0.86 \cdot 10^{-2}$
2450	8	$0.28 \cdot 10^{-4}$	0.13	constant	2	$0.62 \cdot 10^{-8}$	$0.19 \cdot 10^{-2}$
				variable	5	$0.15 \cdot 10^{-8}$	$0.91 \cdot 10^{-3}$
				constant	2	$0.12 \cdot 10^{-8}$	$0.87 \cdot 10^{-3}$
				variable	2	$0.12 \cdot 10^{-8}$	$0.87 \cdot 10^{-3}$
2750	7	0	0	constant	2	0	0
				variable	6	$0.22 \cdot 10^{-10}$	$0.15 \cdot 10^{-3}$
				constant	0	0	0
				variable	0	0	0
3050	8	$0.28 \cdot 10^{-4}$	0.10	constant	2	$0.67 \cdot 10^{-8}$	$0.16 \cdot 10^{-2}$
				variable	4	$0.78 \cdot 10^{-7}$	$0.54 \cdot 10^{-2}$
				constant	2	$0.12 \cdot 10^{-8}$	$0.70 \cdot 10^{-3}$
				variable	2	$0.12 \cdot 10^{-8}$	$0.70 \cdot 10^{-3}$

Table 1 (Contd.)

U [K]	No. of iterations of AM	Φ_m	$e_m(\%)$	Type step	No. iterations of principal algorithm	Φ_s	$e_s(\%)$
3350	7	$0.11 \cdot 10^{-1}$	0.19	constant	2	$0.27 \cdot 10^{-7}$	$0.29 \cdot 10^{-2}$
				variable	6	$0.50 \cdot 10^{-7}$	$0.40 \cdot 10^{-2}$
				constant	2	$0.63 \cdot 10^{-7}$	$0.44 \cdot 10^{-2}$
				variable	3	$0.97 \cdot 10^{-7}$	$0.55 \cdot 10^{-2}$
3650	7	$0.11 \cdot 10^{-3}$	0.17	constant	2	$0.20 \cdot 10^{-9}$	$0.25 \cdot 10^{-3}$
				variable	5	$0.83 \cdot 10^{-9}$	$0.44 \cdot 10^{-3}$
				constant	2	$0.61 \cdot 10^{-7}$	$0.40 \cdot 10^{-2}$
				variable	3	$0.97 \cdot 10^{-7}$	$0.51 \cdot 10^{-2}$
3950	8	$0.28 \cdot 10^{-4}$	$0.79 \cdot 10^{-1}$	constant	2	0	0
				variable	5	$0.11 \cdot 10^{-7}$	$0.15 \cdot 10^{-2}$
				constant	2	$0.90 \cdot 10^{-9}$	$0.54 \cdot 10^{-3}$
				variable	2	$0.90 \cdot 10^{-9}$	$0.54 \cdot 10^{-3}$

The variant of the algorithm with AMM and variable length steps is considered. In this case, a strong value $\epsilon_s = 0.001$ is used for the stopping criterion. The other values of constants are: $NV = 16$, $\epsilon_m = 15.625$, according to the relation in Eq. (39).

These simulations are carried out for 14 values of unknown temperature U .

The results are displayed in Table 2.

Examining Table 2, it is seen that the best way is to use only the data obtained from the temperature sensors, without interpolation.

For this variant, the simulation error e_s is the same as in theoretical running. Values of Φ_s are a little greater for the first variant of real running in comparison with the theoretical running, as was expected. Values of Φ_s and e_s are very big when interpolation is used in comparison with the other situations.

Therefore, in the following, only the first variant of real running will be used.

