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Three Results on Cycle-Wheel Ramsey Numbers

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Abstract Given two graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer N such that, for any graph G of order N, either G_1 is a subgraph of G, or G_2 is a subgraph of the complement of G. We consider the case that G_1 is a cycle and G_2 is a (generalized) wheel. We expand the knowledge on exact values of Ramsey numbers in three directions: large cycles versus wheels of odd order; large wheels versus cycles of even order; and large cycles versus generalized odd wheels.

Keywords Ramsey number · Cycle · Wheel

1 Introduction

In this paper we deal with finite simple graphs only. For any undefined terminology and notation we refer the reader to the textbook of Bondy and Murty [3].

Let $G = (V, E)$ be a graph. We sometimes use $|G|$ instead of $|V|$ to denote the order of G, i.e., the number of vertices of G. For a nonempty proper subset $S \subseteq V(G)$, we let $G[S]$ and $G - S$ denote the subgraphs induced by S and $V(G)\backslash S$, respectively. Let $N_S(v)$ be the set of neighbors of a vertex v that are contained in S, $N_S[v] = N_S(v) \cup \{v\}$ and $d_S(v) = |N_S(v)|$. If $S = V(G)$, we sometimes simply write $N(v) = N_G(v)$, $N[v] = N(v) \cup \{v\}$ and $d(v) = d_G(v)$. For two vertex-disjoint graphs G_1 and G_2, $G_1 \cup G_2$ denotes the disjoint union, and the join $G_1 + G_2$ is the graph obtained from

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$G_1 \cup G_2$ by joining every vertex of G_1 to every vertex of G_2 by an edge. A cycle and a path of order m are denoted by C_m and P_m, respectively. We sometimes use C_m instead of $V(C_m)$ for simplicity. By xP_my we mean a path from x to y with order m. An (X, Y) path is a path that starts at a vertex of X, and terminates at a vertex of Y. We use K_n to denote a complete graph of order n, and $K_{m,n}$ for a complete bipartite graph with bipartition classes of cardinality m and n. A wheel $W_n = K_1 + C_n$ is a graph of order $n+1$ (in the literature, sometimes W_n is used to denote a wheel of order n), and a generalized wheel $W_{m,n} = K_m + C_n$, so that $W_{1,n} = W_n$. As in [3], $\delta(G)$ is the minimum degree, $\Delta(G)$ the maximum degree, $\alpha(G)$ the independence number and $\kappa(G)$ the (vertex) connectivity of G. We use mG to denote m vertex-disjoint copies of G. The length of a longest and shortest cycle of G is denoted by $c(G)$ and $g(G)$, respectively. A graph G is weakly pancyclic if it contains cycles of every length between $g(G)$ and $c(G)$. We say that G is Hamilton-connected if every two distinct vertices of G are connected by a Hamilton path.

Given two graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer N such that, for any graph G of order N, either G contains G_1 as a subgraph or $\overline{G}$ contains G_2 as a subgraph, where $\overline{G}$ is the complement of G. It is easy to check that $R(G_1, G_2) = R(G_2, G_1)$. For specific graphs or graph families, the greatest challenge is to determine the exact values of the Ramsey numbers. This challenging open problem was solved for cycles by Rosta [17], and independently by Faudree and Schelp [13], as shown by the following result. A simpler proof for this result was later provided by Károlyi and Rosta [15].

Theorem 1 (Rosta [17], Faudree and Schelp [13]).

$$R(C_m, C_n) = \begin{cases} 2n-1 & \textit{for } 3 \le m \le n,\ m \textit{ odd},\ n \ne 3, \\ n-1+m/2 & \textit{for } 4 \le m \le n,\ m \textit{ and } n \textit{ even},\ n \ne 4, \\ \max\{n-1+m/2, 2m-1\} & \textit{for } 4 \le m < n,\ m \textit{ even and } n \textit{ odd}. \end{cases}$$

Wheels have also enjoyed quite a lot of attention in the context of Ramsey numbers. In the earliest contribution involving cycle-wheel Ramsey numbers, dating back to 1983, Burr and Erdős [6] determined the Ramsey numbers of a triangle versus wheels of arbitrarily large order.

Theorem 2 (Burr and Erdős [6]). *$R(W_n, C_3) = 2n + 1$ for $n \ge 5$.*

From then on, many papers have been published on cycle-wheel Ramsey numbers. By comparing the order of the two graphs and the parity of the smaller one, we can distinguish four cases: large cycles versus wheels of even (odd) order and large wheels versus cycles of even (odd) order. We recall here that a wheel $W_n = K_1 + C_n$ is a graph of order $n + 1$, so even (odd) wheels correspond to odd (even) n.

For large cycles versus even wheels, Surahmat et al. [22] determined that $R(C_m, W_n) = 3m - 2$ for odd $n \ge 5$ and $m > (5n - 9)/2$. This result was improved by Shi [18] who showed that $R(C_m, W_n) = 3m - 2$ for odd n and $m \ge 3n/2 + 1$. Then Zhang et al. [28] refined that result to $R(C_m, W_n) = 3m - 2$ for odd n, $m \ge n$ and $m \ge 20$. Finally, Chen et al. gave a simpler proof that completely solves this case.

Theorem 3 (Chen et al. [8]). *$R(C_m, W_n) = 3m - 2$ for odd n, $m \geq n \geq 3$ and $(m, n) \neq (3, 3)$.*

For large cycles versus odd wheels, Surahmat et al. [21] proved that $R(C_m, W_n) = 2m - 1$ for even n and $m \geq 5n/2 - 1$. Chen et al. [7] improved this result by reducing the lower bound on m from $m \geq 5n/2 - 1$ to $m \geq 3n/2 + 1$. To completely solve this case, Surahmat et al. [21] proposed the following conjecture.

Conjecture 1 (Surahmat et al. [21]). *$R(C_m, W_n) = 2m - 1$ for even n, $m \geq n$ and $(m, n) \neq (4, 4)$.*

We will confirm the above conjecture for large even m in this paper by proving the following result. We postpone the proof to the final section.

Theorem 4 *$R(C_m, W_n) = 2m - 1$ for even n and $m \geq n + 502$.*

For large wheels versus cycles of odd order, besides Theorem 2 on triangles versus wheels of arbitrarily large order, Zhou [31] showed that $R(W_n, C_m) = 2n + 1$ for m odd and $n \geq 5m - 7$. Even though it has been cited many times, the correctness of the proof in this Chinese language paper is questionable. Recently, it was established by Sun and Chen [19] that $R(W_n, C_5) = 2n + 1$ for $n \geq 6$. And Zhang et al. [30] proved the following more general result.

