Clebsch–Gordan coefficients of finite magnetic groups

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A detailed method is given for the calculation of Clebsch–Gordan coefficients of finite magnetic groups. This method is a generalization of a new method for the calculation of Clebsch–Gordan coefficients of finite nonmagnetic groups which makes use of the fact that the Clebsch–Gordan coefficients may be arranged into vectors which are eigenvectors of certain projection matrices.

1. INTRODUCTION

According to the well-known results of Bargmann¹ and Wigner² the Hilbert space of state vectors of a physical system with a symmetry group G carries a projective unitary–antiunitary (PUA) representation of G . All relevant physical information, such as transition probabilities, is contained in matrix elements of irreducible tensor operators which transform according to irreducible PUA representations of G . An important tool for the calculation of these matrix elements is the Wigner–Eckart theorem, which has been generalized for finite magnetic groups by Aviran and Zak.³ To apply the Wigner–Eckart theorem one needs the Clebsch–Gordan coefficients of G .

Recently a step forward in the theory of Clebsch–Gordan coefficients of nonmagnetic groups has been made by the observation that columns of Clebsch–Gordan coefficients are eigenvectors of some projection matrices. This observation was made first by Schindler and Mirman⁴ for the symmetric groups. A detailed procedure for the construction of the Clebsch–Gordan coefficients was then given by van den Broek and Cornwell⁵ and Dirl.⁶ The advantage of this method over previous methods is that the calculations are straightforward, problems related to multiplicity (overlooked for instance by Cornwell⁷ and Berenson and Birman⁸) are properly taken care of and tedious ad hoc methods (Sakata⁹) and orthogonalization procedures (Koster¹⁰) are avoided.

The present paper extends the method of Ref. 5 to magnetic groups. It will turn out that as in the case of nonmagnetic groups the columns of Clebsch–Gordan coefficients are eigenvectors of projection matrices but that some additional constraints have to be imposed on these eigenvectors which depend on the type (I, II, or III) of UA representations in the decomposition. I will show that these three types of constraints can easily be dealt with.

Relations for the Clebsch–Gordan coefficients of finite magnetic groups have been given by Aviran and Litvin,¹¹ Rudra¹² and Kotsev.¹³ However, none of these authors proves that these relations are sufficient, i.e., that each solution indeed provides a set of Clebsch–Gordan coefficients. Also, a systematic way of solving these relations has not been given.

Different approaches to the problem of calculating Clebsch–Gordan coefficients for finite magnetic groups

have been given by Gard and Backhouse¹⁴ and Sakata.¹⁵ Gard and Backhouse¹⁴ assume that the Clebsch–Gordan coefficients of the nonmagnetic subgroup are known and give a method to construct from them the Clebsch–Gordan coefficients of the whole group. Sakata¹⁵ obtains a collection of matrices which fulfill the requirements to be matrices of Clebsch–Gordan coefficients except for being unitary (or even nonsingular). The problem then remains to find a unitary matrix among this collection of matrices.

In Sec. 3 we will discuss the relations for the Clebsch–Gordan coefficients of finite magnetic groups of Ref. 11 and Ref. 13 and show that these relations are indeed sufficient. Moreover, we will show that the solutions of these relations automatically satisfy the orthogonality relations, which means that each solution gives a matrix of Clebsch–Gordan coefficients which is unitary. The main part of this paper will be Sec. 4, where we will derive a straightforward method to solve the equations of Sec. 3. A summary of this method will be given in Sec. 5, especially for those readers who wish to use the method without bothering for the proofs and the derivations. In Sec. 6 we will give a derivation of the Wigner–Eckart theorem for finite magnetic groups which is different from the derivation of Aviran and Zak³ and which shows the close connection between the form of the Wigner–Eckart theorem and the lemma of Schur for finite magnetic groups.

Although one has in general to consider PUA representation of G , we will restrict ourselves to UA representations in this paper. The generalization to PUA representations, however, is trivial; all results hold for PUA representations as well.

