# On Toughness and Hamiltonicity of $\mathbf{2} \mathbf{K}_{2}$-Free Graphs 

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#### Abstract

The toughness of a (noncomplete) graph $G$ is the minimum value of $t$ for which there is a vertex cut $A$ whose removal yields $|A| / t$ components. Determining toughness is an NP-hard problem for general input graphs. The toughness conjecture of Chvátal, which states that there exists a constant $t$ such that every graph on at least three vertices with toughness at least $t$ is hamiltonian, is still open for general graphs. We extend


[^0]some known toughness results for split graphs to the more general class of $2 K_{2}$-free graphs, that is, graphs that do not contain two vertex-disjoint edges as an induced subgraph. We prove that the problem of determining toughness is polynomially solvable and that Chvátal's toughness conjecture is true for $2 K_{2}$-free graphs. © 2013 Wiley Periodicals, Inc. J. Graph Theory 75: 244-255, 2014

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## 1. INTRODUCTION

Much of the background for this article and references to related work can be found in [1]. A good reference for any undefined terms in graph theory is [4] and complexity theory is [10]. We consider only undirected graphs with no loops or multiple edges. We begin by setting up some standard notation and terminology.

Let $\omega(G)$ denote the number of components of a graph $G$. A vertex cut of a connected graph $G=(V, E)$ is a set $S \subseteq V$ with $\omega(G-S)>1$. A graph $G$ is said to be $t$-tough if $|S| \geq t \omega(G-S)$ for every vertex cut $S$ of $G$. The toughness of $G$, denoted $\tau(G)$, is the maximum value of $t$ for which $G$ is $t$-tough (taking $\tau\left(K_{n}\right)=\infty$ for the complete graph $K_{n}$ on $n \geq 1$ vertices). Hence if $G$ is not a complete graph, $\tau(G)=\min \{|S| / \omega(G-S)\}$, where the minimum is taken over all vertex cuts $S$ of $G$. Following Plummer [16], a vertex cut $S$ of $G$ is called a tough set if $\tau(G)=|S| / \omega(G-S)$, that is, a tough set is a vertex cut $S$ of $G$ for which this minimum is achieved. A graph $G$ is hamiltonian if $G$ contains a Hamilton cycle, that is, a cycle containing every vertex of $G$. A 2-factor of $G$ is a 2 -regular spanning subgraph of $G$. Hence, a Hamilton cycle is a connected 2 -factor.

Historically, most of the research on toughness has been based on a number of conjectures in [6]. The most challenging of these conjectures, which is still open, states that there is a constant $t$ such that every $t$-tough graph is hamiltonian. This conjecture is called Chvátal's Conjecture and has been shown to be true when restricted to a number of graph classes [1], including planar graphs, claw-free graphs, co-comparability graphs, and chordal graphs.

A graph is chordal if every cycle on at least four vertices contains a chord, that is, an edge joining two vertices that are not adjacent on the cycle. Alternatively, one can view a chordal graph as the intersection graph of a family of subtrees of a tree. It was shown in [5] that every 18 -tough graph on at least three vertices is hamiltonian, but this result is probably far from best possible. The best known negative result is from [2] where an infinite class of chordal graphs with toughness close to $7 / 4$ having no Hamilton path is constructed.

There are several subclasses of chordal graphs, however, for which the smallest toughness guaranteeing hamiltonicity is known. A graph is called a split graph if its vertex set can be partitioned into a clique and an independent set; alternatively a split graph can be viewed as the intersection graph of a family of connected subgraphs of a star (and so split graphs are chordal graphs). It was shown in [13] that every 3/2-tough split graph on at least three vertices is hamiltonian, and that this is best possible in the sense that there is a sequence $\left\{G_{n}\right\}_{n=1}^{\infty}$ of split graphs with no 2 -factor and $\tau\left(G_{n}\right) \rightarrow 3 / 2$. This result was generalized by Kaiser, Král, and Stacho [11], who showed that 3/2-tough spiders are hamiltonian; a spider is the intersection graph of a family of connected subgraphs of a subdivision of a star (and so spiders are chordal graphs). Keil [12] showed that
every 1-tough interval graph is hamiltonian (an interval graph is the intersection graph of subpaths of a path), which is clearly best possible. Deogun et al. [7] generalized this by showing that 1 -tough co-comparability graphs (not a subclass of chordal graphs) are hamiltonian.

In this article, we consider a superclass of split graphs called $2 K_{2}$-free graphs. These are graphs that do not contain an induced copy of $2 K_{2}$, the graph on four vertices consisting of two vertex-disjoint edges. It is easy to see that every split graph is a $2 K_{2}$-free graph. One can also easily check that every cochordal graph (i.e., a graph that is the complement of a chordal graph) is $2 K_{2}$-free and so the class of $2 K_{2}$-free graphs is as rich as the class of chordal graphs. In Section 3, we show that Chvátal's Conjecture holds for $2 K_{2}$-free graphs by proving the following theorem.