Theorem 5 (Zhang et al. [30]). *$R(W_n, C_m) = 2n + 1$ for m odd, $n \geq 3(m-1)/2$ and $(m, n) \neq (3, 3), (3, 4)$; $R(W_n, C_m) = 3m - 2$ for m, n odd and $m < n < 3(m-1)/2$.*

For large wheels versus small cycles of even order, even $R(C_4, W_n)$ is quite challenging. Tse [25] determined the value of $R(C_4, W_n)$ for $3 \leq n \leq 12$. Surahmat et al. [23] established an upper bound, which is $R(C_4, W_n) \leq n + \lceil n/3 \rceil + 1$ for $n \geq 6$. Recently, Dybizbański and Dzido [10] refined the upper bound and determined some exact values of $R(C_4, W_n)$.

Theorem 6 (Dybizbański and Dzido [10]). *$R(C_4, W_n) \leq n + \lfloor \sqrt{n-1} \rfloor + 2$ for all $n \geq 10$, and if $q \geq 4$ is a prime power, then $R(C_4, W_{q^2}) = q^2 + q + 1$.*

In the same paper, they proved that $R(C_4, W_n) = n + 5$ for $13 \leq n \leq 16$. Wu et al. [26] went a step further and obtained nine new values of $R(C_4, W_n)$ for $17 \leq n \leq 20$, $34 \leq n \leq 36$ and $n = 26, 43$.

By comparing the Ramsey numbers of $R(C_4, K_{1,n})$ and $R(C_4, W_n)$, we answered a natural question affirmatively by proving the following theorem in a very recent paper [29].

Theorem 7 (Zhang et al. [29]). *$R(C_4, K_{1,n}) = R(C_4, W_n)$ for $n \geq 6$.*

By Theorem 7, we see that the values for $R(C_4, K_{1,n})$ and $R(C_4, W_n)$ are exactly the same for $n \geq 6$. Thus we can use known results on $R(C_4, K_{1,n})$ to obtain new values for $R(C_4, W_n)$ immediately. This yielded the following theorem.

Theorem 8 (Zhang et al. [29]). *If $q \geq 3$, then $R(C_4, W_{q^2+1}) \leq q^2 + q + 2$. If $q \geq 3$ is a prime power, then $R(C_4, W_{q^2+1}) = q^2 + q + 2$.*

A weaker version of Theorem 8 was established by Wu et al. [27] independently. They also obtained some more values.

Theorem 9 (Wu et al. [27]). *If $q \geq 5$ is a prime power, then $R(C_4, W_{q^2-2}) = q^2+q-1$. If $p \geq 3$, $q = 2^p$, $0 \leq k \leq q$ and $k \neq 1, q-1$, then $R(C_4, W_{q^2-k-1}) = q^2+q-k$.*

As far as we know, practically nothing is known about $R(W_n, C_m)$ when m is even and greater than 4. We fill some of this gap by proving the following result in the final section.

Theorem 10 $R(W_n, C_m) = 3m - 2$ *for n odd, m even and $m < n < 3m/2$.*

For large cycles versus generalized even wheels, Surahmat et al. [24] showed that $R(C_m, W_{2,n}) = 3m-2$ for even $n \geq 4$ and $m \geq 9n/2+1$. Shi [18] improved this result by reducing the lower bound on m from $m \geq 9n/2+1$ to $m \geq \max\{3n/2+1, 71\}$. For large cycles versus generalized odd wheels, in the final section we prove the following result that has the same flavor.

Theorem 11 $R(C_m, W_{2,n}) = 4m - 3$ *for n odd, $m \geq 9n/8 + 1$.*

2 Preliminary Lemmas

For our proofs of Theorems 4, 10 and 11 in the next section, we need the following useful lemmas. Except for one (Lemma 15 below), all results are from literature and presented without proofs.

Lemma 1 (Bollobás et al. [1]). $R(C_n, K_5) = 4n - 3$ *for $n \geq 5$.*

Lemma 2 (Bondy [2]). *Let G be a graph of order n. If $\delta(G) \geq n/2$, then either G is pancyclic or n is even and $G = K_{n/2,n/2}$.*

Lemma 3 (Brandt [4]). *Every nonbipartite graph $G = (V, E)$ of order n with $|E| > (n-1)^2/4 + 1$ is weakly pancyclic with $g(G) = 3$.*

Lemma 4 (Brandt et al. [5]). *Every nonbipartite graph G of order n with $\delta(G) \geq (n+2)/3$ is weakly pancyclic with $g(G) = 3$ or 4.*

Lemma 5 (Brandt et al. [5]). *Let G be a 2-connected nonbipartite graph of order n with minimum degree $\delta(G) \geq n/4 + 250$. Then G is weakly pancyclic unless G has odd girth 7, in which case it has cycles of every length from 4 up to its circumference except a 5-cycle.*

Lemma 6 (Dirac [9]). *Let G be a graph with $\delta(G) \geq 2$. Then $c(G) \geq \delta(G) + 1$. If G is 2-connected, then $c(G) \geq min\{2\delta(G), |V(G)|\}$.*

Lemma 7 (Dirac [9]). *Let G be a graph of order n. If $\delta(G) \geq n/2 + 1$, then G is Hamilton-connected.*

Lemma 8 (Erdős and Gallai [11]). *Let $G = (V, E)$ be a graph of order n and $3 \leq c \leq n$. If $|E| \geq ((c-1)(n-1)+1)/2$, then $c(G) \geq c$.*

Lemma 9 (Faudree et al. [12]). *Let G be a graph of order $n \geq 6$. Then* $\max\{c(G), c(\overline{G})\} \geq \lceil 2n/3 \rceil$.

Lemma 10 (Jackson [14]). *Let $G = (X, Y)$ be a bipartite graph with bipartition classes X and Y such that $d(x) \geq t$ for all $x \in X$, where $|X| \geq 2$ and $2 \leq t \leq |Y| \leq 2t - 2$. Then G contains all cycles on $2m$ vertices for $2 \leq m \leq min\{|X|, t\}$.*

Lemma 11 (Károlyi and Rosta [15]). *Suppose that G has a cycle $C = x_1x_2 \ldots x_{2\ell}x_1$, but neither G nor $\overline{G}$ has a $C_{2\ell-1}$. Then $\overline{G}[\{x_1, x_3, \ldots, x_{2\ell-1}\}] = \overline{G}[\{x_2, x_4, \ldots, x_{2\ell}\}] = K_\ell$.*

The following lemma can be found as Proposition 9.4 in [3].

