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Plant zero structure and further order reduction of a singular H_∞ controller

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SUMMARY

A new class of reduced-order controllers is obtained for the H_∞ problem. The reduced-order controller does not compromise the performance attained by the full-order controller. Algorithms for deriving reduced-order H_∞ controllers are presented in both continuous and discrete time. The reduction in order is related to unstable transmission zeros of the subsystem from disturbance inputs to measurement outputs. In the case where the subsystem has no infinite zeros, the resulting order of the H_∞ controller is lower than that of the existing reduced-order H_∞ controller designs which are based on reduced-order observer design. Furthermore, the mechanism of the controller order reduction is analysed on the basis of the two-Riccati equation approach. The structure of the reduced-order H_∞ controller is investigated. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: H_∞ control; reduced-order controller design; transmission zeros

1. INTRODUCTION

Low-order controller design is one of the fundamental though difficult problems in control engineering. In particular, reduced-order H_∞ controller design problems has received quite some attention over the past decade; see, for example, References [1–14]. In References [1–6], fundamental ideas in the controller order reduction lie in the structure of the system: for

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example injectivity or surjectivity of the feedthrough term of a subsystem, and finite or infinite zero structure of a subsystem. A consequence is that we can see *a priori* the resulting order of the reduced-order controller. On the other hand, in References [7–14] the problem is considered in a rather general setup by using linear matrix inequalities (LMIs). After the presentation of an elegant idea of solving control problems using LMIs [8, 9], in the study of reduced-order controller design, the main focus is rather on solving a non-convex optimization problem. Indeed, the tools to find directly a controller of a given order can be based on the numerical method described in References [11–14]. Although the approach is so flexible that many control problems can be treated in a unified manner, the problem is that these algorithms still have numerical difficulties and also the structure of the resulting controller is not well understood.

Our focus is not on solving the optimization problem, but on finding a relationship between the structure of the control problem and the order of the controller. Moreover, we do not investigate optimal or suboptimal controllers of a fixed degree for the H_∞ control problem. Instead, we seek tools to reduce the order of the controller without compromizing performance. For an H_∞ problem, here part of the measurements is not affected by disturbances, reduced-order controller design has been studied in References [1–4, 15]. This was first studied in Kimura, *et al.* [1], then a reduced-order H_∞ controller is derived in continuous-time case on the basis of reduced-order observers by Stoorvogel *et al.* [2]. Also in Reference [15], the continuous-time reduced-order H_∞ controller design has been considered and the structure of the controller is clarified on the basis of a controller parametrization established in Mita *et al.* [16]. On the other hand, in the discrete-time case as well as in the continuous-time case, Xin *et al.* [3] characterized the reduced-order H_∞ controller design problem in terms of LMIs and algorithms to derive a reduced-order controller in this setting are available. The common factor of these approaches is that the resulting order of the H_∞ controller is the same. However, it can be expected that further order reduction is possible.

In this paper, the relationship between these approaches is discussed and the possibility of further order reduction is investigated. Then it is shown that in the continuous-time case there exists a reduced-order H_∞ controller which has lower order than the order of the controllers derived in References [1–4, 15] if the sub-system from disturbances to measurements has unstable transmission zeros on the non-negative real axis but no infinite zeros. Also, in the discrete-time case a similar argument is used to derive a reduced-order H_∞ controller. We present algorithms based on the LMI approach for designing the reduced-order H_∞ controllers for both continuous-time and discrete-time cases. Secondly, the structure of the controller order reduction is analysed on the basis of the classical approach which uses two algebraic Riccati equations. This analysis not only finds new ways to reduce the order of the controller further, but also establishes the plant structure that allows us to reduce the order of the H_∞ controller. From this analysis we also find another algorithm to obtain the reduced-order H_∞ controller. Compared with the former algorithm this one has an advantage that we can obtain a larger class of reduced-order H_∞ controllers in the case where the subsystem has multiple real zeros and in the case where the subsystem has complex zeros.

Notation: I_n denotes the identity matrix of dimension $n \times n$. For a matrix $A \in \mathbb{C}^{n\times n}$, $\lambda(A)$ denotes the set of eigenvalues of A. The conjugate transpose of a matrix A is denoted by A^*. We denote the set of positive real numbers by $\mathbb{R}^+$, the open left half complex plane by $\mathbb{C}^-$ and the open right half complex plane by $\mathbb{C}^+$. In continuous time, if a matrix has all eigenvalues in $\mathbb{C}^-$, the matrix is said to be stable. In discrete time, if a matrix has all eigenvalues inside the unit circle, the matrix is said to be stable. The class of stable real rational transfer functions is denoted by $\mathcal{RH}_\infty$. A generalized inverse of a matrix D is defined by the properties $DD^\dagger D = D$ and $D^\dagger DD^\dagger = D^\dagger$ and is in general not unique. A square matrix D is said to be unitary if $DD^{\mathrm{T}} = I$. An orthogonal complement of a full-rank matrix D is denoted by $D^\perp$, and is chosen in such a way that the following relations hold. If D is of full column rank, $(D\ D^\perp)$ is square and the equation:

$$(D\ D^\perp)\begin{pmatrix} D^\dagger \\ (D^\perp)^{\mathrm{T}} \end{pmatrix} = I$$

holds. If D is of full row rank, $\left(D^{\mathrm{T}}\ (D^\perp)^{\mathrm{T}}\right)^{\mathrm{T}}$ is square and the equation:

$$\left(D^\dagger\ (D^\perp)^{\mathrm{T}}\right)\begin{pmatrix} D \\ D^\perp \end{pmatrix} = I$$

holds. The symbol (A, B, C, D) is used to represent the system with a transfer matrix $D + C(sI - A)^{-1}B$. The symbol $\mathcal{F}_\ell$ is used to represent the linear fractional transform: $\mathcal{F}_\ell(G, Q) = G_{11} + G_{12}Q(I - G_{22}Q)^{-1}G_{21}$, where

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

The H_∞ norm of a stable, continuous-time transfer function $G(s)$ is defined as $\|G(s)\|_\infty := \sup_{\omega\in(-\infty,\infty)} \bar{\sigma}G(j\omega)$ where $\bar{\sigma}$ denotes the largest singular value. Similarly, the H_∞ norm of a stable, discrete-time transfer function $G(z)$ is defined as $\|G(z)\|_\infty := \sup_{\omega\in(-\pi,\pi)} \bar{\sigma}G(e^{j\omega})$. A symbol $\zeta \in \mathbb{C}$ is used to represent s in the continuous time and z in the discrete time.

In this paper, various kinds of zeros are used to present results.

Definition 1 Invariant zero

The invariant zero of a system with realization (A, B, C, D) is defined as $\zeta_0 \in \mathbb{C}$ for which

$$\operatorname{rank}\begin{pmatrix} \zeta_0 I - A & -B \\ C & D \end{pmatrix} < \operatorname{normrank}\begin{pmatrix} \zeta I - A & -B \\ C & D \end{pmatrix}$$

holds [17, 31], where normrank means the rank of a matrix with entries in the field of rational functions.

Definition 2 Transmission zero

The transmission zeros of the system (A, B, C, D) are equal to the zeros of the transfer matrix $G(\zeta)$ as defined through its Smith–McMillan form.

Lemma 1

Suppose $\alpha \in \mathbb{C}$ is not a pole of the transfer matrix $G(\zeta)$. Then the system has a transmission zero in α if and only if $\operatorname{rank}(G(\alpha)) < \operatorname{normrank}(G(\zeta))$.

Moreover, the transmission zeros are a subset of the invariant zeros of a system and in case of a minimal realization these two sets are equal.

2. PROBLEM DESCRIPTION AND BACKGROUND

2.1. Singular H_∞ control

Consider the following linear time-invariant (LTI) system:

$$\Sigma : \begin{cases} \sigma x = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x + D_{21} w + D_{22} u \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $z \in \mathbb{R}^{p_1}$ is the controlled output, $y \in \mathbb{R}^{p_2}$ is the measurement output, $w \in \mathbb{R}^{m_1}$ is the disturbance input and $u \in \mathbb{R}^{m_2}$ is the control input. A symbol σ indicates the derivative, $\sigma x(t) = \mathrm{d}x(t)/\mathrm{d}t$, in continuous time and a shift operation, $\sigma x(k) = x(k+1)$, in discrete time. For this system we make the assumptions that (A, B_2) is stabilizable and (C_2, A) is detectable. The transfer function of Σ is denoted by

$$\Sigma(\zeta) = \begin{pmatrix} \Sigma_{11}(\zeta) & \Sigma_{12}(\zeta) \\ \Sigma_{21}(\zeta) & \Sigma_{22}(\zeta) \end{pmatrix}$$

where Σ_{ij}, $(i, j = 1, 2)$ is the subsystem associated with the transfer matrix $\Sigma_{ij}(\zeta) = D_{ij} + C_j(\zeta I - A)^{-1} B_i$.

The H_∞ suboptimal control problem is to find a stabilizing controller

$$\Sigma_k : \begin{cases} \sigma\eta = A_k \eta + B_k y \\ u = C_k \eta + D_k y \end{cases}$$

where $\eta \in \mathbb{R}^{n_k}$ is the state of the controller, such that the resulting closed-loop system has an H_∞-norm strictly less than an *a priori* given bound γ, if one exists. In this paper we call γ the performance index of the H_∞ control problem. If γ is minimized, the problem is called the H_∞ optimal control problem. It is known that if the suboptimal H_∞ control problem is solvable, then we can always find a suitable controller whose dynamic order is at most equal to the order of system (1). However, in the singular case where the direct feedthrough matrix D_{21} does not have full row rank or in its dual case where D_{12} does not have full column rank we can guarantee *a priori* that a controller of lower order than the order of the system can be found, see References [2, 3, 15].