Theorem 1. Every 25 -tough $2 K_{2}$-free graph on at least three vertices is hamiltonian.
While this establishes Chvátal's Conjecture for a new graph class, like the result for chordal graphs [5], our bound is very likely to be far from extremal. The proof of Theorem 1 relies on the very restrictive structure of triangle-free $2 K_{2}$-free graphs, and we are able to prove a sharp result for such graphs-triangle-free $2 K_{2}$-free graphs are hamiltonian if and only if they are 1 -tough.

Research on toughness has also focused on computational complexity issues. It was shown in [3] that the problem of recognizing $t$-tough graphs is coNP-complete for every fixed positive rational $t$. This implies that it is NP-hard to compute the toughness of a general input graph. On the other hand, toughness can be computed efficiently when the input graph is restricted to certain graph classes; see [1] for more details. In particular, recognizing $t$-tough graphs is polynomially solvable within the classes of claw-free graphs and split graphs [18]. For many other interesting classes, this complexity question is still open, for example, for (maximal) planar graphs and chordal graphs.

In Section 2, we extend the result of [18] on split graphs by showing that the toughness of $2 K_{2}$-free graphs can be computed in polynomial time.

Theorem 2. The toughness of a $2 K_{2}$-free graph can be determined in polynomial time.
We note that while many other problems that are NP-hard in general can be solved in polynomial time on $2 K_{2}$-free graphs, the problem of deciding whether a $2 K_{2}$-free graph is hamiltonian is NP-complete; indeed, the Hamilton cycle problem is even NP-complete on split graphs [15]. We refer the interested reader to [14] for more details and references to other work on $2 K_{2}$-free graphs.

## 2. COMPLEXITY

In this section, we show how to determine the toughness of $2 K_{2}$-free graphs in polynomial time. For a graph $G=(V, E)$, we denote by $N_{G}(I)$ the union of the sets of neighbors of vertices of $I \subseteq V$ in $V \backslash I$. We begin by stating two properties of $2 K_{2}$-free graphs. The first of them follows immediately from the definition and the second one was proved in [9].

Observation 1. A graph $G=(V, E)$ is $2 K_{2}$-free if and only if for every $A \subset V$ at most one component of the graph $G-A$ contains edges.

Lemma 1. [9] A $2 K_{2}$-free graph on $n$ vertices contains at most $n^{2}$ maximal independent sets; moreover, all of them can be listed in time $\mathcal{O}\left(n^{2}\right)$.

Proof of Theorem 2. Given a $2 K_{2}$-free graph $G=(V, E)$ on $n$ vertices as input, use the following algorithm.

Step 1. List all maximal independent sets of $G$ using the (implicit) polynomial-time algorithm from [9]. Denote them by $I_{1}, I_{2}, \ldots, I_{k}$, where $k \leq n^{2}$.
Step 2. For every $i \in\{1,2, \ldots, k\}$ consider the split graph $G_{i}$ obtained from $G$ by adding all necessary edges that turn $V \backslash I_{i}$ into a clique in $G_{i}$. Determine the toughness $\tau_{i}=\tau\left(G_{i}\right)$ using the (implicit) polynomial-time algorithm from [18].
Output. $t=\min \left\{\tau_{i} \mid i \in\{1,2, \ldots, k\}\right\}$.
Clearly, the algorithm outputs $t$ in polynomial time. We show that $t=\tau(G)$. Let $S$ be a tough set of $G$. Then by Observation 1, at most one component of $G-S$ contains edges; the other components induce a nonempty independent set $I$ in $G$. Clearly, $N_{G}(I)$ is also a vertex cut, $N_{G}(I) \subseteq S$ and $\omega\left(G-N_{G}(I)\right) \geq \omega(G-S)$. If $N_{G}(I) \neq S$ then $\left|N_{G}(I)\right|<|S|$ and hence

$$
\frac{\left|N_{G}(I)\right|}{\omega\left(G-N_{G}(I)\right)}<\frac{|S|}{\omega(G-S)},
$$

contradicting that $S$ is a tough set of $G$. Thus, $N_{G}(I)=S$. Let $I_{j}$ be a maximal independent set of $G$ containing $I$. Then $S \cap I_{j}=\emptyset$, since otherwise $I_{j}$ is not independent. Let $G_{j}$ be obtained from $G$ by turning $V \backslash I_{j}$ into a clique, and let $\tau_{j}=\tau\left(G_{j}\right)$. Then $\omega\left(G_{j}-S\right)=$ $\omega(G-S)$, and so

$$
t=\min _{i} \tau_{i} \leq \tau_{j} \leq \frac{|S|}{\omega\left(G_{j}-S\right)}=\frac{|S|}{\omega(G-S)}=\tau(G)
$$

For inequality in the other direction, suppose $t=\tau_{j^{\prime}}=\min _{i} \tau_{i}$, and suppose $S_{j^{\prime}}$ is a tough set of $G_{j^{\prime}}$. Then $\omega\left(G-S_{j^{\prime}}\right) \geq \omega\left(G_{j^{\prime}}-S_{j^{\prime}}\right)$, since adding edges to $G$ cannot increase the number of components of $G-S_{j^{\prime}}$. Hence,

$$
\tau(G) \leq \frac{\left|S_{j^{\prime}}\right|}{\omega\left(G-S_{j^{\prime}}\right)} \leq \frac{\left|S_{j^{\prime}}\right|}{\omega\left(G_{j^{\prime}}-S_{j^{\prime}}\right)}=\tau_{j^{\prime}}=t
$$

We conclude that $t=\tau(G)$, proving Theorem 2 .