Table 2
Computational results for real running

U [K]	Φ_s		$e_s(\%)$	
	Theoretical running	Real running		Theoretical running
		Data from sensors	Data from sensors and interpolation	
50	0	0	$0.178 \cdot 10^{-3}$	$0.404 \cdot 10^{-3}$
350	$0.138 \cdot 10^{-11}$	$0.264 \cdot 10^{-11}$	$0.715 \cdot 10^{-5}$	$0.270 \cdot 10^{-3}$
650	$0.294 \cdot 10^{-11}$	$0.446 \cdot 10^{-11}$	$0.350 \cdot 10^{-3}$	$0.150 \cdot 10^{-3}$
950	$0.150 \cdot 10^{-12}$	$0.485 \cdot 10^{-12}$	$0.121 \cdot 10^{-2}$	$0.642 \cdot 10^{-4}$
1250	0	0	$0.258 \cdot 10^{-2}$	0
1550	$0.148 \cdot 10^{-11}$	$0.199 \cdot 10^{-11}$	$0.446 \cdot 10^{-2}$	$0.158 \cdot 10^{-4}$
1850	$0.325 \cdot 10^{-11}$	$0.364 \cdot 10^{-11}$	$0.686 \cdot 10^{-2}$	$0.726 \cdot 10^{-4}$
2150	$0.113 \cdot 10^{-11}$	$0.139 \cdot 10^{-11}$	$0.977 \cdot 10^{-2}$	$0.568 \cdot 10^{-4}$
2450	$0.230 \cdot 10^{-10}$	$0.278 \cdot 10^{-10}$	$0.132 \cdot 10^{-1}$	$0.130 \cdot 10^{-3}$
2750	0	0	$0.171 \cdot 10^{-1}$	0
3050	$0.170 \cdot 10^{-10}$	$0.203 \cdot 10^{-10}$	$0.216 \cdot 10^{-1}$	$0.104 \cdot 10^{-3}$
3350	$0.976 \cdot 10^{-12}$	$0.128 \cdot 10^{-11}$	$0.266 \cdot 10^{-1}$	$0.292 \cdot 10^{-4}$
3650	$0.709 \cdot 10^{-12}$	$0.808 \cdot 10^{-12}$	$0.320 \cdot 10^{-1}$	$0.268 \cdot 10^{-4}$
3950	$0.880 \cdot 10^{-11}$	$0.120 \cdot 10^{-10}$	$0.380 \cdot 10^{-1}$	$0.804 \cdot 10^{-4}$

Fig. 3. $e_s(\%)$ versus $e_\gamma(\%)$.

5.3. Numerical simulations of the algorithm when disturbed data are used

From a practical point of view, it is possible that some errors appear which are caused by the technical realization of the rod, inexact measuring of time, and so on.

Using the above algorithm, in real running, the simulation error is computed versus the perturbation error, for 3 temperatures: $U=350$ K, 2150 K and 3650 K. These three curves are denoted with **c1**, **c2** and **c3**, respectively.

The following perturbations are carried out:

- Perturbation of the temperature measured by the sensors in the vicinity of the cold end: Consideration is given to a relative error e_γ of the temperature measurements made by all of the sensors, in the range $[0, 2]\%$. The dependencies of the simulation error versus the relative error of the temperature measurement are displayed in Fig. 3.
- Inexact placements of the sensors on the rod: Consideration is given to a deviation of all the sensors Δl in the range $[0, 1.25]$ mm towards the cold end. The dependencies of the simulation error versus the deviation of the sensors are displayed in Fig. 4.
- Inexact measurement of time: Consideration is given to an error of time measurement Δt_0 in the range $[0, 0.1]$ s (delay). The dependencies of the simulation error versus the time delay are displayed in Fig. 5.

Examining Figs. 3–5, we note the nearly linear dependencies between the perturbation factors and the simulation error in all cases. These errors are great at the end of the perturbation range.

According to these simulations, it is necessary to construct carefully the measuring system so that these errors are as small as possible.

6. Conclusions

A new computational method and a device based on it for temperature measurement are presented.

The mathematical model of the device and the new computational method are exposed in detail. At the same time, some possibilities of practical implementation are analyzed and the best variant from them is chosen.