2.2. Background

We can assume, without loss of generality, that the matrix $(C_2\ D_{21})$ has full row rank. For the singular problem where D_{21} does not have full row rank, we then obtain that

$$\operatorname{rank}(C_2\ D_{21}) - \operatorname{rank}(D_{21}) > 0.$$

From References [2, 3, 15], we know that we can obtain a reduced-order H_∞ controller whose order is less than or equal to

$$n_k = n - [\operatorname{rank}(C_2\ D_{21}) - \operatorname{rank}(D_{21})]. \tag{2}$$

A key step in deriving the reduced-order controller is the reduced-order observer-based controller design [2], where the system with partially noise-free measurement outputs can be stabilized with an output feedback controller whose order is equal to the order of the system Σ

minus the number of noise-free measurements. In Reference [15], the continuous-time reduced-order H_∞ controller is also derived from the same standpoint, and the structure of the reduced-order H_∞ controller is clarified on the basis of the classical Riccati-based approach. On the other hand, the reduced-order H_∞ controller design problem can also be characterized in terms of LMIs and algorithms to derive the reduced-order controller of order n_k are presented not only for the continuous-time case but also for the discrete-time case in Reference [3].

Interestingly, from these results it can be established that there exists a reduced-order H_∞ controller which has order lower than n_k if $\Sigma_{21}(\zeta)$ has unstable transmission zeros on the real axis in continuous time and in discrete time as well. We obtain this result by exploiting bilinear transformations to reveal relationships between the continuous-time and discrete-time systems. Motivated by this, we consider an algorithm to obtain reduced-order H_∞ controllers of lower order. In addition, ideas in Reference [15] are utilized to understand the mechanism of the controller order reduction in continuous time as well as to extend the result that is based on the bilinear transformations to the case where $\Sigma_{21}(s)$ has unstable transmission zeros in $\mathbb{C}^+$.

3. RESULTS BASED ON BILINEAR TRANSFORM

3.1. Continuous time

In this section, on the basis of the result presented in Reference [3], we show that there exists a continuous-time reduced-order H_∞ controller which has order less than n_k. Then we present an algorithm, which uses the LMI approach, to obtain the reduced-order H_∞ controller.

First, define bilinear transformations

$$\Gamma_c : s \mapsto z = \frac{s+\alpha}{s-\alpha}, \quad 0 < \alpha \in \mathbb{R} \tag{3}$$

$$\Gamma_{c_0} : s \mapsto \tilde{s} = \frac{1}{s}. \tag{4}$$

Γ_c transforms a continuous-time system into a discrete-time system and Γ_{c_0} transforms a continuous-time system into another continuous-time system. The stability, the order and the H_∞ norm of an LTI system are invariant under these transformations. If α is not an eigenvalue of A, then Γ_c transforms the system Σ to a new system

$$\tilde{\Sigma}_d : \begin{cases} \sigma\tilde{x}_d = \tilde{A}_d\tilde{x}_d + \tilde{B}_{d1}w + \tilde{B}_{d2}u \\ z = \tilde{C}_{d1}\tilde{x}_d + \tilde{D}_{d11}w + \tilde{D}_{d12}u \\ y = \tilde{C}_{d2}\tilde{x}_d + \tilde{D}_{d21}w + \tilde{D}_{d22}u \end{cases} \tag{5}$$

where

$$\begin{aligned} \tilde{x}_d &= (\alpha I - A)x - B_1 w - B_2 u \\ \tilde{A}_d &= -(\alpha I + A)(\alpha I - A)^{-1} \\ \tilde{B}_{di} &= -[(\alpha I + A)(\alpha I - A)^{-1} + I]B_i \\ \tilde{C}_{di} &= C_i(\alpha I - A)^{-1} \\ \tilde{D}_{dij} &= \Sigma_{ij}(\alpha), \quad i, j = 1, 2. \end{aligned}$$

Here, it should be noted that the matrix $\tilde{D}_{d21}$ has a lower rank than the matrix D_{21} if D_{21} has full rank and the following inequality holds:

$$\text{rank}\,(\Sigma_{21}(\alpha)) < \text{normrank}\,(\Sigma_{21}(s)), \quad \alpha > 0 \tag{6}$$

This means that we can choose a bilinear transformation such that $\tilde{D}_{d21}$ has a lower rank than the normal rank of the associated transfer matrix if the system $\Sigma_{21}(s)$ has transmission zeros on the non-negative real axis. Moreover, if 0 is not an eigenvalue of A, Γ_{c_0} transforms the system Σ to a new system $\tilde{\Sigma}_c$, which can be derived similarly as $\tilde{\Sigma}_d$ but of course using a different bilinear transform. On the basis of this observation we deduce the following result.

Theorem 1
Suppose that the system $\Sigma_{21}(s)$ has unstable transmission zeros on the non-negative real axis and let one of these zeros be $\alpha \geqslant 0$. In that case, there exists a continuous-time H_∞ controller with a performance index γ of order

$$n_{k_c} = n - [\text{rank}\,(C_2\ D_{21}) - \text{rank}\,(\Sigma_{21}(\alpha))] \tag{7}$$

if the continuous-time H_∞ problem for the system Σ is solvable with a performance index γ. Moreover, if D_{21} has full rank, then we obtain $n_{k_c} < n_k$.

Before we prove Theorem 1, we state a useful lemma.

Lemma 2
For $\alpha \geqslant 0$, there exists a matrix $N \in \mathbb{R}^{m_2 \times p_2}$ which satisfies

$$\det(\alpha I - A - B_2 N C_2) \neq 0 \tag{8}$$

if (A, B_2, C_2) is stabilizable and detectable.

Proof
Since (A, B_2, C_2) is stabilizable and detectable, for the system Σ we can find a stabilizing controller:

$$\sigma p = Kp + Ly$$
$$u = Mp$$

Then, since $\alpha \geqslant 0$ we have

$$\det \begin{pmatrix} \alpha I - A & -B_2 M \\ -LC_2 & \alpha I - K \end{pmatrix} \neq 0$$

because otherwise the closed-loop system has an unstable pole in α. By taking the Schur complement, we obtain

$$\det(\alpha I - A - B_2 M(\alpha I - K)^{-1} L C_2) \neq 0.$$

Here, we need that $K - \alpha I$ is invertible. If this is not the case, note that when K is replaced by $K + \varepsilon I$ this is still a stabilizing controller for ε small enough. Finally, by replacing $M(\alpha I - K)^{-1}L$ with N we obtain (8). □

By this lemma, even if we have $\det(\alpha I - A) = 0$, by using a static output feedback:

$$u = Ny + v \tag{9}$$

we can make the A-matrix of the new system:

$$\Sigma_F : \begin{cases} \sigma x = (A + B_2 N C_2)x + (B_1 + B_2 N D_{21})w + B_2 v \\ z = (C_1 + D_{12} N C_2)x + (D_{11} + D_{12} N D_{21})w + D_{12} v \\ y = C_2 x + D_{21} w \end{cases} \tag{10}$$

non-singular at α, where we assume $D_{22} = 0$ without loss of generality. Then we can apply the reduced-order controller design in Theorem 1 to Σ_F and obtain a reduced-order controller. The preliminary static output feedback does not change the order of the resultant controller, and zeros of the system Σ_{21} are invariant under this preliminary feedback.

Next, let us move on to the proof of Theorem 1.

Proof of Theorem 1

The condition that (A, B_2, C_2) is stabilizable and detectable is implied by the solvability of the H_∞ control problem. Therefore from Lemma 2, we can assume, without loss of generality, that α is not an eigen-value of A. When $\alpha > 0$, Γ_c preserves the stability, the order and the H_∞ norm of an LTI system. If a controller with transfer matrix Σ_k solves the continuous-time H_∞ control problem and therefore yields an H_∞ norm strictly less than γ, then we have

$$\|\mathscr{F}_\ell(\Sigma(s), \Sigma_k(s))\|_\infty = \|\mathscr{F}_\ell(\tilde{\Sigma}(z), \tilde{\Sigma}_k(z))\|_\infty < \gamma \tag{11}$$

where $\tilde{\Sigma}_k(z)$ is the bilinear transform of $\Sigma_k(s)$. This implies equivalence of existence of H_∞ controllers of an order n' for both the continuous time and discrete time. Using results in Reference [3] we can then obtain a discrete-time controller of order

$$n - [\operatorname{rank}(\tilde{C}_2\, \Sigma_{21}(\alpha)) - \operatorname{rank}(\Sigma_{21}(\alpha))]$$

After applying the inverse bilinear transformation to the discrete-time controller we then obtain a controller of the continuous-time system of the same order. The result follows by noting that

$$\operatorname{rank}(\tilde{C}_2\, \Sigma_{21}(\alpha)) = \operatorname{rank}(C_2\, D_{21}) \tag{12}$$

When $\alpha = 0$, we use Γ_{c_0} to obtain $\tilde{\Sigma}_c$. Similarly Γ_{c_0} preserves the stability, the order and the H_∞ norm of an LTI system. Then a similar argument as in the previous case can be applied.

It remains to show that n_{k_c} is strictly smaller than n_k if D_{21} has full rank. Since α is a transmission zero of the system we know that $\operatorname{rank}(\Sigma_{21}(\alpha)) < \operatorname{normrank}(\Sigma_{21}(s))$. On the other hand, since D_{21} has full rank, we obtain $\operatorname{normrank}(\Sigma_{21}(s)) = \operatorname{rank}(D_{21})$. It is then obvious that $\operatorname{rank}(\Sigma_{21}(\alpha)) < \operatorname{rank}(D_{21})$ and combined with (12) the result follows. □

Remark 1

Using duality, we can also obtain reduced-order controllers when Σ_{12} has unstable zeros with a resulting order of the controller equal to

$$n - \left[\operatorname{rank}\begin{pmatrix} B_2 \\ D_{12} \end{pmatrix} - \operatorname{rank}\Sigma_{12}(\alpha)\right]$$

However due to space limitations we will only discuss the case where Σ_{21} has unstable zeros, although all of our results have dual versions.