## 3. TOUGHNESS AND HAMILTONICITY

In this section, we prove Chvátal's Conjecture for $2 K_{2}$-free graphs by proving Theorem 1 . Note that since all complete graphs on at least three vertices are hamiltonian, we will restrict our attention to noncomplete graphs on at least three vertices.
We will repeatedly use the following easy proposition.
Proposition 1. Let $G$ be a noncomplete $t$-tough graph. Then $\delta(G) \geq 2 t$.
Proof. Let $v$ be a vertex of degree $\delta(G)$. Since $N(v)$ is a vertex cut of $G$ and $G$ is $t$-tough, $|N(v)| \geq 2 t$.

In order to prove Theorem 1, we shall require some structural properties of triangle-free $2 K_{2}$-free graphs.

## A. Triangle-Free and Claw-Free Cases

We begin this subsection by giving a simple, but useful, characterization of triangle-free $2 K_{2}$-free graphs.

We say that a graph $G$ is of $C_{5}^{*}$-type if its vertex set can be partitioned into five disjoint nonempty sets $A_{1}, A_{2}, \ldots, A_{5}$ such that each $A_{i}$ is an independent set, $A_{i} \cup A_{i+1}$ induces a complete bipartite graph in $G$ for all $i$ (taken modulo 5), and the union of any other pair of sets $A_{i} \cup A_{j}$ induces an independent set.
Lemma 2. Let $G=(V, E)$ be a connected triangle-free $2 K_{2}$-free graph. Then either $G$ is bipartite or $G$ is of $C_{5}^{*}$-type.

Proof. Suppose $G$ is not bipartite. Then $G$ contains an odd cycle. Since $G$ is trianglefree and $2 K_{2}$-free, this implies $G$ contains an induced 5-cycle $C=x_{1} x_{2} x_{3} x_{4} x_{5} x_{1}$, and every vertex of $G$ has at most two neighbors in $C$. If $x \in V \backslash V(C)$ has one neighbor in $C$, say $x_{1}$, then $x x_{1}$ and $x_{3} x_{4}$ induce a copy of $2 K_{2}$; if $x$ has no neighbors in $C$, then by connectivity $x$ has at least one neighbor $y$ in $G$. Since $y$ has at most two neighbors in $C$, there is some edge $x_{i} x_{i+1}$ in $C$ such that $y$ is not adjacent to $x_{i}$ and $x_{i+1}$. But then $x_{i} x_{i+1}$ and $x y$ induce a copy of $2 K_{2}$. Thus, every vertex of $G$ has exactly two neighbors in $C$. Since $G$ is triangle-free, every vertex $v$ of $G$ has neighbors $x_{i-1}$ and $x_{i+1}$ in $C$ for some $i$ (taken modulo 5). Let $A_{i}$ denote the set of vertices adjacent to $x_{i-1}$ and $x_{i+1}$ so that the $A_{i}$ 's form a partition of the vertices. By triangle-freeness, each $A_{i} \cup A_{i+2}$ is an independent set, and by $2 K_{2}$-freeness, each $A_{i} \cup A_{i+1}$ induces a complete bipartite graph in $G$ (if $y_{i} \in A_{i}$ and $y_{i+1} \in A_{i+1}$ are nonadjacent, then $x_{i-1} y_{i}$ and $y_{i+1} x_{i+2}$ induce a copy of $\left.2 K_{2}\right)$. Thus, $G$ is of $C_{5}^{*}$-type.

Lemma 2 implies the following corollary. We write $\alpha(G)$ for the size of the largest independent set of $G$.

Corollary 3. Let $G$ be a connected $2 K_{2}$-free graph on $n$ vertices.

- If $G$ is triangle-free, then $\alpha(G) \geq 2 n / 5$.
- If $G$ is noncomplete and 4-tough, then $G$ contains at least two vertex-disjoint triangles.