Lemma 12 *Let G be a k-connected graph, and let X and Y be subsets of $V(G)$ of cardinality at least k. Then there exists a family of k pairwise disjoint (X, Y) paths in G.*

Lemma 13 (Zhang et al. [30]) *Let C be a longest cycle in a graph G and $v_1, v_2 \in V(G)\setminus V(C)$ with $t = |N_C(v_1) \cup N_C(v_2)|$. Then $t \leq \lfloor |C|/2 \rfloor + 1$ and if $v_1v_2 \in E(G)$, then $t \leq \lfloor |C|/2 \rfloor$.*

Lemma 14 (Zhang et al. [30]) *For a graph G, let (X, Y) be a partition of $V(G)$. Suppose that for some odd $n \geq 5$, $|Y| \geq (n + 1)/2$ and any two vertices of Y have at least $(n - 1)/2$ common nonadjacent vertices in X. If $\overline{G}$ contains no C_n, then $G[Y]$ is a complete graph.*

Lemma 15 *Let $C = x_1x_2 \ldots x_rx_1$ be a longest cycle in a graph G with $r \geq n$, and let $Y = \{y_1, y_2, \ldots, y_{d-r}\} = V(G)\setminus V(C)$ with $d - r \geq (n + 1)/2$. Suppose that $\overline{G}$ contains no C_n and $G[Y]$ is a complete graph. Then $\overline{G}$ is bipartite.*

Proof If $E(V(C), Y) = \emptyset$, then $G[V(C)]$ is a complete graph by Lemma 14. Hence, in this case it is easy to deduce that $\overline{G}$ is bipartite. Now let $E(V(C), Y) \neq \emptyset$, and let $x_1y_1 \in E(G)$. Then, since C is a longest cycle in G and $G[Y]$ a complete graph, it follows that $E(X_1, Y\setminus\{y_1\}) = \emptyset$, where $X_1 = \{x_2, x_3, \ldots, x_{d-r+1}\} \cup \{x_r, x_{r-1}, \ldots, x_{2r-d+1}\}$. Because $d - r - 1 \geq (n - 1)/2$, using Lemma 14 again, we obtain that $G[X_1]$ is a complete graph. Let $H \subseteq G[V(C)]$ be a maximal complete graph containing x_2. We claim that $V(C)\setminus\{x_1\} \subseteq V(H)$. If not, there is an $x_i \in V(C)\setminus V(H)$ such that x_i is adjacent to some vertex of H on the cycle and nonadjacent to some vertex of H. Then $E(\{x_i\}, Y\setminus\{y_1\}) = \emptyset$; otherwise there clearly is a longer cycle than C. Furthermore, $\overline{G}[V(H) \cup (Y\setminus\{y_1\}) \cup \{x_i\}]$ contains a C_n, a contradiction. For the same reason, we have $V(C)\setminus\{x_1\} \subseteq N(x_1)$ or $Y\setminus\{y_1\} \subseteq N(x_1)$. Therefore, $\overline{G}$ is bipartite. □

Lemma 16 (Surahmat et al. [20]). $R(C_n, W_4) = 2n - 1$ *for* $n \geq 5$.

At the end of this section, we list some known small Ramsey numbers that we need.

Lemma 17 ([16]).

(i) $R(W_n, C_3) = 2n + 3$ *for* $n = 3, 4$;
(ii) $R(W_5, C_4) = 10$.

3 Proofs of the Main Results

3.1 Proof of Theorem 4

Let G be a graph of order $2m-1$ with n even and $m \geq n+502$. Suppose to the contrary that neither G contains W_n nor $\overline{G}$ contains C_m.

Assume that $v \in V(G)$ with $d(v) = d = \Delta(G)$ and $H = G[N(v)]$. We are first going to show that $d \geq (3m+1)/2-252$. To the contrary, assume that $d \leq 3m/2-252$. Then $\delta(\overline{G}) \geq 2m-2-(3m/2-252) = m/2+250$. By Theorem 2 and Lemma 17, we have $g(\overline{G}) = 3$, and so $\overline{G}$ is nonbipartite. If $\kappa(\overline{G}) \geq 2$, then $\overline{G}$ contains C_m by Lemmas 6 and 5, a contradiction. So, we assume next that $\kappa(\overline{G}) \leq 1$. Then there exists some $u \in V(G)$ such that $G-u$ contains a spanning complete bipartite graph with bipartite sets V_1, V_2 and $|V_1| \geq |V_2|$. Obviously, $|V_1| \geq m-1$ and $|V_2| \geq \delta(\overline{G})$. If $\Delta(G[V_1]) \geq n/2$, let $x \in V_1$ with $d(x) = \Delta(G[V_1])$. Then x together with $n/2$ neighbors in V_1 and $n/2$ neighbors in V_2 form a W_n with x as its hub, a contradiction. This implies that $\delta(\overline{G}[V_1]) \geq |V_1| - n/2 > |V_1|/2+1$. If $|V_1| \geq m$, then by Lemma 2, $\overline{G}[V_1]$ contains C_m, a contradiction. Thus, we conclude that $|V_1| = |V_2| = m-1$. Since $\delta(\overline{G}) > 250$, we may assume that $x_1, x_2 \in N_{\overline{G}}(u) \cap V_1$. By Lemma 7, $\overline{G}[V_1]$ has a Hamilton (x_1, x_2)-path which together with u forms C_m in $\overline{G}$, our final contradiction. Therefore, henceforth we assume that $d \geq (3m+1)/2-252$.

By the assumptions, H has no C_n. If m is even, then since $m \geq n+502$, we get that $R(C_n, C_m) \leq 3m/2-252$ by Theorem 1, a contradiction. Thus, m is odd. Next, we first prove the following claim.

Claim A. *H contains no $2K_{(m+1)/2}$.*

Proof Suppose that H contains $2K_{(m+1)/2}$ with V_1, V_2 as their vertex sets.