Remark 2

If $\alpha < 0$, Γ_c no longer preserves the stability of the system and inequality (11) is not satisfied. This idea therefore only works when $\Sigma_{21}(s)$ has transmission zeros on the non-negative real axis. On the other hand, in the case where $\Sigma_{21}(s)$ has stable zeros, the controller order reduction problem is discussed in References [5, 6].

Remark 3

This theorem states that if $\Sigma_{21}(s)$ has a transmission zero on the non-negative real axis, we can obtain a reduced-order H_∞ controller of order n_{k_c}. Based on the above proof we obtain the following algorithm to design the reduced-order H_∞ controller.

I. Transform the system Σ by Γ_c or by Γ_{c_0} to obtain $\tilde{\Sigma}_d$ or $\tilde{\Sigma}_c$, where the parameter α is chosen such that inequality (6) is satisfied. If the matrix A has an eigenvalue at α, apply the preliminary feedback (9) for Σ first, and then apply the transformation.
II. Solve the discrete-time H_∞ problem for $\tilde{\Sigma}_d$ or the continuous-time H_∞ problem for $\tilde{\Sigma}_c$ by using the LMI algorithm presented in Reference [3], and obtain an H_∞ controller of order n_{k_c}.
III. Apply the inverse of the bilinear transformation that is used in step I to this controller to obtain an n_{k_c}th order continuous-time H_∞ controller with realization (A_k, B_k, C_k, D_k). The resultant controller is obtained as $(A_k, B_k, C_k, D_k + N)$ where if the preliminary feedback is not used, N is put to zero.

Remark 4

If the matrix D_{21} has full column rank, the difference between n_k and n_{k_c} is

$$\begin{aligned} n_k - n_{k_c} &= \text{rank}\,(D_{21}) - \text{rank}\,(\Sigma_{21}(\alpha)) \\ &= \text{normrank}\,(\Sigma_{21}(s)) - \text{rank}\,(\Sigma_{21}(\alpha)) \end{aligned}$$

This indicates that the difference in the order between the reduced order H_∞ controller obtained here and the n_kth order H_∞ controller is equal to the geometric multiplicity [17] of an unstable transmission zero of $\Sigma_{21}(s)$ at α. Thus, the order reduction of the controller depends on the selection of the zero. If the geometric multiplicity of the zero is higher, then we can obtain a lower order H_∞ controller.

3.2. Discrete time

For the discrete-time case we use, similar to the continuous time, a bilinear transformation:

$$\Gamma_d : z \mapsto \bar{z} = \frac{\beta z - 1}{\beta - z}, \quad |\beta| > 1, \quad \beta \in \mathbb{R}. \tag{13}$$

This transformation transforms a discrete-time system into another discrete-time system, and the unit circle and the unit disk are invariant under this transformation. The stability, the order and the H_∞ norm of an LTI system are therefore invariant under this transformation. If β is not an eigenvalue of A; this can be assumed, without loss of generality, from Lemma 2, $\Gamma_{\rm d}$ transforms the system Σ to a new system:

$$\bar{\Sigma}_{\rm d}: \begin{cases} \sigma\bar{x}_{\rm d} = \bar{A}_{\rm d}\bar{x}_{\rm d} + \bar{B}_{\rm d1}w + \bar{B}_{\rm d2}u \\ z = \bar{C}_{\rm d1}\bar{x}_{\rm d} + \bar{D}_{\rm d11}w + \bar{D}_{\rm d12}u \\ y = \bar{C}_{\rm d2}\bar{x}_{\rm d} + \bar{D}_{\rm d21}w + \bar{D}_{\rm d22}u \end{cases} \tag{14}$$

where

$$\begin{aligned} \bar{x}_{\rm d} &= (\beta I - A)x - B_1 w - B_2 u \\ \bar{A}_{\rm d} &= -(I - \beta A)(\beta I - A)^{-1} \\ \bar{B}_{{\rm d}i} &= -[-(I - \beta A)(\beta I - A)^{-1} + \beta I]B_i \\ \bar{C}_{{\rm d}i} &= C_i(\beta I - A)^{-1} \\ \bar{D}_{{\rm d}ij} &= \Sigma_{ij}(\beta), \quad i, j = 1, 2 \end{aligned}$$

Here it should be noted that the matrix $\bar{D}_{\rm d21}$ has a lower rank than the matrix D_{21} if the following inequality:

$$\text{rank}\,(\Sigma_{21}(\beta)) < \text{normrank}\,\Sigma_{21}(z) \tag{15}$$

holds. This means that $\bar{D}_{\rm d21}$ has a lower rank if the system $\Sigma_{21}(z)$ has a transmission zero $\{\beta \in \mathbb{R} \| \beta| > 1\}$.

On the other hand, when $\Sigma_{21}(z)$ has a transmission zero at $z = 1$ or -1, we may introduce one of the following bilinear transformations:

$$\Gamma_{{\rm d}_1}: z \mapsto s = \eta\frac{z+1}{z-1}, \quad 0 < \eta \in \mathbb{R}, \quad \text{if } \Sigma_{21}(z) \text{ has a zero at 1}$$

$$\Gamma_{{\rm d}_{-1}}: z \mapsto s = \eta\frac{z-1}{z+1}, \quad 0 < \eta \in \mathbb{R}, \quad \text{if } \Sigma_{21}(z) \text{ has a zero at } -1$$

These transformations transform a discrete-time system into a continuous-time system. It can also be verified that these transformation preserve the stability, the order and the H_∞ norm of an LTI system. If 1 is not an eigenvalue of A, $\Gamma_{{\rm d}_1}$ transforms the system Σ to a new system $\check{\Sigma}$. On the other hand, if -1 is not an eigenvalue of A, $\Gamma_{{\rm d}_{-1}}$ transforms the system Σ to a new system $\hat{\Sigma}$.

On the basis of these observations, the following result can be deduced.

Theorem 2

Suppose that the system $\Sigma_{21}(z)$ has a transmission zero

$$\beta \in \{z \in \mathbb{R} \| z| \geqslant 1\}$$

Then, there exists a discrete-time H_∞ controller of order

$$n_{k_c} = n - [\text{rank}\,(C_2\ D_{21}) - \text{rank}\,(\Sigma_{21}(\beta))] \tag{16}$$

which solves the H_∞ control problem with a performance index γ if the discrete-time H_∞ problem for the system Σ is solvable with a performance index γ. Moreover, if D_{21} has full rank, then we obtain $n_{k_c} < n_k$.

Proof

Since the parallel argument for the continuous-time case can be applied to this case, we omit the details. If one of the transmission zeros is at $z=1$ or -1, transform the discrete-time system Σ into a continuous-time system by using one of the bilinear transformations of Γ_{d_1} and $\Gamma_{d_{-1}}$. By the parallel argument with the continuous case, we can obtain a discrete-time H_∞ controller of order n_{k_c} for Σ. In the case where no zero is at $z=1$ nor at $z=-1$, but one of the zeros is $\beta \in \{z \in \mathbb{R} \| z| > 1\}$, we obtain a discrete-time system $\bar{\Sigma}_d$ by using Γ_d. Since stability, the order and the H_∞ norm are invariant under this transformation, it is verified that we can obtain a discrete-time H_∞ controller of order n_{k_c} for Σ. It remains to show that n_{k_c} is strictly smaller than n_k if D_{21} has full rank. Since β is a transmission zero of the system we know that rank $(\Sigma_{21}(\beta)) <$ normrank $(\Sigma_{21}(s))$. On the other hand, since D_{21} has full rank we obtain normrank$(\Sigma_{21}(s)) =$ rank(D_{21}). It is then obvious that rank $(\Sigma_{21}(\beta)) <$ rank (D_{21}), hence the result follows. □

Remark 5

This theorem says that if $\Sigma_{21}(z)$ has transmission zeros on $\{z \in \mathbb{R} \| z| \geqslant 1\}$, we can obtain a reduced-order H_∞ controller of order n_{k_c} by following the algorithm:

I. If $\Sigma_{21}(z)$ has a transmission zero on $\{z \in \mathbb{R} \| z| > 1\}$, transform the system Σ by Γ_d to obtain the system $\bar{\Sigma}$, where the parameter β is chosen as the transmission zero in $\{z \in \mathbb{R} \| z| > 1\}$. Then proceed to step II. If $\Sigma_{21}(z)$ has a transmission zero at 1 or at -1, transform the system Σ by Γ_{d_1} $\Gamma_{d_{-1}}$ respectively, where the parameter η is chosen arbitrary such that $0 < \eta \in \mathbb{R}$ is satisfied. Then proceed to step III.

II. Solve the discrete-time H_∞ problem for $\bar{\Sigma}_d$ using the algorithm based on LMIs as presented in Reference [3], and obtain a discrete-time H_∞ controller of order n_{k_c}. Then proceed to step IV.

III. Solve the continuous-time H_∞ problem for $\check{\Sigma}$ or $\hat{\Sigma}$ using the algorithm based on LMIs as presented in Reference [3], and obtain a continuous-time H_∞ controller of order n_{k_c}.

IV. Transform the controller by the inverse of the bilinear transformation that is applied in step I to obtain a discrete-time H_∞ controller of order n_{k_c}.

Remark 6

If the matrix D_{21} has full column rank, the difference between n_k and n_{k_c} is

$$\begin{aligned} n_k - n_{k_c} &= \text{rank}\,(D_{21}) - \text{rank}\,(\Sigma_{21}(\beta)) \\ &= \text{normrank}\,(\Sigma_{21}(z)) - \text{rank}\,(\Sigma_{21}(\beta)) \end{aligned}$$

where $\beta \in \{z \in \mathbb{R} \| z| \geqslant 1\}$. This implies that the difference in the order between the reduced-order H_∞ controller obtained here and the n_kth order H_∞ controller is equal to the geometric multiplicity [17] of an unstable transmission zero of $\Sigma_{21}(z)$ on $\{z \in \mathbb{R} \| z| \geqslant 1\}$. Thus, the number in order reduction of the controller depends on the selection of the zero. If the geometric multiplicity of the zero is higher, then we can obtain a lower order H_∞ controller.