Proof. Statement (i) follows immediately from Lemma 2.
Suppose that, contrary to (ii), $G=(V, E)$ is a noncomplete 4-tough $2 K_{2}$-free graph with no pair of vertex-disjoint triangles. Thus, we can make $G$ triangle-free by removing at most three vertices, and so, by (i) and using Proposition $1, G$ has an independent set $I$ with $|I| \geq 2(n-3) / 5 \geq 2$. Thus, $V \backslash I$ is a vertex cut with $|V \backslash I| \leq \frac{3(n-3)}{5}+3$. Since $G$ is 4-tough, we have

$$
4 \leq \frac{|V \backslash I|}{\omega(G-(V \backslash I))} \leq \frac{3(n-3) / 5+3}{2(n-3) / 5} \leq \frac{3}{2}+\frac{3}{2}=3
$$

a contradiction.
Now we prove that Chvátal's Conjecture is true for triangle-free $2 K_{2}$-free graphs; moreover, we give the extremal lower bound on toughness that guarantees hamiltonicity.


FIGURE 1. The special graphs.

While the proof is conceptually simple, it is technically awkward, and this makes it longer than one might expect.
Theorem 4. Let $G=(V, E)$ be a triangle-free $2 K_{2}$-free graph on at least three vertices. Then $G$ is hamiltonian if and only if $G$ is 1 -tough.

Proof. Since 1-toughness is clearly a necessary condition for hamiltonicity, it suffices to prove that $G$ is hamiltonian if $G$ is 1-tough. For this, let $G$ be a 1-tough triangle-free $2 K_{2}$-free graph on $n \geq 3$ vertices. Then $G$ is 2-connected and by Lemma 2, $G$ is either bipartite or of $C_{5}^{*}$-type.

First, suppose $G$ is bipartite. Since $G$ is 1-tough, the bipartition must be into two equalsized sets $X=\left\{x_{1}, \ldots, x_{k}\right\}$ and $Y=\left\{y_{1}, \ldots, y_{k}\right\}$. Since $G$ is $2 K_{2}$-free, the neighborhoods of the vertices in $X$ are nested: we can order them so that $N\left(x_{i}\right) \subseteq N\left(x_{j}\right)$ for all $i \leq j$. Since $G$ is 1-tough, $\left|N\left(x_{i}\right)\right|>i$ for $i=1, \ldots, k-1$ and $N\left(x_{k}\right)=Y$. Thus, we can order the vertices of $Y$ such that $N\left(x_{i}\right) \supseteq\left\{y_{1}, \ldots, y_{i+1}\right\}$ for $i=1, \ldots, k-1$. This immediately gives us the Hamilton cycle $y_{1} x_{1} y_{2} x_{2} \cdots y_{k} x_{k} y_{1}$.

Next suppose $G$ is of $C_{5}^{*}$-type with sets $A_{i}$ of cardinality $a_{i}$ as in Lemma 2, where indices are taken modulo 5 . We choose an indexing of the sets $A_{i}$ in such a way that $a_{1}-a_{5}=\max _{i}\left|a_{i+1}-a_{i}\right|$. Note that $G$ is 1-tough if and only if

$$
\begin{gather*}
a_{i-2}-a_{i-1}+a_{i}-a_{i+1}+a_{i+2} \geq 0  \tag{1}\\
a_{j-1}-a_{j}+a_{j+1} \geq 1 \tag{2}
\end{gather*}
$$

for all $i, j \in\{1, \ldots, 5\}$, because any tough set $S$ of $G$ is of the form $S=A_{i-2} \cup A_{i} \cup A_{i+2}$ for some $i$ or of the form $S=A_{j-1} \cup A_{j+1}$ for some $j$.

We prove by induction on $n$ that $G$ is hamiltonian. It is straightforward to verify that the graphs in Figure 1 (we call them the special graphs) as well as the graphs with $a_{1}=a_{5}$ are hamiltonian. This gives us the base of the induction.

For the induction step, assume that $G$ is any 1-tough not special $C_{5}^{*}$-type graph with $a_{1} \geq a_{5}+1 \geq 2$. Then $a_{2} \geq 2$ since otherwise, by our choice of cyclic indexing, we have $a_{2}=a_{5}=1$ and $a_{1} \geq 2$, which contradicts $j=1$ in (2).

First, we show that the inequalities (1) and (2) remain true if we replace $a_{1}$ and $a_{2}$, respectively, with $a_{1}^{\prime}=a_{1}-1$ and $a_{2}^{\prime}=a_{2}-1$. It is clearly true for $i=1,2,3,5$ in (1) and $j=1,2,4$ in (2). Since $a_{2}-a_{3}+a_{4} \geq 1$ and $a_{1}-a_{5} \geq 1$ we have $a_{2}^{\prime}-a_{3}+$ $a_{4}-a_{5}+a_{1}^{\prime}=a_{2}-a_{3}+a_{4}-a_{5}+a_{1}-2 \geq 0$, proving $i=4$ of (1). Since $a_{4} \geq 1$ and $a_{1}-a_{5} \geq 1$ we have $a_{4}-a_{5}+a_{1}^{\prime}=a_{4}-a_{5}+a_{1}-1 \geq 1$, proving $j=5$ in (2).