If there exist two independent edges between V_1 and V_2, then H contains a C_n, a contradiction. If there is at least one edge between V_1 and V_2, then V_1 or V_2 contains a vertex w such that $E(V_1\backslash\{w\}, V_2\backslash\{w\}) = \emptyset$. For any $u \notin V_1 \cup V_2\backslash\{w\}$, we have $V_1\backslash\{w\} \subseteq N(u)$ or $V_2\backslash\{w\} \subseteq N(u)$, for otherwise $\overline{G}[V_1 \cup V_2 \cup \{u\}]$ contains C_m. Let $U_i = \{u \mid V_i\backslash\{w\} \subseteq N(u), u \notin V_1 \cup V_2\backslash\{w\}\}$ for $i = 1, 2$. Assume that $|(V_1\backslash\{w\}) \cup U_1| \geq |(V_2\backslash\{w\}) \cup U_2|$. Then $|(V_1\backslash\{w\}) \cup U_1| \geq m \geq n+502$. Taking $n+1$ vertices from $(V_1\backslash\{w\}) \cup U_1$ such that at least $n/2+1$ of them are in $V_1\backslash\{w\}$, we obtain W_n in $G[(V_1\backslash\{w\}) \cup U_1]$, a contradiction. If there is no edge between V_1 and V_2, we can derive a contradiction in a similar way. □

We next show that H is nonbipartite. To the contrary, suppose H is a bipartite graph with $V(H) = (X, Y)$ and $|X| \geq |Y|$. Since $\overline{G}$ contains no C_m, we have $|X| \leq m-1$ and $|Y| = d - |X| \geq (m+1)/2 - 251 > n/2$. If there exists two independent edges between X and Y in $\overline{G}$, then since both $\overline{G}[X]$ and $\overline{G}[Y]$ are complete graphs, $\overline{H}$ contains C_m, a contradiction. Thus, there exists some vertex $z \in X \cup Y$ such that $H-z$ with bipartition $(X\backslash\{z\}, Y \setminus \{z\})$ is a complete bipartite graph. But then $H-z$ contains $K_{n/2,n/2}$, and so H contains C_n, a contradiction. Therefore, henceforth we assume that H is nonbipartite.

We now show that $\overline{H}$ is also nonbipartite. To the contrary, suppose $\overline{H}$ is a bipartite graph with $V(\overline{H}) = (X, Y)$ and $|X| \geq |Y|$. Since $G[X]$ contains no C_n, we have $|X| \leq n-1 \leq m-503$ and hence $|Y| = d - |X| \geq (m+1)/2 + 251$. Clearly,

H contains $2K_{(m+1)/2}$, contradicting Claim A. Thus, henceforth we assume that $\overline{H}$ is nonbipartite.

If $|E(\overline{H})| > (d-1)^2/4+1$, then by Lemma 3, $\overline{H}$ is weakly pancyclic with $g(\overline{H}) = 3$. By Theorem 1, $R(C_n, C_{m+1}) = m + n/2 \le (3m+1)/2 - 252$ for m is odd and $m \ge n+502$. Since H contains no C_n, $\overline{H}$ contains C_{m+1}, which implies that $\overline{H}$ contains C_m, a contradiction. Thus, we have $|E(H)| \ge d(d-1)/2-(d-1)^2/4-1 > (d-1)^2/4+1$. By Lemma 3, H is weakly pancyclic with $g(H) = 3$. Since H contains no C_n, we have $c(H) < n$ and so H has no C_m. Because $\overline{H}$ contains C_{m+1} and has no C_m, H contains $2K_{(m+1)/2}$ by Lemma 11, contradicting Claim A.

Since $2K_{m-1}$ contains no C_m and its complement contains no W_n, we obtain that $R(C_m, W_n) \ge 2m-1$. By the above arguments, $R(C_m, W_n) \le 2m-1$ for n even and $m \ge n+502$. This completes the proof of Theorem 4. □

3.2 Proof of Theorem 10

By Lemma 17, we may assume that $m \ge 6$. The lower bound $R(W_n, C_m) \ge 3m-2$ follows from the fact that a complete tripartite graph $K_{m-1,m-1,m-1}$ contains no W_n and its complement contains no C_m. To prove $R(W_n, C_m) \le 3m-2$, let G be graph of order $3m-2$, and suppose that neither G contains W_n nor $\overline{G}$ contains C_m.

We first show that G contains no K_n. If G contains a K_n, then every other vertex in G has at most two neighbors in K_n; otherwise G contains a W_n. Since $n-2 \ge m/2$, by Lemma 10, $\overline{G}$ contains a C_m, a contradiction. Thus, G contains no K_n.

We next show that $\delta(\overline{G}) = m-1$. Let $v \in V(G)$ with $d(v) = \Delta(G) = d$, let $H = G[N(v)]$ and let $Z = V(G)\backslash N[v]$. If $\overline{G}$ is a bipartite graph, say with $V(\overline{G}) = (X, Y)$ and $|X| \ge |Y|$, then $|X| \ge 3m/2-1 \ge n$, which implies that $G[X]$ contains a K_n, a contradiction. Thus, $\overline{G}$ is nonbipartite. If $\delta(\overline{G}) \ge m$, then by Lemmas 4 and 6, $\overline{G}$ contains C_m, a contradiction. If $\delta(\overline{G}) \le m-2$, then $d(v) \ge 2m-1$, that is, $|H| \ge 2m-1$. Since $\overline{H}$ has no C_m, H contains a C_n by Theorem 1, which together with v forms a W_n in G, a contradiction. Thus, we have $\delta(\overline{G}) = m-1$ and $d = 2m-2$.

We next show that H and $\overline{H}$ are both nonbipartite. First assume that H is a bipartite graph, say with $V(H) = (X, Y)$. Then, since $\overline{G}$ contains no C_m, we have $|X| = |Y| = m-1$ and $e(X, Y) \ge |X||Y|-1$. Because $\delta(\overline{G}) = m-1 \ge 5$, there exist two distinct vertices $x_1, x_2 \in X$ such that $x_1z_1, x_2z_2 \in E(\overline{G})$, where $z_1, z_2 \in Z$. If $z_1 = z_2$, then $\overline{G}[X \cup \{z_1\}]$ contains a C_m; and if $z_1 \ne z_2$, then noting that $z_1, z_2 \in N_{\overline{G}}(v)$, we see that $\overline{G}[X \cup \{z_1, z_2, v\}]$ has a C_m. Hence, H is nonbipartite.

Now suppose $\overline{H}$ is a bipartite graph. Let $V(\overline{H}) = (X, Y)$ and $|X| \ge |Y|$. Since G has no K_n, we get $|X \cup \{v\}| \le n-1$, hence $|X| \le n-2$ and $|Y| \ge 2m-n \ge m/2+1$. If $\kappa(H) \ge 2$, then H has two independent edges between X and Y. Since both $G[X]$ and $G[Y]$ are complete graphs, H contains a C_n, a contradiction. Let now $\kappa(H) \le 1$. Then there exists a vertex w such that $\overline{H} - w$ is a complete bipartite graph in which each partite set has at least $m/2$ vertices. Since m is even, $\overline{H}$ contains a C_m, a contradiction. Therefore, $\overline{H}$ is also nonbipartite.