3.3. Discussions

We have thus characterized a new class of reduced-order H_∞ controllers by using one unstable real zero of Σ_{21}. However, in general, Σ_{21} might have more unstable zeros and they might be complex. Moreover, by using a continuity argument, we can argue that if a zero is slightly changed by some perturbation, we may still be able to reduce the order. Thus, we have issues as

1. In the case where Σ_{21} has multiple zeros, we have to choose one of the zeros as the parameter α even when the zeros are close to each other. However, from a continuity argument we can expect that both of the zeros can render to reduce the order of the controller. Our algorithm can handle only one zero, however, it is not clear whether further reduction is possible in case of multiple zeros.
2. In the case where the system Σ_{21} has zeros in the open right half-plane, we can also apply the bilinear transformation Γ_c to the system Σ. Here, α is a complex number such that $\alpha \in \mathbb{C}^+$. Then Γ_c still preserves the stability of the system and inequality (11) is also satisfied. In this case, we have to solve an H_∞ control problem for a system with complex coefficients. Thus, whether this technique generally works for the system that has a non-minimum-phase subsystem Σ_{21} or not depends on whether the LMI techniques from Reference [3] work for complex valued systems.

Algorithms presented in this section relies on a reduction in rank of the transfer matrix $\Sigma_{12}(\zeta)$ at its transmission zeros. Clearly, computing numerically a rank is intrinsically difficult. However, computing transmission zeros can be done in a reliable fashion. Therefore, we will, in general, apply this transformation and work with system which have singular values which are very small in a point α but not exactly equal to zero. If we perturb the system to make the singular values equal to zero then we can rely on a continuity argument to conclude that for the obtained reduced-order system will also work for the original system. This is mathematically not a precise statement but in examples we have seen that these algorithms are numerically more robust then one might think after noting all the rank evaluations.

4. ANALYSIS OF THE CONTROLLER ORDER REDUCTION

In the previous section we have established the existence of a new class of reduced-order H_∞ controllers, and obtained an algorithm to design the controller. The structure of the reduced-order H_∞ controller obtained there is not cleat because it is based on the bilinear transformation. Therefore, it is not easy in our view to analyse the above questions on continuity and multiple zeros based on this technique of bilinear transforms. Moreover, we know from previous papers that the structure of the reduced-order H_∞ controller of order n_k can be explained by a reduced-order observer. Hence, we will present a different approach which will allow us to better understand the structure of the reduced-order H_∞ controller. This will then allow us to obtain results for the cases of multiple zeros and for continuity arguments.

Thus, motivated by the results presented in the previous section, we analyse the mechanism of the controller order reduction in the continuous-time singular H_∞ problem. We will see that the results based on the bilinear transformation can be extended to more general cases. The analysis is based on a solution obtained by using the fundamental two-Riccati equation approach [1, 15, 16, 18, 19]. We assume that Σ_{21} has unstable zeros in $\mathbb{C}^+$ and, in addition, that D_{21} has full column rank. To this end, we introduce our recent work [15], which uses the two-ARE approach presented by Mita *et al.* [16], and extend its idea to analyse the controller order reduction.

4.1. Preliminaries

Consider the continuous-time system described in (1). In addition to the conditions that (A, B_2) is stabilizable and (C_2, A) is detectable, we consider the H_∞ problem under the following assumptions:

A1. D_{12} is of full column rank.
A2. D_{21} is of full column rank.
A3. Σ_{21} has invariant zeros in $\mathbb{C}^+$
A4. Σ_{12} and Σ_{21} do not have invariant zeros on the imaginary axis.

Assumptions A1, A2 and A4 are made to simplify our analysis. Assumption A3 captures the feature of the singular H_∞ problem we consider. The more general singular problem when these assumptions are not satisfied has been studied in References [3, 20–24]. However, there is no study of the controller order reduction exploiting the existence of invariant zeros in $\mathbb{C}^+$.

Without loss of generality, we can put assumptions on the matrices C_2, D_{11}, D_{21} and D_{22} as follows:

B1. $D_{11}=0$ and $D_{22}=0$.
B2.

$$(C_2\, D_{21}) = \begin{pmatrix} C_{21} & 0 \\ C_{22} & I_{m_1} \end{pmatrix},$$

where $C_{21} \in \mathbb{R}^{(p_2-m_1)\times n}$ is of full row rank.

Assumption B1 can be relaxed by using some standard techniques as described in References [21, 25]. Assumption B2 basically amounts to choosing a suitable basis for the input and output spaces.

The invariant zeros of the system Σ_{21} can be made explicit by a suitable state-space transformation for the system Σ, i.e. $x = T\bar{x}$ with T invertible.

Lemma 3

There exists a suitable basis such that the matrices $A - B_1 D_{21}^{\dagger} C_2$, $(D_{21}^{\perp})^{\mathrm{T}}\, C_2$ and $D_{21}^{\dagger} C_2$ have the following block decomposition with A_- stable and $A_+ \in \mathbb{R}^{r\times r}$ antistable

$$\begin{pmatrix} A - B_1 D_{21}^{\dagger} C_2 \\ \hdashline (D_{21}^{\perp})^{\mathrm{T}} C_2 \\ \hdashline D_{21}^{\dagger} C_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{13} & A_{14} & 0 & 0 \\ A_{31} & A_{32} & A_- & 0 \\ A_{33} & A_{34} & 0 & A_+ \\ \hdashline I_{p_2-m_1} & 0 & 0 & 0 \\ \hdashline C_{22ll} & C_{22lr} & C_{22rl} & C_{22rr} \end{pmatrix} \tag{17}$$

where the pair (A_+, C_{22rr}) is observable.

By the definition of the invariant zero, it can be verified that eigenvalues of A_+ are the unstable invariant zeros of Σ_{21}.

Proof
See Reference [26]. □

Furthermore, for notational ease we partition C_1 accordingly with $D_{21}^{\dagger} C_2$ as

$$C_1 = (C_{1ll}\; C_{1lr}\; C_{1rl}\; C_{1rr})$$

4.2. Parameterization of the H_∞ controller

In this section, we derive the full-order H_∞ controller for the system Σ on the basis of the two-ARE approach. Here full order means the same order as the system Σ has. First, we introduce two AREs:

$$X(A - B_2 D_{12}^{\dagger} C_1) + (A - B_2 D_{12}^{\dagger} C_1)^{\mathrm{T}} X + X\left[\gamma^{-2} B_1 B_1^{\mathrm{T}} - B^2 D_{12}^{\dagger}(B_2 D_{12}^{\dagger})^{\mathrm{T}}\right] X + C_1^{\mathrm{T}} D_{12}^{\perp}(D_{12}^{\perp})^{\mathrm{T}} C_1 = 0 \tag{18}$$

and

$$Y A_{ZH}^{\mathrm{T}} + A_{ZH} Y + Y\left[\gamma^{-2} C_1^{\mathrm{T}} C_1 - (D_{12}^{\dagger} C_2)^{\mathrm{T}} D_{12}^{\dagger} C_2\right] Y = 0 \tag{19}$$

Here, A_{ZH} is defined by $A_{ZH} = A - B_1 D_{21}^{\dagger} C_2 + L_H (D_{21}^{\perp})^{\mathrm{T}} C_2$ where a matrix L_H is selected such that

$$\{\lambda(A_{ZH})\} - \{\lambda(A_+)\} \subset \mathbb{C}^- \tag{20}$$

is satisfied.[¶] This condition implies that the observable subspace of the pair $(A - B_1 D_{21}^{\dagger} C_2, (D_{21}^{\perp})^{\mathrm{T}} C_2)$ is stabilized by L_H.

If these AREs have solutions that satisfy

$$\lambda\left(A - B_2 D_{12}^{\dagger} C_1 + \left[\gamma^{-2} B_1 B_1^{\mathrm{T}} - B_2 D_{12}^{\dagger}(B_2 D_{12}^{\dagger})^{\mathrm{T}}\right] X\right) \subset \mathbb{C}^-$$

$$\lambda\left(A_{ZH} + Y\left[\gamma^{-2} C_1^{\mathrm{T}} C_1 - \left(D_{21}^{\dagger} C_2\right)^{\mathrm{T}} D_{21}^{\dagger} C_2\right]\right) \subset \mathbb{C}^-$$

we call these solutions the stabilizing solutions. It is easy to show that the stabilizing solution of these AREs is unique and that the solution Y is independent of a specific choice for L_H, provided that (20) is satisfied. These AREs are the key equations for analysing the mechanism of the controller order reduction as well as for deriving the H_∞ controller.

By following the result presented in Reference [16], we can obtain the class of all suboptimal H_∞ controllers, which are parametrized with two free parameters.

[¶] The set difference is defined by $\mathcal{S} - \mathcal{T} := \{x \in \mathcal{S} | x \notin \mathcal{T}\}$.