Finally, it remains to prove $j=3$ in (2). If $a_{1} \geq a_{5}+2$, then by (1) we have $a_{1}+a_{3} \leq$ $a_{2}+a_{4}+a_{5} \leq a_{2}+a_{4}+a_{1}-2$, and rearranging gives $a_{2}^{\prime}-a_{3}+a_{4} \geq 1$, as required. Thus, we may assume $a_{1}=a_{5}+1$; by our choice of $a_{1}$ and $a_{5}$ we have that for every $i$

$$
\begin{equation*}
\left|a_{i}-a_{i-1}\right| \leq 1, \tag{3}
\end{equation*}
$$

and in particular, $a_{3}-a_{4} \leq 1$. If $a_{2} \geq 3$ or $a_{3} \leq a_{4}$, then $a_{2}^{\prime}-a_{3}+a_{4}=a_{2}-a_{3}+a_{4}-$ $1 \geq 1$, as required. The only case remaining is when $a_{5}=\alpha, a_{1}=\alpha+1, a_{2}=2, a_{3}=$ $\beta, a_{4}=\beta-1$ for some integers $\alpha, \beta$, and due to (3), we must have $\alpha \in\{1,2\}$ and $\beta \in\{2,3\}$. But this is impossible since the case $\alpha=1, \beta=2$ corresponds to the special graph a), the case $\alpha=2, \beta=3$ corresponds to the special graph c ), and the two other cases yield the special graph b).
So, if we pick arbitrary $x_{1} \in A_{1}$ and $x_{2} \in A_{2}$ then the graph $G^{\prime}=G-\left\{x_{1}, x_{2}\right\}$ is 1-tough and of $C_{5}^{*}$-type. Then by induction, $G^{\prime}$ has a Hamilton cycle $H^{\prime}$. Taking two arbitrary vertices $y_{1} \in A_{1} \backslash\left\{x_{1}\right\}$ and $y_{2} \in A_{2} \backslash\left\{x_{2}\right\}$ on $H^{\prime}$, if $y_{1} y_{2}$ is an edge of $H^{\prime}$, then we can remove $y_{1} y_{2}$ and include $x_{1} y_{2}, x_{2} y_{1}$, and $x_{1} x_{2}$ to form a Hamilton cycle $H$ of $G$. If $y_{1} y_{2}$ is not an edge of $H^{\prime}$ for any $y_{1} \in A_{1} \backslash\left\{x_{1}\right\}$ and $y_{2} \in A_{2} \backslash\left\{x_{2}\right\}$, then let $z_{1}, z_{1}^{\prime} \in A_{5}$ and $z_{2}, z_{2}^{\prime} \in A_{3}$ be the two neighbors in $H^{\prime}$ of $y_{1}$ and $y_{2}$, respectively. We remove the edges $y_{1} z_{1}$ and $y_{2} z_{2}$ from $H^{\prime}$ and replace them with $x_{1} z_{1}$ and $x_{2} z_{2}$ to form $H^{\prime \prime} ; H^{\prime \prime}$ is the union of two paths that together span every vertex of $G$. We obtain a Hamilton cycle $H$ of $G$ either by adding the edges $x_{1} x_{2}$ and $y_{1} y_{2}$, or by adding the edges $x_{1} y_{2}$ and $y_{1} x_{2}$ to $H^{\prime \prime}$.

We require the following refinement of the previous theorem.
Lemma 3. Let $G$ be a connected triangle-free $2 K_{2}$-free graph on $n$ vertices and suppose that $\alpha(G)<n / 2$. Then $G$ is of $C_{5}^{*}$-type and for every edge $e$ of $G$, there is a Hamilton cycle of $G$ that includes $e$.

Proof. Since $\alpha(G)<n / 2, G$ cannot be bipartite, and so, by Lemma 2, $G$ is of $C_{5}^{*}$-type with independent sets $A_{1}, \ldots, A_{5}$. Recall that $G$ is 1-tough if and only if the cardinalities of $A_{i}$ satisfy (1) and (2). Using $\alpha(G)<n / 2$, one can easily check that all 10 inequalities (1) and (2) hold (and so $G$ is 1-tough), and that in fact the inequalities (1) are strict. By Theorem 4, $G$ has a Hamilton cycle $H$.
Suppose the given edge $e$ is not in $H$. Without loss of generality, let $e=x y$ with $x \in A_{1}$ and $y \in A_{5}$. If $H$ has no edges connecting $A_{1}$ with $A_{5}$, then every edge of $H$ connects $A_{2} \cup A_{4}$ with $A_{1} \cup A_{3} \cup A_{5}$, implying that $\left|A_{2}\right|+\left|A_{4}\right|=\left|A_{1}\right|+\left|A_{3}\right|+\left|A_{5}\right|$. This contradicts that (1) holds strictly, and so $H$ contains an edge $a b$ with $a \in A_{1}$ and $b \in A_{5}$. By symmetry, we may simply switch the roles of $a, b$ and $x, y$, respectively, to give a Hamilton cycle of $G$ passing through $e=x y$.

Before moving on to our main result, we digress to prove the following result, which stands independently from the rest of the article.