The behaviour of the algorithm in real conditions using disturbed data is shown at the end.

Generally, the temperature sensors work in the stationary state. The described system can be

Fig. 4. $e_s(\%)$ versus Δl [mm]. Fig. 5. $e_s(\%)$ versus Δt_0 [s].

considered as a temperature sensor operating in the transient state. It uses a small part of the beginning of the transient state of the heat transfer in the rod.

The proposed device can measure any values of temperature in difficult places. In fact, it works as an attenuator of high temperatures and as an amplifier of low temperatures.

Some observations could be noted at the end.

- Placing the temperature sensors equidistantly is not necessary. This type of placement is chosen to make the numerical implementation easier. The distances between the sensors could be varied but it is necessary to know the exact distances.
- It is not necessary to place the temperature sensors in the vicinity of the cold end. This type of placement is chosen to ease the explanation of the theoretical part. As a function of the real working conditions, they could be placed on another part of the rod (e.g., in the middle of it). It is necessary to know these placements precisely. Otherwise, an important error could appear, as shown in Fig. 4.
- It is not necessary that time be considered from 0 to T in the cost function in Eq. (5). From a practical point of view, we must avoid measuring very small variations of temperature, when

important errors can occur. Therefore, it is possible to use a sub-range $[T_1, T] \subset [0, T]$ in the cost function without any other modifications of the algorithm.

- To make the numerical implementation easier we considered that all material coefficients do not depend on temperature. It appears that this hypothesis is not true in real conditions because there are some slow dependencies between material coefficients and temperature. Some minor modifications in the algorithm are necessary to take these dependencies into account when the device works in certain conditions.

References

- [1] A.D. Helfrick and W.D. Cooper, *Modern Electronic Instrumentation and Measurement Techniques* (Prentice Hall, New York, 1991).
- [2] L.D. Jones and A.F. Chin, *Electronic Instruments and Measurements* (Prentice Hall, New York, 1991).
- [3] F. Pavese, Cryogenics, in: J. Scholz and T. Ricolfi, eds., *Sensors 4* (VCH, Weinheim, 1990) 345–357.
- [4] OMEGA, *The Temperature Handbook*, 1993.
- [5] F.P. Incropera and D.P. DeWitt, *Fundamentals of Heat and Mass Transfer* (Wiley, London and New York, 1990).
- [6] V. Barbu, *Optimal Control of Variational Inequalities* (Pitman, London, 1984).
- [7] G. Knowles, *An Introduction to Applied Optimal Control* (Academic Press, New York, 1981).
- [8] I. Lasiecka and R. Triggiani, *Differential and Algebraic Riccati Equations with Application to Boundary/Point Control Problems: Continuous Theory and Approximation Theory* (Springer-Verlag, Berlin, 1991).
- [9] N. Damean, A new method and a new device for temperature measurement using one-dimensional heat equation, in: I. Kosuke, ed., *Proceedings of 14th Annual International Computers in Engineering Conference*, Sept. 11–14, Minneapolis, (1994) 763–766.
- [10] A. Kirsch, Some remarks concerning the computation of bang-bang controls, in: K.H. Hoffmann and W. Krabs, eds., *Optimal Control of Partial Differential Equations* (Birkhäuser Verlag, Basel, 1984) 144–156.
- [11] S. Anița and N. Damean, A problem of measurement of temperature solved using the one-dimensional heat equation, *Int. J. Heat Mass Trans.* 36 (1993) 1043–1048.
- [12] V. Arnăutu, On Approximation of the inverse one-phase Stefan problem, *Int. Ser. Numer. Math.* 99 (1991) 69–81.
- [13] M. Sibony, Sur l'approximation d'équations et inéquations aux dérivées partielles non linéaires de type monotone, *J. Math. Anal. Appl.* 34 (1971) 502–564.
- [14] W.D. Ames, *Numerical Methods for Partial Differential Equations* (Academic Press, New York and London, 1992).