Lemma 4

Suppose that (A, B_2) is observable and (A, C_2) is detectable, and the assumptions A1–A4 are satisfied. Then an H_∞ controller with performance index γ exists if and only if the two AREs in (18) and (19) have stabilizing solutions $X \geqslant 0$, $Y \geqslant 0$ and these matrices satisfy

$$\gamma^2 I - XY > 0 \tag{21}$$

If there exists an H_∞ controller, the class of all H_∞ controllers is represented by

$$K_\infty = \{K_\infty(s) | N(s),\ W(s) \in \mathcal{RH}_\infty,\ \|N(s)\|_\infty < \gamma\} \tag{22}$$

with

$$K_\infty(s) = K_1^{-1}(s) K_2(s) \tag{23}$$

where parameters are defined by

$$\begin{aligned}
K_1(s) &= C_K(s)(sI - A_Y)^{-1}\hat{B}_2 + \Pi^{-1} \\
K_2(s) &= C_K(s)(sI - A_Y)^{-1}H_\infty + N(s)D_{21}^{\dagger} + W(s)(D_{21}^{\perp})^{\mathrm{T}} \\
A_Y &= A + \gamma^{-2} Y C_1^{\mathrm{T}} C_1 + H_\infty C_2 \\
C_K(s) &= -\Pi^{-1} F_\infty Z + N(s) D_{21}^{\dagger} \hat{C}_2 Z + W(s)(D_{21}^{\perp})^{\mathrm{T}} C_2 \\
\hat{B}_2 &= B_2 + \gamma^{-2} Y C_1^{\mathrm{T}} D_{12} \\
\hat{C}_2 &= \gamma^{-2} D_{21} B_1^{\mathrm{T}} X + C_2 \\
F_\infty &= -D_{12}^{\dagger} C_1 - D_{12}^{\dagger}(B_2 D_{12}^{\dagger})^{\mathrm{T}} X \\
H_\infty &= -B_1 D_{21}^{\dagger} - Y(D_{21}^{\dagger} C_2)^{\mathrm{T}} D_{21}^{\dagger} + L_H (D_{21}^{\perp})^{\prime\mathrm{T}} \\
z &= (I - \gamma^{-2} YX)^{-1} \\
\Pi &= (D_{12}^{\mathrm{T}} D_{12})^{-1/2}.
\end{aligned}$$

Here, $N(s)$ and $W(s)$ are called free parameters.

Proof

See Reference [16]. □

Remark 7

1. If we put free parameters equal to zero, we obtain an nth order H_∞ controller which is called the central solution. The free parameters might be utilized to improve the controller performance. But also, the free parameters can be used to reduce the order of the controller. Controller order reduction using this philosophy is discussed in the next section.
2. It has been reported for the regular H_∞ optimal control in References [7, 27] that the matrix Z might be ill conditioned as γ tends to the optimal value. Thus, there exists a numerical difficulty in calculating the controller by the formula in (22). However, this difficulty depends on the realization of the controller, and using a descriptor form can be

circumvented [7, 19, 22, 25, 28]. On the other hand, in the limit the infinite zeros of the controller are cancelled by the poles of the controller at infinity. Thus the controller order can be further reduced by exploiting this property.

4.3. n_kth order H_∞ controller

4.3.1. Deflation of the Riccati equation

By using the formula in (17), the unique stabilizing solution of the ARE in (19) can be represented by the unique stabilizing solution of the reduced-order ARE:

$$Y_r A_+^{\mathrm{T}} + A_+ Y_r + Y_r(\gamma^{-2} C_{1rr}^{\mathrm{T}} C_{1rr} - C_{22rr}^{\mathrm{T}} C_{22rr}) Y_r = 0 \tag{24}$$

where the solution Y_r stabilizes the matrix A_+ as

$$A_{Y_r} := A_+ + Y_r(\gamma^{-2} C_{1rr}^{\mathrm{T}} C_{1rr} - C_{22rr}^{\mathrm{T}} C_{22rr})$$

i.e. $\lambda(A_{Y_r}) \subset \mathbb{C}^-$.

Lemma 5

A positive-semi-definite matrix

$$Y = \begin{pmatrix} 0 & 0 \\ 0 & Y_r \end{pmatrix} \in \mathbb{R}^{n \times n} \tag{25}$$

with $Y_r > 0$ is a stabilizing solution of the ARE in (19) if and only if $Y_r > 0$ is a stabilizing solution of the reduced-order ARE in (24).

Proof

First, to make the proof as easy to understand as possible we represent L_H as

$$L_H = \begin{pmatrix} L_{H_{11}} \\ L_{H_{13}} \\ L_{H_{31}} \\ L_{H_{33}} \end{pmatrix} \in \mathbb{R}^{n \times (p_2 - m_1)}. \tag{26}$$

Since L_H satisfies condition (20), submatrices $L_{H_{11}}$ and $L_{H_{13}}$ can be chosen arbitrarily as long as

$$\lambda \begin{pmatrix} A_{11} + L_{H_{11}} & A_{12} \\ A_{13} + L_{H_{13}} & A_{14} \end{pmatrix} \subset \mathbb{C}^- \tag{27}$$

is satisfied and $L_{H_{31}}$ and $L_{H_{33}}$ are completely free.

Necessity: Suppose the ARE in (19) has the stabilizing solution $Y \geqslant 0$. The ARE can be represented as

$$Y A_{ZH}^{\mathrm{T}} + A_Y Y = 0 \tag{28}$$

Let U be a row basis of the stable subspace of the matrix A_{ZH}. Then we can choose $U = (I_{(n-\text{rank }(A+))}\ 0)$, which satisfies the following equation:

$$UA_{ZH} = \Lambda U \tag{29}$$

where

$$\Lambda = \begin{pmatrix} A_{11} + L_{H_{11}} & A_{12} & 0 \\ A_{13} + L_{H_{13}} & A_{14} & 0 \\ A_{31} + L_{H_{31}} & A_{32} & A_{-} \end{pmatrix}$$

is a stable matrix. After post-multiplication by U^{T}, Equation (28) becomes

$$(YU^{\mathrm{T}})\Lambda^{\mathrm{T}} + A_Y(YU^{\mathrm{T}}) = 0$$

Since $\lambda(\Lambda) \subset \mathbb{C}^-$, $\lambda(A_Y) \subset \mathbb{C}^-$, we can deduce that

$$YU^{\mathrm{T}} = 0$$

Since Y is a symmetric matrix it must be of the form in (25). By substituting this Y into the ARE (19), the reduced-order ARE (24) can be derived. Hence, it is necessary that the ARE has the positive-semi-definite stabilizing solution Y_r. Let a vector x be in $\ker Y_r$. Then by pre-multiplying the ARE (24) with x we have

$$Y_r A_+^{\mathrm{T}} x = 0$$

This implies $A_+^{\mathrm{T}} x \in \mathrm{Y_r}$, in other words $\ker Y_r$ is A_+^{T}-invariant. Here, we denote the restriction of A_+^{T} to $\ker Y_r$ as $A_+^{\mathrm{T}}|_{\ker Y_r}$. Then we also have

$$(A_+^{\mathrm{T}} + RY_r)|_{\ker Y_r} = A_+^{\mathrm{T}}|_{\ker Yr}$$

where

$$R := \gamma^{-2} C_{1rr}^{\mathrm{T}} C_{1rr} - C_{22rr}^{\mathrm{T}} C_{22rr}$$

However, we know

$$\begin{aligned} \mathbb{C}^+ \supset \lambda(A_+) = \lambda(A_+^{\mathrm{T}}) &\supset \lambda(A_+^{\mathrm{T}}|_{\ker Yr}) \\ &= \lambda(A_+^{\mathrm{T}} + RY_r|_{\ker Y_r}) \subset \lambda(A_+^{\mathrm{T}} + RY_r) \subset \mathbb{C}^- \end{aligned}$$

This implies that $\ker Y_r = \{0\}$, in other words, Y_r is invertible.

Sufficiency: Suppose that the reduced-order ARE in (24) has the stabilizing solution $Y_r > 0$. If we select Y as in (25), it can be verified that

$$\begin{aligned} \lambda(A_Y) &= \lambda\left(A_{ZH} + Y\left[\gamma^{-2} C_1^{\mathrm{T}} C_1 - \left(D_{21}^{\dagger} C_2\right)^{\mathrm{T}} D_{21}^{\dagger} C_2\right]\right) \\ &= \lambda \begin{pmatrix} A_{11} + L_{H_{11}} & A_{12} & 0 & 0 \\ A_{13} + L_{H_{13}} & A_{14} & 0 & 0 \\ * & A_{32} & A_{-} & 0 \\ * & * & * & A_{Y_r} \end{pmatrix} \subset \mathbb{C}^- \end{aligned}$$

where $*'$s are matrices of less interest, and also that the matrix Y satisfies the ARE in (19). Hence Y is a positive-semi-definite stabilizing solution of the ARE. $\square$

4.3.2. A controller order reduction

Utilizing the free parameter $W \in \mathcal{RH}_\infty$, we derive a class of H_∞ controllers whose central solution has an order n_k.