Theorem 5. Let $G$ be a claw-free $2 K_{2}$-free graph on at least three vertices. Then $G$ is hamiltonian if and only if $G$ is 1-tough.

We use a fact proved in [17], namely that every 2 -connected $2 K_{2}$-free graph $G$ has a dominating cycle, that is, a cycle $C$ such that $G-V(C)$ is an independent set. For a vertex $x$ on $C$, we write $x^{+}$and $x^{-}$for the successor and predecessor of $x$ along a fixed orientation of $C$. For vertices $a, b$ on $C$, we write $P(a, b)$ for the path from $a$ to $b$ along the fixed orientation of $C$.

Proof of Theorem 5. Let $G$ be a 1-tough (and so 2-connected) claw-free, $2 K_{2}$-free graph on at least three vertices, and suppose $G$ is not hamiltonian. Let $C$ be a longest dominating cycle of $G$, and fix an orientation of $C$. We show that $C$ is a Hamilton cycle. If not, let $v \in V(G) \backslash V(C)$ and let $x_{1}$ and $x_{2}$ be two distinct neighbors of $v$ on $C$. Obviously, by the choice of $C, v x_{i}^{+} \notin E(G)$ and $v x_{i}^{-} \notin E(G)$. Hence, since $G$ is claw-free, $x_{i}^{-} x_{i}^{+} \in$ $E(G)$. Considering the edges $v x_{1}$ and $x_{2}^{-} x_{2}^{+}$, and using $2 K_{2}$-freeness, at least one of $x_{2}^{-}, x_{2}^{+}$ is a neighbor of $x_{1}$, say $x_{2}^{-} x_{1} \in E(G)$. Then the cycle $x_{1} v x_{2} P\left(x_{2}, x_{1}^{-}\right) x_{1}^{-} x_{1}^{+} P\left(x_{1}^{+}, x_{2}^{-}\right) x_{2}^{-} x_{1}$ is a longer dominating cycle than $C$, a contradiction.

## B. General Case

The proof of Theorem 1 follows immediately from Lemmas 4 and 5. Let $G$ be a $2 K_{2}$-free graph. We say that $G$ has a path-triangle factor (PT-factor for short) if $G$ has a spanning subgraph in which each component is either a triangle (a T-component) or a pair of vertex-disjoint triangles connected by a path that has exactly one vertex in common with each of the triangles (a TPT-component). In a TPT-component, the edge of a triangle that is not incident with a vertex of the connecting path is called the free edge of that triangle.

Lemma 4. Let $G$ be a $2 K_{2}$-free graph. If $G$ has a $P T$-factor, then $G$ is hamiltonian.

Proof. Suppose $G$ is a $2 K_{2}$-free graph with a PT-factor, and let $F$ be a PT-factor with a minimum number of components. We first show that $F$ has only one component.
If $F$ has two T-components, then there is an edge between them since $G$ is $2 K_{2}$-free. Thus, the two T-components can be reduced to a single TPT-component, contradicting the choice of $F$.
Suppose $F$ contains two TPT-components $H_{1}$ and $H_{2}$ where $H_{i}$ consists of a path $Q_{i}$ connecting the triangles $T_{i}$ and $T_{i}^{\prime}$ for $i=1,2$. Let $e_{i}$ be the free edge of $T_{i}^{\prime}$. Since $G$ is $2 K_{2}$-free, there must be an edge $e$ connecting one end of $e_{1}$ to one end of $e_{2}$. We can now construct a long path $P$ from $T_{1}$ to $T_{2}$ using all the edges of $Q_{1}$ and $Q_{2}$, the edges $e$, $e_{1}$, and $e_{2}$, and using another (suitable) edge from $T_{1}^{\prime}$ and another (suitable) edge from $T_{2}^{\prime}$. Together, $T_{1}, P$, and $T_{2}$ form a single TPT-component that includes all vertices of $H_{1}$ and $H_{2}$. This contradicts our choice of $F$. In a similar way, a T-component and a TPT-component in $F$ can be reduced to a single TPT-component.
Thus $F$ has a single component, which must be a TPT-component in case $G$ is not a complete graph. Let $x_{1} x_{2}$ and $y_{1} y_{2}$ be the free edges of the two triangles in this TPTcomponent. It is easy to see that for every $i \in\{1,2\}$ and $j \in\{1,2\}$ there is a Hamilton path in $G$ connecting $x_{i}$ and $y_{j}$. Since $G$ is $2 K_{2}$-free, there is an edge $x_{i} y_{j}$ in $G$ for some $i, j \in\{1,2\}$. Then the Hamilton path from $x_{i}$ to $y_{j}$ mentioned above combined with the edge $x_{i} y_{j}$ forms a Hamilton cycle of $G$.

Lemma 5. Every 25 -tough $2 K_{2}$-free graph on at least three vertices has a PT-factor.