2. DEFINITIONS AND PRELIMINARIES

Let G be a finite group and G_0 a subgroup of G of index 2. A unitary–antiunitary (UA) representation of G with respect to G_0 is a homomorphic mapping T from G into the group of unitary and antiunitary operators on some Hilbert space $\mathcal{H}$ such that $T(g)$ is unitary if $g \in G_0$ and $T(g)$ is antiunitary if $g \notin G_0$. In the following we shall drop the phrase “with respect to G_0 .” Suppose $\mathcal{H}$ is finite dimensional and let $\{\phi_1, \dots, \phi_n\}$ be an orthonormal basis of $\mathcal{H}$. For each operator $T(g)$ a unitary $n \times n$ matrix $D(g)$ is defined by $D_{ij}(g) = (\phi_i, T(g)\phi_j)$. The matrices $D(g)$ satisfy

$$D(g)D(g') = D(gg'), \quad \forall g, g' \in G, \quad (2.1)$$

where D^* is defined by

$$D^g = \begin{cases} D & \text{if } g \in G_0 \\ D^* & \text{if } g \notin G_0 \end{cases} \quad (2.2)$$

A UA representation of G can also be defined to be a mapping from G into the unitary matrices of some dimension n such that Eq. (2.1) holds. The connection with the previous definition lies in the choice of the basis $\{\phi_1, \dots, \phi_n\}$ of $\mathcal{H}$. In the sequel T and D will always denote an operator and a matrix, respectively.

A UA representation T of G is reducible if there exists a proper subspace of $\mathcal{H}$ which is invariant under $T(G)$; otherwise T is irreducible. Two UA representations T_1 and T_2 of G in the Hilbert spaces H_1 and H_2 respectively are equivalent if there exists a unitary mapping $U: H_2 \rightarrow H_1$ such that

$$T_1(g)U = UT_2(g), \quad \forall g \in G, \quad (2.3a)$$

or, equivalently, in terms of matrices, two UA representations D_1 and D_2 of G are equivalent if there exists a unitary matrix U such that

$$D_1(g)U = UD_2(g), \quad \forall g \in G. \quad (2.3b)$$

According to Wigner¹⁶ the irreducible UA representations are divided into three types: the restriction $D \downarrow G_0$ of an irreducible UA representation D of G to G_0 , which is a unitary representation of G_0 , is irreducible (Type I), is reducible into two equivalent irreducible components (Type II), or is reducible into two inequivalent irreducible components (Type III).

Let a be a fixed element of $G \setminus G_0$. An irreducible UA representation of G of type II is equivalent with a UA representation which has the form

$$D(g) = \begin{pmatrix} \Delta(g) & 0 \\ 0 & \Delta(g) \end{pmatrix} \quad \forall g \in G_0; \quad D(a) = \begin{pmatrix} 0 & U \\ -U & 0 \end{pmatrix}, \quad (2.4)$$

where Δ is an irreducible unitary representation of G_0 and U satisfies

$$UU^* = -\Delta(a^2) \quad (2.5)$$

and

$$U\Delta^*(a^{-1}ga)U^{-1} = \Delta(g) \quad \forall g \in G_0. \quad (2.6)$$

An irreducible UA representation of G of type III is equivalent with a UA representation which has the form

$$D(g) = \begin{pmatrix} \Delta(g) & 0 \\ 0 & \Delta^*(a^{-1}ga) \end{pmatrix} \quad \forall g \in G_0; \\ D(a) = \begin{pmatrix} 0 & \Delta(a^2) \\ 1 & 0 \end{pmatrix}, \quad (2.7)$$

where Δ is an irreducible unitary representation of G_0 which is not equivalent with $\bar{\Delta}$ defined by $\bar{\Delta}(g) = \Delta^*(a^{-1}ga)$. An irreducible UA representation is said to be in standard form if it is of type I, if it is of type II and satisfies Eq. (2.4) or if it is of type III and satisfies Eq. (2.7). So each irreducible UA representation is equivalent with an irreducible UA representation which is in standard form. From now on we will only consider one fixed chosen representative which is in standard form of each class of equivalent irreducible UA representations. These representatives will be denoted by D^α , D^β , ..., and their dimensions by $d_\alpha, d_\beta, \dots$.

The lemma of Schur may be generalized for irreducible UA representations as follows. Let D be an irreducible UA representation of G which is in standard form and let U be a matrix which satisfies

$$D(g)U^g = UD(g) \quad \forall g \in G. \quad (2.8)$$

Then

$$U = \lambda 1; \lambda \in \mathbb{R} \quad \text{if } D \text{ is of type I}, \quad (2.9a)$$

$$U = \begin{pmatrix} \lambda 1 & \mu 1 \\ -\mu^* 1 & \lambda^* 1 \end{pmatrix}; \lambda, \mu \in \mathbb{C} \quad \text{if } D \text{ is of type II}, \quad (2.9b)$$

and

$$U = \begin{pmatrix} \lambda 1 & 0 \\ 0 & \lambda^* 1 \end{pmatrix}; \lambda \in \mathbb{C} \quad \text{if } D \text{ is of type III} \quad (2.9c)$$

(1 always denotes a unit matrix).