If $|E(\overline{H})| \ge d(d-1)/4+1/2$, then by Lemmas 3 and 8, $\overline{H}$ contains a C_m, a contradiction. Thus, $|E(\overline{H})| < d(d-1)/4+1/2$. Since m is even and $d = 2m-2$, we have $d \equiv 2 \pmod 4$. Thus, $|E(\overline{H})| \le d(d-1)/4 - 1/2$, and hence $|E(H)| \ge d(d-1)/4 + 1/2$. By Lemmas 3 and 8, H is weakly pancyclic with $g(H) = 3$ and $c(H) \ge m$. Let $C = x_1x_2 \dots x_rx_1$ be a longest cycle in H and $V(H)\backslash V(C) = Y = \{y_1, y_2, \dots, y_{d-r}\}$. Then $m \le r \le n-1$ and $d - r \ge 2m - n - 1 \ge m/2$. By Lemma 13, $|N_C(y_i) \cup N_C(y_j)| \le \lfloor |C|/2 \rfloor + 1$, and so $|N_{\overline{G}}(y_i) \cap N_{\overline{G}}(y_j) \cap V(C)| \ge \lceil |C|/2 \rceil - 1 \ge m/2 - 1$ for any two distinct vertices $y_i, y_j \in Y$. If $|N_{\overline{G}}(y_i) \cap N_{\overline{G}}(y_j) \cap V(C)| \ge m/2$ for two distinct vertices $y_i, y_j \in Y$, say $y_1, y_{m/2}$ are two such vertices. Then we can choose $m/2$ vertices $x_{i_1}, x_{i_2}, \dots, x_{i_{m/2}}$ from $V(C)$ such that $x_{i_j} \in N_{\overline{G}}(y_j) \cap N_{\overline{G}}(y_{j+1})$ for $1 \le j \le m/2 - 1$ and $x_{i_{m/2}} \in N_{\overline{G}}(y_{m/2}) \cap N_{\overline{G}}(y_1)$, implying that $y_1x_{i_1}y_2x_{i_2}y_3 \dots y_{m/2}x_{i_{m/2}}y_1$ is a C_m in $\overline{G}$, a contradiction. Thus, we have $|N_{\overline{G}}(y_i) \cap N_{\overline{G}}(y_j) \cap V(C)| = m/2 - 1$ for any two distinct vertices $y_i, y_j \in Y$. By Lemma 13, we have $r = m$, $d - r = m - 2$ and $\overline{G}[Y] = K_{m-2}$. Let $x' \in N_{\overline{G}}(y_1) \cap N_{\overline{G}}(y_2) \cap V(C)$ and $x'' \in N_{\overline{G}}(y_2) \cap N_{\overline{G}}(y_3) \cap V(C) - \{x'\}$. Then $\overline{G}[Y \cup \{x', x''\}]$ contains a C_m, our final contradiction. This completes the proof of Theorem 10. □

3.3 Proof of Theorem 11

Since $W_{2,3} = K_5$, using Lemma 1 we see that Theorem 11 holds for $n = 3$, and so we may assume that $n \ge 5$. Because neither $4K_{m-1}$ contains a C_m nor its complement contains a $W_{2,n}$, we get that $R(C_m, W_{2,n}) \ge 4m - 3$. So it suffices to show $R(C_m, W_{2,n}) \le 4m - 3$. Let G be a graph of order $4m - 3$ with $m \ge 9n/8 + 1$. Suppose that G contains no $W_{2,n}$ and that $\overline{G}$ contains no C_m.

We distinguish the following two cases.

Case 1. $\overline{G}$ contains no $2K_{\lceil m/2 \rceil}$.

If $\overline{G}$ is bipartite, then $\alpha(\overline{G}) \ge 2m - 1 \ge n + 2$, which implies that G contains $W_{2,n}$, a contradiction. So $\overline{G}$ is nonbipartite. If $\delta(\overline{G}) \ge (4m-1)/3$, then $\overline{G}$ contains C_m by Lemmas 4 and 6, a contradiction. Hence, $\delta(\overline{G}) \le (4m-2)/3$ and so $\Delta(G) \ge (8m-10)/3$. Let u be a vertex with $d_G(u) = \Delta(G) = d$ and let $G_u = G[N(u)]$.

Since $|G_u| \ge 2n + 3$ and G has no $W_{2,n}$,$\overline{G_u}$ is nonbipartite. We first show that $\delta(\overline{G_u}) \le (d+1)/3$. To the contrary, assume $\delta(\overline{G_u}) \ge (d+2)/3$. Then by Lemma 4, $\overline{G_u}$ is weakly pancyclic with $g(\overline{G_u}) \le 4$. If $\kappa(\overline{G_u}) \ge 2$, then by Lemma 6, $c(\overline{G_u}) \ge 2\delta(\overline{G_u}) \ge m$, so then $\overline{G_u}$ contains a C_m, a contradiction. Assume $\kappa(\overline{G_u}) \le 1$. Then for some $w \in V(G_u)$, $V(G_u)\backslash\{w\}$ can be partitioned into two parts V', V'' such that $e_G(V', V'') = |V'||V''|$. Assume that $|V'| \ge |V''|$ and choose w under these restrictions such that $|V'| - |V''|$ is as large as possible. It is obvious that $|V''| \ge \delta(\overline{G_u}) \ge \lceil m/2 \rceil$. Noting that $m \ge n + 2 \ge 7$ and $d \ge \lceil (8m-10)/3 \rceil$, we get that $d \ge 2m + 2$, and hence $|V'| \ge m + 1$. If $\delta(\overline{G}[V']) \ge (|V'|+1)/2$, then by Lemma 2, $\overline{G}[V']$ contains a C_m, a contradiction. Thus there exists some $u' \in V'$ such that $d_{V'}(u') = \Delta(G[V']) > (|V'|-3)/2 \ge n/2$. If $G[V'']$ has at least one edge, then G has a C_n in $N(u')$ which together with u, u' forms a $W_{2,n}$ in G. So $\overline{G}[V'']$ is a complete graph. In this case, $|V''| \le m - 1$, since $\overline{G}$ has no C_m. Thus, $|V'| = d - 1 - |V''| \ge m + 2$ and $d_{V'}(u') > (|V'|-3)/2$ and $d_{V'}(u') \ge \lceil m/2 \rceil \ge n/2 + 1$.Since $\overline{G}$ has no $2K_{\lceil m/2 \rceil}$, G has at least one edge in $N_{V'}(u')$, and so G has also a C_n in $N(u')$ which together with u, u' forms a $W_{2,n}$ in G. Therefore, $\delta(\overline{G_u}) \leq (d+1)/3$, implying that $\Delta(G_u) \geq (2d-4)/3$.

Let v be a vertex of G_u with $d_{G_u}(v) = \Delta(G_u) = h$, and set $H = G_u[N(v)]$. Then $h \geq 16(m-2)/9$. If H contains a C_n, then this C_n together with u, v forms a $W_{2,n}$ in G, a contradiction. Hence, H contains no C_n.