Proof. Let $G=(V, E)$ be a $t$-tough $2 K_{2}$-free graph with $t \geq 25$. By Proposition 1 , $\delta(G) \geq 50$, and by Corollary 3 (ii), $G$ has at least two vertex-disjoint triangles, say $T_{1}$ and $T_{2}$. So $G$ has vertex-disjoint triangles $T_{1}, T_{2}, \ldots, T_{k}$ such that the (possibly empty) set of vertices $X=V \backslash \cup_{j} T_{j}$ induces a triangle-free subgraph $G[X]$ of $G$. If $X$ is empty,
then we have a PT-factor as required. Otherwise, let $p=|X|$ and let $\mathbf{T}$ denote the set of triangles. We consider two cases.
Case 1. There is an independent set $I$ of size at least $p / 2$ in $G[X]$.
First, we show that $k \geq 4 p$. If $p \leq 3$, then since $\delta(G) \geq 50$, one easily obtains $k \geq$ $4 p$. Otherwise, the removal from $G$ of all vertices not in $I$ produces at least $p / 2 \geq 2$ components, and therefore $(3 k+p / 2) /(p / 2) \geq t \geq 25$. This yields $k \geq 4 p$.

We say that a vertex of $X$ is adjacent to a triangle $T_{j}$ of $\mathbf{T}$ if it is adjacent to at least one vertex of $T_{j}$. We partition $X$ into two sets $A$ and $B$ as follows: given $x \in X$ we include $x$ in $A$ if it is adjacent to fewer than $2 p$ triangles from $\mathbf{T}$ and include $x$ in $B$ otherwise. We observe that the set $A$ is independent: if $x y$ is an edge in $G[A]$, then since both $x$ and $y$ are adjacent to fewer than $2 p$ of at least $4 p$ triangles in $\mathbf{T}$, there is a triangle $T^{*}$ that is neither adjacent to $x$ nor to $y$. But then any edge of $T^{*}$ together with $x y$ induces a copy of $2 K_{2}$, a contradiction. Set $a=|A|$ so that $|B|=p-a$.
Next, we show how to cover the set $A$ by a set $\mathcal{P}(A)$ of disjoint paths of length 2 in such a way that
(a) each vertex of $A$ is an inner vertex of a path;
(b) all end vertices of the paths are either in $B$ or in triangles of $\mathbf{T}$; and
(c) each triangle of $\mathbf{T}$ contains an end vertex of at most one path.

In order to do this, consider an auxiliary bipartite graph $H$ with a bipartition into sets $A$ and $B \cup T$, where $T=\left\{t_{1}, t_{2}, \ldots, t_{k}\right\}$ is a set of vertices corresponding to the triangles of $\mathbf{T}$. The edges of $H$ consist of all edges joining vertices of $A$ and $B$ in $G$ together with all edges $x t_{j}$ for which $x \in A$ is adjacent to the triangle $T_{j} \in \mathbf{T}$. It is not hard to see that having vertex-disjoint paths of length 2 satisfying (a), (b), and (c) is equivalent to having a subgraph of $H$ in which every vertex in $A$ has degree 2 and all other vertices have degree at most 1 . Such a subgraph exists, for if not, then by the defect form of Hall's Theorem, there is a subset $A^{\prime} \subset A$ whose neighborhood $N_{H}\left(A^{\prime}\right)$ in $H$ has cardinality less than $2\left|A^{\prime}\right|$. But then the neighborhood $N_{G}\left(A^{\prime}\right)$ of $A^{\prime}$ in $G$ has cardinality less than $6\left|A^{\prime}\right|$. This violates the toughness condition on $G$, since taking $N_{G}\left(A^{\prime}\right)$ as our vertex cut, we find that $G-N_{G}\left(A^{\prime}\right)$ has at least $\left|A^{\prime}\right|$ components (recall $A$ is independent) and so

$$
25 \leq t \leq \frac{\left|N_{G}\left(A^{\prime}\right)\right|}{\omega\left(G-N_{G}\left(A^{\prime}\right)\right)} \leq \frac{6\left|A^{\prime}\right|}{\left|A^{\prime}\right|}=6,
$$

a contradiction. (Note that if $\left|A^{\prime}\right|=1$, then $N_{G}\left(A^{\prime}\right)$ may not be a vertex cut, but the fact that $\left|N_{G}\left(A^{\prime}\right)\right|<6\left|A^{\prime}\right|$ then violates the minimum degree condition on $G$.) Thus, $G$ contains a set $\mathcal{P}(A)$ of vertex-disjoint paths of length 2 satisfying conditions (a), (b), and (c) above.

Call a triangle of $\mathbf{T}$ taken if it contains an end vertex of one of the paths in $\mathcal{P}(A)$, and call it nontaken otherwise. Note that there are at most $2 a$ taken triangles. By the definition of $B$, each vertex of $B$ is adjacent to at least $2 p$ triangles of $\mathbf{T}$ and so adjacent to at least $2 p-2 a$ nontaken triangles of $\mathbf{T}$. Since $|B|=p-a$, we can greedily match each vertex $b \in B$ to two nontaken triangles adjacent to $b$ such that each nontaken triangle is matched with at most one vertex of $B$. Thus, we have a set $\mathcal{P}(B)$ of vertex-disjoint paths of length 2 such that

- each vertex of $B$ is an inner vertex of a path;
- all end vertices of the paths are in nontaken triangles of $\mathbf{T}$; and
- each triangle of $\mathbf{T}$ contains an end vertex of at most one path.