Theorem 3

A class of H_∞ controllers whose central solution has an order n_k is represented by

$$\mathcal{K}_\infty^{n_k} = \{K_\infty^{n_k}(s) | N \in \mathcal{RH}_\infty, \|N\|_\infty < \gamma\} \tag{30}$$

with

$$K_\infty^{n_k}(s) = \tilde{K}_1^{-1}(s)\tilde{K}_2(s) \tag{31}$$

where

$$\begin{aligned}
\tilde{K}_1(s) &= \tilde{C}_K(s)(sI - \tilde{A}_Y)^{-1}\tilde{B}_2 + \Pi^{-1}\\
\tilde{K}_2(s) &= \tilde{C}_K(s)(sI - \tilde{A}_Y)^{-1}\tilde{H}_\infty + \Omega\\
\tilde{A}_Y &= (V^\perp)^{\mathrm{T}} A_Y V^\perp \in \mathbb{R}^{n_k \times n_k}\\
\tilde{C}_K(s) &= (-\Pi^{-1}F_\infty Z + N(s)D_{21}^\dagger \hat{C}_2 Z)V^\perp\\
\tilde{B}_2 &= (V^\perp)^{\mathrm{T}}\hat{B}_2\\
\tilde{H}_\infty &= (V^\perp)^{\mathrm{T}} H_\infty\\
\Omega &= \Pi^{-1}F_\infty Z V (D_{21}^\perp)^{\mathrm{T}} + N(s)D_{21}^\dagger (I - \hat{C}_2 Z V (D_{21}^\perp)^{\mathrm{T}}).
\end{aligned}$$

Proof

Since the matrices $L_{H_{31}}$ and $L_{H_{33}}$ in (26) are arbitrary we can choose these as

$$\begin{aligned}
L_{H_{31}} &= -A_{31}\\
L_{H_{33}} &= -A_{33} - Y_r(\gamma^{-2}C_{1rr}^{\mathrm{T}}C_{1ll} \text{-} C_{22rr}^{\mathrm{T}}C_{22ll})
\end{aligned}$$

Also, $L_{H_{11}}$ and $L_{H_{13}}$ can be chosen arbitrarily, provided (27) is satisfied. We choose these matrices such that

$$\begin{aligned}
\lambda(A_{11} + L_{H_{11}}) &\subset \mathbb{C}^-\\
L_{H_{13}} &= -A_{13}
\end{aligned}$$

are satisfied. Then, A_Y is represented by

$$A_Y = \begin{pmatrix} A_{11} + L_{H_1} & A_{12} & 0 & 0\\ 0 & A_{14} & 0 & 0\\ 0 & A_{32} & A_- & 0\\ 0 & \tilde{A}_{34} & \tilde{A}_{43} & A_{Y_r} \end{pmatrix}$$

where

$$\tilde{A}_{34} = A_{34} + Y_r(\gamma^{-2}C_{1rr}^{\mathrm{T}}C_{1lr} - C_{22rr}^{\mathrm{T}}C_{22lr})$$
$$\tilde{A}_{43} = Y_r(\gamma^{-2}C_{1rr}^{\mathrm{T}}C_{1rl} - C_{22rr}^{\mathrm{T}}C_{22rl}).$$

The structure of the controller in (23) can be seen as an observer-based feedback [15]. From Lemma 4, it can be seen that the matrix $C_K(s)$ is parametrized by $W(s) \in \mathcal{RH}_\infty$, and that a full column rank matrix

$$V = \begin{pmatrix} I_{p_2-m_1} \\ 0 \end{pmatrix} \in \mathbb{R}^{n\times(p_2-m_1)}$$

satisfies

$$C_K(s)V = (-\Pi^{-1}F_\infty Z + N(s)D_{21}^{\dagger}\hat{C}_2 Z)V + W(s).$$

Hence, if we select the parameter $W(s)$ as

$$W(s) = -(-\Pi^{-1}F_\infty Z + N(s)D_{21}^{\dagger}\hat{C}_2 Z)V \tag{32}$$

we can see that the following equations:

$$A_Y V = V(A_{11} + L_{H_1})$$
$$C_K(s)V = 0$$

hold. Therefore, the parameter that satisfies (32) gives a reduced-order H_∞ controller whose central solution has an order n_k.

Substituting (32) into the formula in (23) reduces the order of the controller and we obtain (31). Since $W(s) \in \mathcal{RH}_\infty$ the controller is an H_∞ controller. □

Remark 8

1. The controller order reduction is as a consequence of a finite pole-zero cancellation in the controller. The number of the controller order reduction matches the number of measurements which are not affected by the disturbances.
2. If the free parameter $N(s)$ is put to zero, the order of the controller becomes n_k. The same order of the singular H_∞ controller is attained in Stoorvogel *et al.* [2] and Xin *et al.* [3]. On the other hand, the result obtained in this paper gives a nice interpretation of the structure of the reduced-order H_∞ controller. It still preserves the structure of the observer-based controller, which has the same order as the reduced-order observer as we also have an observer structure in Reference [2].
3. If we utilize the free parameter $N(s)$, we might obtain further order reduction. The next section discusses this topic.

4.4. Further order reduction of the n_kth order H_∞ controller

As we have seen in Section 3, the reduced-order H_∞ controller obtained in the previous section may further reduce its order. This section investigates whether the reduced-order controller which is derived by the ARE approach can be further reduced in order. We will find that the further order reduction is related to unstable zeros of Σ_{21} and also to the structure of the zeros.

4.4.1. A basic result corresponding to Section 3

First, begin with an n_kth order H_∞ control law with transfer matrix given by

$$K_\infty^{n_k}(s) = -\Pi\tilde{C}_K(s)(sI - \tilde{A}_Y + \tilde{B}_2\Pi\tilde{C}_K(s))^{-1}(\tilde{B}_2\Pi\Omega - \tilde{H}_\infty) + \Pi\Omega. \tag{33}$$

The matrix $\tilde{A}_Y \in \mathbb{R}^{n_k\times n_k}$ can be represented by

$$\tilde{A}_Y = \begin{pmatrix} A_{Y_1} & 0 \\ A_{Y_3} & A_{Y_r} \end{pmatrix}$$

and its components are written as

$$A_{Y_1} = \begin{pmatrix} A_{14} & 0 \\ A_{32} & A_- \end{pmatrix}, \quad A_{Y_3} = (\tilde{A}_{34}\,\tilde{A}_{43})$$

$$A_{Y_r} = A_+ + Y_r R \in \mathbb{R}^{r\times r}$$

It should be noted that the matrix $\tilde{C}_K(s)$ contains a free parameter $N \in \mathcal{RH}_\infty$, $\|N\|_\infty < \gamma$. If it is put to zero, the controller becomes an n_kth order H_∞ controller. On the other hand, we have possibilities of utilizing the parameter for improving performance of the controller. One of the possibilities is the controller order reduction which we will study in detail. For notational ease, we define a matrix $\hat{Z}$ by

$$\hat{Z} = (I - \gamma^{-2} Y_r X_4)^{-1} \tag{34}$$

where the matrix $X_4 \in \mathbb{R}^{r\times r}$ is a diagonal block of the matrix $X \in \mathbb{R}^{n\times n}$ which is decomposed as

$$X = \begin{pmatrix} * & * \\ * & X_4 \end{pmatrix}$$

Here, *'s represent submatrices of less interest.

We state a lemma which is central to our method to obtain a reduced-order controller. Note that in the above we have a free parameter N which is a rational, stable transfer matrix with H_∞ norm less than γ. In the following lemma, we constrain N to be a static transfer matrix, i.e. a matrix. Hence, we choose N from the set

$$\{N \in \mathbb{R}^{m_2\times m_1} | \bar{\sigma}(N) < \gamma\} \tag{35}$$

Lemma 6

Suppose that the H_∞ control problem for Σ with assumptions A1–A4 and (A, B_2, C_2) is stabilizable and detectable is solvable. Furthermore, suppose that Σ is equivalent to a realization which allows the formula in (17) where $\lambda(A_+)$ is the invariant zero of Σ_{21}. Let $V_m \in \mathbb{R}^{r\times m}$ be an arbitrary matrix which satisfies rank $V_m = m$ and

$$A_+^{\mathrm{T}} V_m = V_m J, \quad J \in \mathbb{R}^{m\times m} \tag{36}$$

where m is an arbitrary number such that $m \leqslant r$. Then the following properties hold:

1. If the following rank condition

$$\operatorname{rank}\left(D_{21}^{\dagger}\hat{C}_2\begin{pmatrix}0\\ \hat{Z}Y_rV_m\end{pmatrix}\right)=m \tag{37}$$

   is satisfied, there exists a matrix N in set (35) which satisfies the equation

$$\tilde{C}_K(s)\begin{pmatrix}0\\ Y_rV_m\end{pmatrix}=0 \tag{38}$$

   where $\tilde{C}_K(s)$ is defined in Theorem 3 and depends on N (constrained to be a static transfer matrix).
2. If the solution of (38) satisfies condition (35), the solution reduces the order of the H_∞ controller in (33) to $n_k - m$, while preserving the closed loop performance γ.

Proof
From the solvability of the H_∞ problem, the ARE in (24) has a solution. Hence, the equation

$$Y_rA_+^{\mathrm{T}}+A_{Y_r}Y_r=0$$

is satisfied. Post-multiplying the above equation by the matrix V_m and using (36) we obtain

$$A_{Y_r}Y_rV_m=-Y_rV_mJ$$

hence we also have

$$\tilde{A}_Y\begin{pmatrix}0\\ Y_rV_m\end{pmatrix}=-\begin{pmatrix}0\\ Y_rV_m\end{pmatrix}J$$

Hence, if there exists a solution N which satisfies Equation (38) and since the matrix Y_rV_m has full rank, the controller in (33) reduces its order by the number m, and the controller satisfies the closed-loop performance γ provided that condition (35) is satisfied. It remains to show that property 1 holds.

By a simple calculation, we can see that the following equation holds:

$$\begin{aligned}\tilde{C}_K(s)\begin{pmatrix}0\\ Y_rV_m\end{pmatrix}&=(-\Pi^{-1}F_\infty+ND_{21}^{\dagger}\hat{C}_2)ZV^{\perp}\begin{pmatrix}0\\ Y_rV_m\end{pmatrix}\\&=(-\Pi^{-1}F_\infty+ND_{21}^{\dagger}\hat{C}_2)\begin{pmatrix}0\\ \hat{Z}Y_rV_m\end{pmatrix}.\end{aligned} \tag{39}$$

Since the second term on the right-hand side of Equation (39) contains the free parameter N we can choose the parameter in such a way that it satisfies Equation (38) if the rank condition in (37) holds. □

Remark 9

1. This controller reduction can be considered as a consequence of a finite pole-zero cancellation in the controller. The finite mode is originally from the finite unstable zero

mode of the system Σ_{21}. Hence, how much the order of the controller is reduced is bounded by the number of the unstable zeros.
2. The assumption that the zeros of the system Σ_{21} are on the positive real axis corresponds to the assumption made in Theorem 1 excluding zeros at the origin. If the system Σ_{21} has multiple distinct unstable zeros on the real axis, the technique of Section 3 (bilinear transform) is based on a specific choice of one of these unstable invariant zeros. On the other hand, the technique of Lemma 6 does not depend on such an arbitrary choice.
3. Questions left here are whether conditions (37) and (35) are satisfied or not, or under what condition these are satisfied.