We are now going to show that H and $\overline{H}$ are both nonbipartite. First assume that H is bipartite, say with $V(H) = (X, Y)$ and $|X| \geq |Y|$. Then $|X| \leq m-1$; otherwise $\overline{H}$ has a C_m. Thus, $|Y| \geq h-(m-1) \geq (7m-23)/9$. If $m \geq 8$, then we have $|Y| \geq \lceil m/2 \rceil$; if $m = 7$, then since $d \geq (8m-10)/3$ and $h \geq (2d-4)/3$, we have $h \geq 10$ and so $|Y| \geq 4 = \lceil m/2 \rceil$. Thus, $\overline{H}$ always contains $2K_{\lceil m/2 \rceil}$, a contradiction. Therefore, H is nonbipartite.

Next suppose that $\overline{H}$ is bipartite, say with $V(\overline{H}) = (X, Y)$ and $|X| \geq |Y|$. Since $h \geq 16(m-2)/9 \geq 16((9n/8+1)-2)/9 = 2n-16/9$, then $h \geq \lceil 2n-16/9 \rceil = 2n-1$. Hence, $|X| \geq n$ and $G[X]$ contains a C_n, a contradiction. Thus, $\overline{H}$ is also nonbipartite.

Noting that $m \geq 9n/8+1$, $n \geq 5$ and $h \geq 16(m-2)/9$, we have $\lfloor h/2 \rfloor + 1 \geq n$ and $h \geq 7$. If $|E(H)| \geq (h+1)(h-1)/4-1$, then $|E(H)| \geq (((\lfloor h/2 \rfloor+1)-1)(h-1)+1)/2$. Thus, by Lemmas 3 and 8, H contains a C_n, a contradiction. This implies that $|E(\overline{H})| > (h-1)^2/4+1$. By Lemma 3, $\overline{H}$ is weakly pancyclic with $g(\overline{H}) = 3$. If $c(\overline{H}) \leq \lfloor h/2 \rfloor$, then $|E(\overline{H})| < (h+1)(h-1)/4-1$ by Lemma 8, which implies that $|E(H)| > (h-1)^2/4+1$, and $c(H) \geq n$ by Lemma 9. Thus, H contains a C_n by Lemma 3, a contradiction. Therefore, $\overline{H}$ is weakly pancyclic with $g(\overline{H}) = 3$ and $c(\overline{H}) \geq \lfloor h/2 \rfloor + 1$.

Let $C = x_1x_2 \ldots x_rx_1$ be a longest cycle in $\overline{H}$, and let $Y = \{y_1, y_2, \ldots, y_{h-r}\} = V(\overline{H}) \backslash V(C)$. Then $\lfloor h/2 \rfloor + 1 \leq r \leq m-1$ and $h-r \geq h-m+1$. If $n \geq 7$, then it is easy to check that $|Y| \geq (n+1)/2$; if $n = 5$, we have $m \geq 7$ and $|Y| \geq \lceil (7m-23)/9 \rceil \geq 3$, and we also get that $|Y| \geq (n+1)/2$. By Lemma 13, any two vertices of Y have at least $\lceil r/2 \rceil - 1$ common nonadjacent vertices in C. It is easy to check that $\lceil r/2 \rceil - 1 \geq r/2 - 1 \geq (\lfloor h/2 \rfloor + 1)/2 - 1 \geq h/4 - 3/4 > (n-3)/2$. Since n is odd, $\lceil r/2 \rceil - 1 \geq (n-1)/2$. By Lemmas 14 and 15, H is bipartite, a contradiction. This completes Case 1.

Case 2. $\overline{G}$ contains $2K_{\lceil m/2 \rceil}$.

We first deal with the subcase that $\kappa(\overline{G}) \leq 2$. We assume that $\{u, w\}$ is a cut set and that $V(\overline{G}) \backslash \{u, w\} = X \cup Y$ with $|X| \geq |Y|$ and $E_{\overline{G}}(X, Y) = \emptyset$. Obviously, $|X| \geq 2m-2$. If $G[X]$ contains W_n, then $G[X \cup \{y\}]$ contains $W_{2,n}$ for any $y \in Y$, and hence $|X| \leq 3m-3$ by Theorem 3. Thus, we have $2m-2 \leq |X| \leq 3m-3$ and $|Y| \geq m-2$. If $|X| \geq 2m-1$, then $G[X]$ contains C_n by Theorem 1. If $y'y'' \in E(G[Y])$, then $G[X \cup \{y', y''\}]$ contains $W_{2,n}$, and so $\overline{G}[Y]$ is a complete graph. Since $\overline{G}$ has no C_m, $|Y| \leq m-1$. If $|Y| = m-2$, then $\overline{G}[Y] = K_{m-2}$. Since $\overline{G}[Y \cup \{u, w\}]$ has no C_m, we may assume $uy \in E(G)$ for some $y \in Y$. By Theorem 3, $G[X \cup \{u\}]$ contains W_n, which implies that $G[X \cup \{u, y\}]$ has $W_{2,n}$, a contradiction. If $|Y| = m-1$, then $\overline{G}[Y] = K_{m-1}$. Since $\overline{G}[Y \cup \{u, w\}]$ contains no C_m, we can choose $y \in Y$ such that $uy, wy \in E(G)$. By Theorem 3, $G[X \cup \{u, w\}]$ contains W_n, which implies that $G[X \cup \{u, w, y\}]$ contains $W_{2,n}$, again a contradiction. Therefore, we conclude that $|X| = 2m-2$ and $|Y| = 2m-3$. By Lemma 16, $G[Y \cup \{u, w\}]$ contains W_4. Assuming that w is not the hub of the W_4, then $G[Y \cup \{u\}]$ has a triangle

$vy'y''$, where $y', y'' \in Y$. If $G[X]$ contains C_n, then $G[X \cup \{y', y''\}]$ contains $W_{2,n}$, and hence $G[X]$ contains no C_n. By Theorem 1, $G[X \cup \{u\}]$ contains C_n, which implies that $G[X]$ contains P_{n-1} and any C_n in $G[X \cup \{u\}]$ contains u. If $v = u$, then any C_n in $G[X \cup \{u\}]$ together with y', y'' forms $W_{2,n}$, and if $v \neq u$, then any P_{n-1} in $G[X]$ together with v, y', y'' forms $W_{2,n}$ in G, a contradiction. Henceforth, we may assume that $\kappa(\overline{G}) \geq 3$.