Now we are ready to construct a PT-factor of $G$. Consider the subgraph $F$ of $G$ formed from the edges of $\mathcal{P}(A), \mathcal{P}(B)$, and $\mathbf{T}$. Note that the edges of $\mathcal{P}(A) \cup \mathcal{P}(B)$ form a forest $F^{\prime}$, and each leaf of that forest is a vertex of exactly one triangle from $\mathbf{T}$. Note also that in $F^{\prime}$, each $a \in A$ has degree 2 , each $b \in B$ has degree 2 or 3 , and each $b \in B$ is adjacent to exactly two leaves. We construct a subgraph $F^{*}$ of $F$ as follows: for each $b \in B$ that has degree 3 in $F$, we remove one edge of $F$ between $b$ and a triangle of $\mathbf{T}$. It is easy to see that $F^{*}$ is a PT-factor of $G$.

Case 2. $\alpha(G[X])<p / 2$
By Observation 1, $G[X]$ is the union of one nontrivial component $J$ and an independent set $I$. Since $\alpha(G[X])<p / 2=|X| / 2$, we have $\alpha(J)<|J| / 2$. By Lemma 3, $J$ is of $C_{5}^{*}$-type and for every edge $e$ in $J$ there is a Hamilton cycle $C_{e}$ of $J$ that includes $e$.

Recall that $k \geq 2$. We say that a vertex of $J$ is $b a d$ if it is nonadjacent to all of the triangles in T. Since $G$ is $2 K_{2}$-free, the set of all bad vertices is independent. Since $J$ contains an odd cycle, there must be an edge $e=x y$ in $J$ such that neither $x$ nor $y$ is bad. Furthermore, every triangle in $\mathbf{T}$ is adjacent to $x$ or to $y$. Since these vertices are not bad, we can choose two different triangles $T_{1}, T_{2}$ in $\mathbf{T}$ such that $x$ is adjacent to $T_{1}$ and $y$ is adjacent to $T_{2}$. Combined with the Hamilton path $C_{e}-e$ they form a TPT-component covering all vertices of $J$.
To finish the construction of a PT-factor, we have to match every vertex $v \in I$ with two distinct triangles from $\mathbf{T} \backslash\left\{T_{1}, T_{2}\right\}$ such that no triangle is matched with two different vertices. As in Case 1, by the defect form of Hall's Theorem, this fails if and only if there is a subset $I^{\prime} \subseteq I$ adjacent to fewer than $2\left|I^{\prime}\right|$ triangles, that is, $\left|N_{G}\left(I^{\prime}\right)\right|<$ $6\left|I^{\prime}\right|+6$. If $\left|I^{\prime}\right|=1$, this violates that $\delta(G) \geq 50$. If $\left|I^{\prime}\right|>1$ then $N_{G}\left(I^{\prime}\right)$ is a vertex cut with $\omega\left(G-N_{G}\left(I^{\prime}\right)\right) \geq\left|I^{\prime}\right|$, violating that $G$ is 25 -tough. This completes the proof of Lemma 5.

## 4. CONCLUSION

Recall that in [13] it was shown that every 3/2-tough split graph on at least three vertices is hamiltonian, and that this result is best possible. Since split graphs are $2 K_{2}$-free, this shows that we cannot decrease the bound in Theorem 1 below $3 / 2$. Thus, there is a large gap between our upper bound of 25 and the lower bound of $3 / 2$. We are not sure whether the lower bound is extremal, but we are almost certain that the upper bound is not! We believe it may be possible to extend the method of PT-factors to obtain a bound close to 6 (note that Case 1 in Lemma 5 is the only place where toughness 25 is used), but we believe a different approach will be needed to obtain the extremal bound.

For 2-tough $2 K_{2}$-free graphs, we can use the existence of a 2 -factor (guaranteed by a result in [8]) to obtain a fairly long cycle. It is not difficult to see how to combine the cycles of a 2 -factor in a $2 K_{2}$-free graph to obtain a long cycle that misses at most one vertex of each cycle in the 2 -factor (and that picks up all the vertices of the triangles of the 2 -factor). In particular one can prove the following result. We leave the details to the reader.

Theorem 6. Let $G$ be a $2 K_{2}$-free graph on $n \geq 3$ vertices and let $k$ be the number of cycles of length at least 4 in a 2-factor of $G$. Then $G$ has a cycle of length at least $n-k$.

It is not clear whether we can use the above result as a starting point for proving that 2-tough $2 K_{2}$-free graphs are hamiltonian, let alone for obtaining a best possible toughness result for general $2 K_{2}$-free graphs.

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