Now, we are ready to state a result of the H_∞ controller reduction by using the solution of the two-ARE approach.

Theorem 4
Suppose that the H_∞ control problem for Σ with assumptions A1–A4 and (A, B_2, C_2) is stabilizable and detectable is solvable. Furthermore, suppose that Σ_{21} has an invariant zero on the positive real axis with the geometric multiplicity m. Then there exists an $(n_k - m)$th order H_∞ controller with the performance index γ.

Proof
Since the H_∞ problem is solvable, both AREs in (18) and (19) have stablizing solutions. A straightforward calculations with the ARE in (18) yields

$$\begin{aligned} & X(A - B_1 D_{21}^{\dagger} C_2) + (A - B_1 D_{21}^{\dagger} C_2)^{\mathrm{T}} X \\ & \quad + C_1^{\mathrm{T}} C_1 - \gamma^2 C_2^{\mathrm{T}} (D_{21}^{\dagger})^{\mathrm{T}} D_{21}^{\dagger} C_2 - Q = 0 \end{aligned} \tag{40}$$

where Q is defined by

$$Q := \begin{pmatrix} -\Pi^{-1} F_\infty \\ D_{21}^{\dagger} \hat{C}_2 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I \end{pmatrix} \begin{pmatrix} -\Pi^{-1} & F_\infty \\ D_{21}^{\dagger} & \hat{C}_2 \end{pmatrix}.$$

From the ARE (19) and Equation (40) we obtain

$$\begin{aligned} & YX(A - B_1 D_{21}^{\dagger} C_2) Y + Y(A - B_1 D_{21}^{\dagger} C_2)^{\mathrm{T}} XY \\ & -\gamma^2 (Y A_{ZH}^{\mathrm{T}} + A_{ZH} Y) - YQY = 0 \end{aligned}$$

Since Y has the form in (25), representing Q as

$$Q = \begin{pmatrix} * & * \\ * & Q_4 \end{pmatrix}, \; Q_4 \in \mathbb{R}^{r \times r}$$

yields the equation

$$(Y_r X_4 - \gamma^2 I) A_+ Y_r + Y_r A_+^{\mathrm{T}} (Y_r X_4 - \gamma^2 I)^{\mathrm{T}} = Y_r Q_4 Y_r.$$

Since both Y_r and $\hat{Z}$ are non-singular, we obtain

$$V_m^{\mathrm{T}} A + \hat{Z} Y_r V_m + V_m^{\mathrm{T}} Y_r \hat{Z}^{\mathrm{T}} A_+^{\mathrm{T}} V_m = -\gamma^{-2} V_m^{\mathrm{T}} Y_r \hat{Z}^{\mathrm{T}} Q_4 \hat{Z} Y_r V_m \tag{41}$$

where $\hat{Z}$ is defined in (34). Since Σ_{21} has an invariant zero on the positive real axis with the geometric multiplicity m and $\{\lambda(A_+)\}$ is the set of unstable invariant zeros of Σ_{21}, we can choose V_m in (36) such that

$$J = \alpha I_m \tag{42}$$

holds. Hence from Equations (36), (41) and (42), we obtain

$$V_m^{\mathrm{T}} Y_r \hat{Z}^{\mathrm{T}} Q_4 \hat{Z} Y_r V_m = -2\gamma^2 \alpha V_m^{\mathrm{T}} Y_r \hat{Z}^{\mathrm{T}} V_m$$

where $\alpha > 0$. Since $Y_r > 0$ and $\gamma^2 I - Y_r X_4 > 0$, we have

$$Y_r \hat{Z}^{\mathrm{T}} = \gamma^2 Y_r^{1/2} (\gamma^2 I - Y_r^{1/2} X_4 Y_r^{1/2})^{-1} Y_r^{1/2} > 0$$

Hence, we obtain

$$V_m^{\mathrm{T}} Y_r \hat{Z}^{\mathrm{T}} Q_4 \hat{Z} Y_r V_m < 0. \tag{43}$$

This inequality is equivalent to

$$\begin{pmatrix} 0 \\ \hat{Z} Y_r V_m \end{pmatrix}^{\mathrm{T}} Q \begin{pmatrix} 0 \\ \hat{Z} Y_r V_m \end{pmatrix} < 0 \tag{44}$$

hence we have

$$\begin{pmatrix} 0 \\ \hat{Z} Y_r V_m \end{pmatrix}^{\mathrm{T}} (\Pi^{-1} F_\infty)^{\mathrm{T}} \Pi^{-1} F_\infty \begin{pmatrix} 0 \\ \hat{Z} Y_r V_m \end{pmatrix}$$
$$< \gamma^2 \begin{pmatrix} 0 \\ \hat{Z} Y_r V_m \end{pmatrix}^{\mathrm{T}} \left(D_{21}^{\dagger} \hat{C}_2 \right)^{\mathrm{T}} D_{21}^{\dagger} \hat{C}_2 \begin{pmatrix} 0 \\ \hat{Z} Y_r V_m \end{pmatrix}.$$

Here, noting that the following inequality

$$\begin{pmatrix} 0 \\ \hat{Z} Y_r V_m \end{pmatrix}^{\mathrm{T}} (\Pi^{-1} F_\infty)^{\mathrm{T}} \Pi^{-1} F_\infty \begin{pmatrix} 0 \\ \hat{Z} Y_r V_m \end{pmatrix} \geqslant 0$$

holds, we have

$$\gamma^2 \begin{pmatrix} 0 \\ \hat{Z} Y_r V_m \end{pmatrix}^{\mathrm{T}} (D_{21}^{\dagger} \hat{C}_2)^{\mathrm{T}} D_{21}^{\dagger} \hat{C}_2 \begin{pmatrix} 0 \\ \hat{Z} Y_r V_m \end{pmatrix} > 0.$$

This means that the rank condition in (37) holds. Thus Equation (38) is solvable. Moreover, since the solution to (38) satisfies the equation

$$\begin{pmatrix} 0 \\ \hat{Z} Y_r V_m \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} -\Pi^{-1} F_\infty \\ D_{21}^{\dagger} \hat{C}_2 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} I & 0 \\ 0 & -N^{\mathrm{T}} N \end{pmatrix} \begin{pmatrix} -\Pi^{-1} F_\infty \\ D_{21}^{\dagger} \hat{C}_2 \end{pmatrix} \begin{pmatrix} 0 \\ \hat{Z} Y_r V_m \end{pmatrix} = 0$$

from (44) it is straightforward to verify that (35) holds. Thus, from Lemma 6, it is shown that we obtain an H_∞ controller of order $n_k - m$ by using the solution of (38).

Remark 10
Since α is an unstable invariant zero of Σ_{21}, we can see that the maximal number m which satisfies (36) and (42) amounts to the geometric multiplicity [17] of an unstable invariant zero of Σ_{21}. Therefore, the order of the controller given in this theorem coincides with the result given in Theorem 1, except for the case $\alpha = 0$, as

$$n_{k_c} = n_k - m$$

provided that the matrix D_{21} has the full column rank.

4.4.2. Beneficial results of the two-ARE approach
The following theorem gives a condition for the H_∞ controller reduction in the case where two distinct zero modes in positive real number are included in J.

Theorem 5
Suppose that the H_∞ control problem for Σ with assumptions A1–A4 and (A, B_2, C_2) is stabilizable and detectable is solvable. Furthermore, suppose that we choose the matrix V_m in (36) such that the matrix J is represented by

$$J = \begin{pmatrix} \alpha_i I_{m_i} & 0 \\ 0 & \alpha_j I_{m_j} \end{pmatrix} \tag{45}$$

where $0 < \alpha_i < \alpha_j$ are unstable invariant zeros of Σ_{21} and $m = m_i + m_j$. Then, the H_∞ controller can be reduced to the order $n_k - m$ while preserving the closed-loop performance γ if the following condition

$$\alpha_j - \alpha_i \in \left\{ \varepsilon > 0 | 2\alpha_i F + \varepsilon \tilde{F} > 0 \right\} \tag{46}$$

is satisfied. Here F is a positive-definite matrix defined by

$$F = V_m^{\mathrm{T}} Y_r \hat{Z}^{\mathrm{T}} V_m$$

By using submatrices of

$$F = \begin{pmatrix} F_1 & F_2 \\ F_2^{\mathrm{T}} & F_4 \end{pmatrix}$$

the matrix $\tilde{F}$ is defined by

$$\tilde{F} = \begin{pmatrix} 0 & F_2 \\ F_2^{\mathrm{T}} & 2F_4 \end{pmatrix}.$$

Proof
As in the proof of Theorem 4, to prove this result it is sufficient to show that inequality (43) holds if condition (46) is satisfied under the assumption that the H_∞ problem is solvable.

As is obtained in the proof of Theorem 4, if the H_∞ problem is solvable, Equation (41) holds. From (36) and (41) we have

$$V_m^{\mathrm{T}} Y_r \hat{Z}^{\mathrm{T}} Q_4 \hat{Z} Y_r V_m = -\gamma^2 (JF + FJ) \tag{47}$$

where we put F as

$$F := V_m^{\mathrm{T}} Y_r \hat{Z}^{\mathrm{T}} V_m.$$

From $Y_r > 0$ and $\gamma^2 I - Y_r X_4 > 0$, we have $F > 0$. Since J can be represented by

$$J = \left(\alpha_i I + (\alpha_j - \alpha_i) \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \right)$$

substituting J into Equation (47) we have

$$V_m^{\mathrm{T}} Y_r \hat{Z}^{\mathrm{T}} Q_4 \hat{Z} Y_r V_m = -\gamma^2 (2\alpha_i F + (\alpha_j - \alpha_i)\tilde{F}).$$

If condition (46) is satisfied, we therefore obtain (43). □

Remark 11

1. Condition (46) can be easily checked by solving an LMI feasibility problem [29]. Since $F > 0$, we note that if the distance between α_i and α_j is small enough, then condition (46) is satisfied. This means that if the unstable zeros are located close each other we can obtain a lower order H_∞ controller.
2. Compared with the algorithm presented in Section 3, this result has an advantage that if the system Σ_{21} has distinct zeros on the positive real axis, we can further investigate lower order H_∞ controllers.