From this the following orthogonality relations may be derived^{13,17}: If D^γ is of type I,

$$\frac{d_\gamma}{|G|} \sum_{g \in G_0} D^{\gamma*}(g)_{pq} D^\gamma(g)_{kl} = \frac{1}{2} \delta_{pk} \delta_{ql} \delta_{\gamma\eta}, \quad (2.10a)$$

$$\frac{d_\gamma}{|G|} \sum_{g \in G_0} D^{\gamma*}(g)_{pq} D^\gamma(g)_{kl} = \frac{1}{2} \delta_{pk} \delta_{ql} \delta_{\gamma\eta}. \quad (2.10b)$$

If D^γ is of type II,

$$\frac{d_\gamma}{|G|} \sum_{g \in G_0} D^{\gamma*}(g)_{pq} D^\gamma(g)_{kl} = \delta_{\gamma\eta} (\delta_{pk} \delta_{ql} + \delta_{p,k+(d_\gamma/2)} \delta_{q,l+(d_\gamma/2)}), \quad (p \sim q), \quad (2.11a)$$

$$\frac{d_\gamma}{|G|} \sum_{g \in G_0} D^{\gamma*}(g)_{pq} D^\gamma(g)_{kl} = \delta_{\gamma\eta} (\delta_{pk} \delta_{ql} - \delta_{p,k+(d_\gamma/2)} \delta_{q,l+(d_\gamma/2)}), \quad (p \not\sim q). \quad (2.11b)$$

If D^γ is of type III

$$\frac{d_\gamma}{|G|} \sum_{g \in G_0} D^{\gamma*}(g)_{pq} D^\gamma(g)_{kl} = \delta_{pk} \delta_{ql} \delta_{\gamma\eta}, \quad (p \sim q), \quad (2.12a)$$

$$\frac{d_\gamma}{|G|} \sum_{g \in G_0} D^{\gamma*}(g)_{pq} D^\gamma(g)_{kl} = \delta_{pk} \delta_{ql} \delta_{\gamma\eta}, \quad (p \not\sim q). \quad (2.12b)$$

Here $|G|$ is the order of G ; $p \sim q$ means $1 \leq p, q \leq (d_\gamma/2)$ or $(d_\gamma/2) + 1 \leq p, q \leq d_\gamma$. In Eq. (2.11) and in the remainder of this paper indices of the type $k + d_\gamma/2$ should be taken modulo d_γ . It should be noted that the orthogonality relations in this form do not hold in general if the UA representations are not in standard form.

The direct product $D^\alpha \otimes D^\beta$ of two irreducible UA representations D^α and D^β of G is defined by

$$(D^\alpha \otimes D^\beta)(g)_{ij,kl} = D^\alpha(g)_{ik} D^\beta(g)_{jl}, \quad \forall g \in G. \quad (2.13)$$

In general $D^\alpha \otimes D^\beta$ is a reducible UA representation of dimension $d_\alpha d_\beta$; suppose it is equivalent with the direct sum $\Sigma_\gamma \oplus m_\gamma D^\gamma$. Then there exists a unitary matrix U with the property

$$(D^\alpha \otimes D^\beta)(g)U^g = U \sum_\gamma \oplus m_\gamma D^\gamma(g), \quad \forall g \in G. \quad (2.14)$$

The elements of the matrix U are the Clebsch–Gordan coefficients. We label the row of U by the pairs (i, j) ;

$i = 1, 2, \dots, d_\alpha; j = 1, 2, \dots, d_\beta$ and the columns of U by the triples (γ, τ, k) with $m_\gamma \neq 0; \tau = 1, 2, \dots, m_\gamma$ and $k = 1, 2, \dots, d_\gamma$. The Clebsch–Gordan coefficients may then be denoted by

$$\begin{pmatrix} \alpha & \beta & | & \gamma, \tau \\ i & j & | & k \end{pmatrix},$$