We set A, B as the vertex sets of the $2K_{\lceil m/2 \rceil}$. Since $m \geq 9n/8 + 1$, $m \geq n + 2$ and $\lceil m/2 \rceil \geq \lceil (n+2)/2 \rceil = (n+3)/2 \geq 4$. By Lemma 12, $\overline{G}$ contains three disjoint paths joining A and B, denoted by $Q_i = a_i c_{i1} c_{i2} \ldots c_{ip_i} b_i$, where $1 \leq i \leq 3$, $a_i \in A$, $b_i \in B$ and $c_{ij} \notin A \cup B$ for $1 \leq j \leq p_i$. It is obvious that $p_i \geq 0$ and $p_i + 2$ is the order of Q_i for $1 \leq i \leq 3$. We choose three such disjoint paths Q_1, Q_2, Q_3 from $\overline{G}$ in such a way that $p_1 + p_2 + p_3$ is as small as possible. Without loss of generality, we may assume $p_1 \geq p_2 \geq p_3$.

If $p_2 + p_3 \leq m - 4$, then it is easy to check that $\overline{G}$ contains a C_m, a contradiction, implying that $p_2 + p_3 \geq m - 3 \geq 4$ and $p_2 \geq 2$. If $p_1 \geq 7$, then $c_{12}c_{14} \in E(G)$; otherwise $Q_1' = a_1 c_{11} c_{12} c_{14} \ldots c_{1p_1} b_1$ is a path shorter than Q_1 in $\overline{G}$ and Q_1', Q_2, Q_3 are also three disjoint paths joining A and B, contradicting the choice of Q_1, Q_2, Q_3. For the same reason, to avoid a path Q_1' which is shorter than Q_1 and together with Q_2, Q_3 forms three disjoint paths joining A and B in $\overline{G}$, G contains a complete multipartite graph with five partite sets: $\{c_{12}\}, \{c_{14}\}, \{c_{16}\}, A \backslash \{a_2, a_3\}, B \backslash \{b_2, b_3\}$. Since both $|A \backslash \{a_2, a_3\}|$ and $|B \backslash \{b_2, b_3\}|$ are at least $(n-1)/2$, then G contains a $W_{2,n}$, a contradiction. This implies that $p_1 \leq 6$.

By the choice of Q_1, Q_2, Q_3, we see that every vertex of $A \backslash \{a_2, a_3\}$ is adjacent to every vertex of $B \backslash \{b_2, b_3\}$. Furthermore, if $p_i \geq 1$, then a_i is adjacent to every vertex of $V(Q_i) \backslash \{a_i, c_{i1}\}$, c_{i1} is adjacent to every vertex of $V(Q_i) \backslash \{a_i, c_{i1}, c_{i2}\}$, c_{i2} is adjacent to every vertex of $V(Q_i) \backslash \{c_{i1}, c_{i2}, c_{i3}\}$, ..., b_i is adjacent to every vertex of $V(Q_i) \backslash \{c_{ip_i}, b_i\}$, where $1 \leq i \leq 3$. If $p_i \geq 2$, then for $j \geq 2$, c_{ij} is adjacent to every vertex of $A \backslash \{a_1, a_2, a_3\}$; for $j \leq p_i - 1$, c_{ij} is adjacent to every vertex of $B \backslash \{b_1, b_2, b_3\}$. For $1 \leq i < s \leq 3$, if $\lfloor m/2 \rfloor \leq j + t \leq m - 2$, then c_{ij} is adjacent to c_{st}. This is because, if $c_{ij}c_{st} \in E(\overline{G})$, then $a_i c_{i1} \ldots c_{ij} c_{st} \ldots c_{s1} a_s$ is a path which together with $m - 2 - j - t$ vertices of $A \backslash \{a_i, a_s\}$ forms a C_m in $\overline{G}$, a contradiction. For $1 \leq i < s \leq 3$, if $\lfloor m/2 \rfloor \leq (p_i - j + 1) + (p_s - t + 1) \leq m - 2$, then c_{ij} is adjacent to c_{st}. This is because, if $c_{ij}c_{st} \in E(\overline{G})$, then $b_i c_{ip_i} \ldots c_{ij} c_{st} \ldots c_{sp_s} b_s$ is a path which together with $m + j + t - p_i - p_s - 4$ vertices of $B \backslash \{b_i, b_s\}$ forms a C_m in $\overline{G}$, a contradiction. We can also determine whether c_{ij} is adjacent to a_s or b_s under similar conditions. In the following, through a tedious but straightforward case distinction, we will always find a $W_{2,n}$ in G, which is a contradiction and confirms our claim. Unless specifically mentioned, the existence of the edges of the $W_{2,n}$ that we will find each time is validated by the above arguments.

Set $A \backslash \{a_1, a_2, a_3\} = \{a_4, a_5, \ldots, a_{\lceil m/2 \rceil}\}$ and $B \backslash \{b_1, b_2, b_3\} = \{b_4, b_5, \ldots, b_{\lceil m/2 \rceil}\}$. If $(p_1, p_2) = (6, 6), (6, 5)$, then $7 \leq m \leq p_2 + p_3 + 3 \leq 2p_2 + 3 \leq 15$. We see that G contains a $W_{2,n} = \{c_{12}\} + \{c_{14}\} + C_n$, where $C_n = a_4 c_x b_4 a_5 b_5 \ldots a_{(n+3)/2} b_{(n+3)/2} a_1 b_1 a_4$, where $c_x = c_{23}$ for $10 \leq m \leq 11$, and $c_x = c_{22}$ for $8 \leq m \leq 9$ or $12 \leq m \leq 15$. For $m = 7$, either $a_1 c_{21} \in E(G)$ or $c_{14}c_{22} \in E(G)$; otherwise $\overline{G}$ contains a $C_7 = a_1 c_{11} c_{12} c_{13} c_{14} c_{22} c_{21} a_1$, a contradiction. Thus, G contains a $W_{2,5} = \{c_{12}\} + \{c_{14}\} + C_5$, where $C_5 = a_1c_{21}b_4a_4b_1a_1$ if $a_1c_{21} \in E(G)$, and $C_5 = a_4c_{22}b_4a_1b_1a_4$ if $c_{14}c_{22} \in E(G)$. If $(p_1, p_2) = (6, 4), (6, 3), (5, 5), (5, 4), (5, 3)$, then $7 \le m \le 2p_2+3 \le 13$. We see that G contains a $W_{2,n} = \{c_{12}\} + \{c_{14}\} + C_n$, where $C_n = a_4c_{22}b_4a_5b_5 \dots a_{(n+3)/2}b_{(n+3)/2}a_1b_1a_4$ for $8 \le m \le 13$. For $m = 7$, we can obtain the same $W_{2,5}$ as in the previous case. If $(p_1, p_2) = (6, 2), (5, 2)$, then $m = 7$ and $p_3 = 2$. In this case, G contains a $W_{2,5} = \{c_{12}\} + \{c_{14}\} + C_5$, where $C_5 = a_4c_{22}c_{32}c_{21}b_4a_4$. For the remainder we may assume that $p_1 \le 4$.