Next, we consider the case where the system Σ_{21} has complex zeros on the right half-plane and the complex mode is included in J.

Theorem 6

Suppose that the H_∞ control problem for Σ with assumptions A1–A4 and (A, B_2, C_2) is stabilizable and detectable is solvable. Furthermore, suppose that we choose the matrix V_m in (36) such that the matrix J is represented by

$$J = \mathrm{blockdiag}\left(\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \ldots, \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \right) \tag{48}$$

where α and β with $\alpha > 0$, α, $\beta \in \mathbb{R}$ are, respectively, the real part and the imaginary part of an invariant zero of Σ_{21}. Then, the H_∞ controller can be reduced to the order $n_k - m$ while preserving the closed-loop performance γ if the following condition

$$\beta \in \{\varepsilon > 0 | \alpha F + \varepsilon \tilde{F} > 0\} \tag{49}$$

is satisfied. Here, F is a positive-definite Hermitan matrix and is defined as

$$F = \hat{V}_m^* V_m^{\mathrm{T}} Y_r \hat{Z}^{\mathrm{T}} V_m \hat{V}_m$$

where $\hat{V}_m \in \mathbb{C}^{m\times m}$ is a non-singular matrix which satisfies

$$J\hat{V}_m = \hat{V}_m\hat{\Lambda}_m$$
$$\hat{\Lambda}_m = \begin{pmatrix} \lambda I_{m/2} & 0 \\ 0 & \bar{\lambda} I_{m/2} \end{pmatrix} \tag{50}$$
$$\lambda = \alpha + \mathrm{j}\beta.$$

By using submatrices of

$$F = \begin{pmatrix} F_1 & F_2 \\ F_2^* & F_4 \end{pmatrix}$$

we define the matrix $\tilde{F}$ by

$$\tilde{F} = \begin{pmatrix} 0 & -\mathrm{j}F_2 \\ \mathrm{j}F_2^* & 0 \end{pmatrix}.$$

Proof

As in the proof of Theorem 5, we show that inequality (43) holds if condition (49) is satisfied under the assumption that the H_∞ problem is solvable.

As is obtained in the proof of Theorem 4, if the H_∞ problem is solvable, Equation (41) holds. From (36) and (41) we have

$$V_m^{\mathrm{T}} Y_r \hat{Z}^{\mathrm{T}} Q_4 \hat{Z} Y_r V_m = -\gamma^2 (J^{\mathrm{T}} V_m^{\mathrm{T}} Y_r \hat{Z}^{\mathrm{T}} V_m + V_m^{\mathrm{T}} \hat{Z} Y_r V_m J).$$

Also from (50) and the above equation we obtain

$$\hat{V}_m^* (V_m^{\mathrm{T}} Y_r \hat{Z}^{\mathrm{T}} Q_4 \hat{Z} Y_r V_m) \hat{V}_m = -\gamma^2 (\hat{\Lambda}_m^* F + F\hat{\Lambda}_m). \tag{51}$$

Since $\hat{\Lambda}_m$ is represented by

$$\hat{\Lambda}_m = \alpha I + \beta \begin{pmatrix} jI & 0 \\ 0 & -jI \end{pmatrix}$$

substituting $\hat{\Lambda}_m$ into Equation (51) we have

$$\hat{V}_m^* (V_m^{\mathrm{T}} Y_r \hat{Z}^{\mathrm{T}} Q_4 \hat{Z} Y_r V_m) \hat{V}_m = -2\gamma^2 (\alpha F + \beta \tilde{F}).$$

Since $\hat{V}_m$ is non-singular, if condition (49) is satisfied, we obtain (43). □

Remark 12

1. Condition (49) can be easily checked by solving an LMI feasibility problem [29]. Since $F > 0$, we note that if the imaginary part of the zero is small enough then condition (49) is satisfied. This means that if the complex unstable zeros are located closer to the real axis, we can obtain a lower order H_∞ controller.
2. Compared with the algorithm presented in Section 3, this result has an advantage that if the system Σ_{21} has complex zeros in $\mathbb{C}^+$, we can further investigate a lower order H_∞ controller.

5. NUMERICAL EXAMPLES

Consider the system

$$\Sigma \begin{cases} \dot{x} = 3x + (0.5 \quad 0.5)\tilde{w} + u \\ z = 2x + u \\ y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{w} \end{cases}$$

The signal $\tilde{w}$ is the output of the system

$$\Sigma_{\mathrm{d}} \begin{cases} \dot{\xi} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \xi + \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix} w \\ \tilde{w} = \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix} \xi + \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} w \end{cases}$$

where E, F, G and H are

$$E = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix}, \; F = \begin{pmatrix} -\lambda_1 - \lambda_2 - 3 \\ \lambda_1 \lambda_2 - 2 \end{pmatrix}, \; G = (1 \quad 0), \; H = 1.$$

Since the system matrix of Σ_{d} loses its normal rank at the eigenvalues of the matrix

$$\begin{pmatrix} E - FG & 0 \\ 0 & E - FG \end{pmatrix}$$

we can see that Σ_{d} has zeros at λ_1 and λ_2. The whole system is illustrated in Figure 1. The system is an example where the subsystem from w to y has zeros at λ_1 and λ_2, and it has the direct-feedthrough term of column full rank. In a state-space form, the system is represented by a fifth-order dynamical equation. For this system we consider the H_∞ control problem where the parameter γ is given as $\gamma = 3.0$.

By using our technique given in this paper, we obtain the following result:

1. In the case where the zeros are given by $\lambda_1 = \lambda_2 = 3$, we obtain a first order H_∞ controller

$$\dot{\eta} = -3\eta + (0 \quad 0.12 \quad 0.12)y$$

$$u = 3.4\eta + (-4.5 \quad -0.83 \quad -0.83)y$$

The H_∞ norm of the closed-loop system is 2.37.

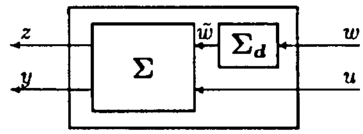

Figure 1. The complete system.

2. In the case where the zeros are given by $\lambda_1 = 3$ and $\lambda_2 \leqslant 5.3$, we can obtain a second-order H_∞ controller. In particular, given $\lambda_2 = 5.3$ the H_∞ controller is

$$\dot{\eta} = \begin{pmatrix} -3.7 & 1.1 \\ 1.1 & -4.6 \end{pmatrix} \eta + \begin{pmatrix} 0 & -0.4 & 0.13 \\ 0 & 0.6 & 0.084 \end{pmatrix} y$$

$$u = \begin{pmatrix} 0 & 9.4 \end{pmatrix} \eta + (\,-6.8 \quad -2.5 \quad -1.6\,) y$$

Then the H_∞ norm of the closed-loop system is 2.99. On the other hand, if the zero λ_2 is more than 5.3, we cannot find a second-order H_∞ controller with our technique.
3. In the case where the zeros are given by $\lambda_1 = 3 + \mathrm{j}\beta$ and $\lambda_2 = \bar{\lambda}_1$ where $\beta \leqslant 2.8$, we can obtain a second-order H_∞ controller. In particular, given $\beta = 2.8$ the H_∞ controller is

$$\dot{\eta} = \begin{pmatrix} -3.6 & -1.7 \\ 4.9 & -2.4 \end{pmatrix} \eta + \begin{pmatrix} 0 & 0.34 & 0.11 \\ 0 & -0.49 & -1.3 \end{pmatrix} y$$

$$u = \begin{pmatrix} 0 & 11 \end{pmatrix} \eta + (\,-11 \quad -2.5 \quad 0.48\,) y$$

Then the H_∞ norm of the closed-loop system is 2.99. On the other hand, if β is more than 2.8, we cannot find a second-order H_∞ controller with our technique.

6. CONCLUSION

In this paper, we have established the existence of a new class of reduced-order H_∞ controllers for the cases of continuous time and discrete time. The reduced-order H_∞ controllers are characterized by unstable invariant zeros of the system Σ_{21}. An algorithm to obtain the reduced-order H_∞ controllers is presented in both cases on the basis of the LMI approach. This algorithm uses a bilinear transformation and depends on choosing one real unstable zero.

In order to better understand the relation between unstable zeros and the controller order reduction, we use a controller parametrization obtained from the fundamental two-ARE approach in continuous-time case. The mechanism of the controller order reduction is explained with finite pole-zero cancellations in the parametrized controller. Moreover, in the cases where the unstable zeros are distinct and are located in $\mathbb{R}^+$, or are located in $\mathbb{C}^+$ we obtain some conditions under which the order of the H_∞ controller is further reduced.

On the other hand, there are some open problems which should be further investigated. In our state-space analysis, we have assumed that the matrix D_{21} has full column rank. However, this assumption is not necessary, hence an analysis to remove this assumption is left as the subject of further research. Stable invariant zeros have already been discussed in References [5, 6] from the view point of avoiding stable pole-zero cancellation between the plant and the H_∞ controller. These analyses are related to the present analysis. Hence, it is also interesting to analyse stable invariant zeros by using the presented state-space analysis. We know that as the control performance reaches the optimal level γ^* the classical central controller reduces its order [7, 27, 28, 30]. This is because we have a pole-zero cancellation at infinity. It is not clear how to combine this with a reduced-order observer.

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