If $(p_1, p_2, p_3) = (4, 4, 4)$ and $m \ne 8$, or if $(p_1, p_2, p_3) = (4, 4, 3)$, then $m \le 11$ and G contains a $W_{2,n} = \{c_{12}\} + \{c_{22}\} + C_n$, where $C_n = a_4c_{32}c_{14}c_{31}b_4a_5b_5 \dots a_{(n+3)/2}b_{(n+3)/2}a_4$. If $(p_1, p_2, p_3) = (4, 4, 4)$ and $m = 8$, then $n = 5$ and G contains a $W_{2,5} = \{a_4\} + \{b_4\} + C_5$, where $C_5 = c_{12}c_{22}c_{32}c_{23}c_{33}c_{12}$. If $(p_1, p_2, p_3) = (4, 4, 2)$, then $m \le 9$. We see that G contains a $W_{2,n} = \{c_{12}\}+\{c_{22}\}+C_n$, where $C_n = a_4c_{32}c_{14}c_{31}b_4a_5b_5 \dots a_{(n+3)/2}b_{(n+3)/2}a_4$ for the cases $m = 8, 9$ or the case that $m = 7$ and $c_{14}c_{32} \in E(G)$. If $m = 7$ and $c_{24}c_{32} \in E(G)$, since $c_{14}c_{32}$ and $c_{24}c_{32}$ are symmetrical, we can also obtain a $W_{2,n}$ in G. Thus, $c_{14}c_{32}, c_{24}c_{32} \in E(\overline{G})$, and then $\overline{G}$ contains a $C_7 = b_1c_{14}c_{32}c_{24}b_2b_3b_4b_1$, a contradiction. If $(p_1, p_2, p_3) = (4, 4, 1)$, then $m \le 8$ and $n = 5$. For $m = 7$, G contains a $W_{2,5} = \{c_{21}\} + \{c_{31}\} + C_5$, where $C_5 = c_{14}c_{23}c_{13}c_{24}c_{12}c_{14}$. For $m = 8$, G contains a $W_{2,5} = \{a_3\} + \{b_3\} + C_5$, where $C_5 = c_{13}c_{22}c_{12}c_{23}c_{11}c_{13}$. If $(p_1, p_2, p_3) = (4, 4, 0)$, then $m = 7$ and G contains a $W_{2,5} = \{c_{21}\} + \{b_3\} + C_5$, where $C_5 = c_{14}c_{23}c_{13}c_{24}c_{12}c_{14}$. If $(p_1, p_2, p_3) = (4, 3, 3), (3, 3, 3)$, then $m \le 9$ and G contains a $W_{2,n} = \{c_{12}\} + \{c_{22}\} + C_n$, where $C_n = a_4c_{33}c_{31}b_4a_5b_5 \dots a_{(n+3)/2}b_{(n+3)/2}c_{32}a_4$. If $(p_1, p_2, p_3) = (4, 3, 2)$, then $m \le 8$ and $n = 5$. For the case $m = 8$ or the case $m = 7$ and $c_{11}c_{31} \in E(G)$, G contains a $W_{2,5} = \{c_{13}\} + \{c_{22}\} + C_5$, where $C_5 = c_{31}c_{11}c_{32}a_4b_4c_{31}$. For $m = 7$ and $c_{11}c_{31} \in E(\overline{G})$, we have $b_3c_{14} \in E(G)$; otherwise $c_{11}c_{12}c_{13}c_{14}b_3c_{32}c_{31}c_{11}$ is a C_7 in $\overline{G}$, a contradiction. Then G contains a $W_{2,5} = \{c_{12}\} + \{c_{22}\} + C_5$, where $C_5 = b_4c_{31}c_{14}b_3a_4b_4$. If $(p_1, p_2, p_3) = (4, 3, 1)$, then $m = 7$. If $a_3c_{11}, b_3c_{14} \in E(\overline{G})$, then $c_{11}c_{12}c_{13}c_{14}b_3c_{31}a_3c_{11}$ is a C_7 in $\overline{G}$, a contradiction. By symmetry, we may assume that $a_3c_{11} \in E(G)$, and G contains a $W_{2,5} = \{c_{13}\} + \{c_{22}\} + C_5$, where $C_5 = a_3c_{11}b_3a_4b_4a_3$. If $(p_1, p_2, p_3) = (4, 2, 2)$, then $m = 7$. If $a_3c_{12}, a_4c_{21} \in E(\overline{G})$, then $a_1c_{11}c_{12}a_3a_4c_{21}a_2a_1$ is a C_7 in $\overline{G}$, a contradiction. Hence, either $a_3a_{12} \in E(G)$ or $a_4a_{21} \in E(G)$. Thus, G contains a $W_{2,5} = \{c_{12}\} + \{c_{21}\} + C_5$, where $C_5 = a_xc_{32}b_2c_{31}b_3a_x$, $a_x = a_3$ if $a_3a_{12} \in E(G)$, and $a_x = a_4$ if $a_4a_{21} \in E(G)$. If $(p_1, p_2, p_3) = (3, 3, 2)$, then $m \le 8$ and $n = 5$. For the case $m = 8$ or the case $m = 7$ and $a_3c_{21} \in E(G)$, G contains a $W_{2,5} = \{c_{12}\} + \{c_{21}\} + C_5$, where $C_5 = a_3c_{32}c_{23}c_{31}b_3a_3$. For $m = 7$ and $a_3c_{21} \in E(\overline{G})$, we have $b_3c_{23} \in E(G)$; otherwise $a_3c_{31}c_{32}b_3c_{23}c_{22}c_{21}a_3$ is a C_7 in $\overline{G}$, a contradiction. Since a_3c_{21} and b_3c_{23} are symmetrical, we can also obtain a $W_{2,5}$ if $b_3c_{23} \in E(G)$. If $(p_1, p_2, p_3) = (3, 2, 2)$, then $m = 7$ and G contains a $W_{2,5} = \{c_{12}\} + \{c_{21}\} + C_5$, where $C_5 = a_3c_{32}b_2c_{31}b_3a_3$. If $(p_1, p_2, p_3) = (2, 2, 2)$, then $m = 7$ and G contains a $W_{2,5} = \{c_{11}\}+\{c_{21}\}+C_5$, where $C_5 = a_3c_{32}b_2c_{31}b_3a_3$. Since $p_2 + p_3 \ge 4$, we have considered all the possible combinations of values for (p_1, p_2, p_3), and each time we derived a contradiction. This completes the proof of Case 2 and of Theorem 11.

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