Another definition of the Clebsch–Gordan coefficients, which is easily shown to be completely equivalent with the preceding definition is the following. Let T be a homomorphic mapping from G into the operators on a $d_\alpha d_\beta$ -dimensional Hilbert space $\mathcal{H}$ such that $T(g)$ is unitary if $g \in G_0$ and $T(g)$ is antiunitary if $g \notin G_0$; let $\{\phi_{ij}^{\alpha\beta}\}$ ($i = 1, 2, \dots, d_\alpha; j = 1, 2, \dots, d_\beta$) be an orthonormal basis of $\mathcal{H}$ which transforms under T according to

$$T(g)\phi_{ij}^{\alpha\beta} = \sum_{kl} (D^\alpha \otimes D^\beta)(g)_{kl,ij} \phi_{kl}^{\alpha\beta}, \quad \forall g \in G. \quad (2.15)$$

Then there exists in $\mathcal{H}$ an orthonormal basis $\{\psi_k^{\gamma,\tau}\}$ ($m_\gamma \neq 0; \tau = 1, 2, \dots, m_\gamma; k = 1, 2, \dots, d_\gamma$) such that

$$T(g)\psi_k^{\gamma,\tau} = \sum_l D^\gamma(g)_{lk} \psi_l^{\gamma,\tau}, \quad \forall g \in G. \quad (2.16)$$

The Clebsch–Gordan coefficients are then defined by

$$\psi_k^{\gamma,\tau} = \sum_{ij} \begin{pmatrix} \alpha & \beta & | & \gamma, \tau \\ i & j & | & k \end{pmatrix} \phi_{ij}^{\alpha\beta}. \quad (2.17)$$

Since both bases are orthonormal it follows

$$\phi_{ij}^{\alpha\beta} = \sum_{\gamma,\tau,k} \begin{pmatrix} \alpha & \beta & | & \gamma, \tau \\ i & j & | & k \end{pmatrix}^* \psi_k^{\gamma,\tau} \quad (2.18)$$

Now we define for each γ with $m_\gamma > 0$ and for $p, q = 1, 2, \dots, d_\gamma$ the operators P_{pq}^γ and Q_{pq}^γ by

$$P_{pq}^\gamma = \frac{d_\gamma}{|G|} \sum_{g \in G_0} D^\gamma(g)_{pq}^* T(g) \quad (2.19)$$

and

$$Q_{pq}^\gamma = \frac{d_\gamma}{|G|} \sum_{g \in G_0} D^\gamma(g)_{pq}^* T(g). \quad (2.20)$$

If D^γ is of type II or type III, P_{pq}^γ is equal to zero if $p \neq q$ and Q_{pq}^γ is equal to zero if $p \sim q$.

From the definitions the following expressions are derived immediately:

$$T(g)P_{pq}^\gamma = \sum_k D^\gamma(g)_{kp} P_{kq}^\gamma, \quad \forall g \in G_0, \quad (2.21a)$$

$$T(g)P_{pq}^\gamma = \sum_k D^\gamma(g)_{kp} Q_{kq}^\gamma, \quad \forall g \in G \setminus G_0, \quad (2.21b)$$

$$T(g)Q_{pq}^\gamma = \sum_k D^\gamma(g)_{kp} Q_{kq}^\gamma, \quad \forall g \in G_0, \quad (2.21c)$$

$$T(g)Q_{pq}^\gamma = \sum_k D^\gamma(g)_{kp} P_{kq}^\gamma, \quad \forall g \in G \setminus G_0. \quad (2.21d)$$

Using the orthogonality relations one can derive in a straightforward way the following expressions. If D^γ is of type I,

$$P_{pq}^\gamma P_{rs}^\gamma = \frac{1}{2} \delta_{\gamma\eta} \delta_{qr} P_{ps}^\gamma \quad (2.22a)$$

$$Q_{pq}^\gamma Q_{rs}^\gamma = \frac{1}{2} \delta_{\gamma\eta} \delta_{qr} P_{ps}^\gamma \quad (2.22b)$$

$$P_{pq}^\gamma Q_{rs}^\gamma = \frac{1}{2} \delta_{\gamma\eta} \delta_{qr} Q_{ps}^\gamma \quad (2.22c)$$

$$Q_{pq}^\gamma P_{rs}^\gamma = \frac{1}{2} \delta_{\gamma\eta} \delta_{qr} Q_{ps}^\gamma \quad (2.22d)$$

if D^γ is of type II,

$$P_{pq}^\gamma P_{rs}^\gamma = \delta_{\gamma\eta} (\delta_{qr} P_{ps}^\gamma + \delta_{q,r+(d_\gamma/2)} P_{p+(d_\gamma/2),s}^\gamma), \quad (p \sim q), \quad (2.23a)$$

$$Q_{pq}^\gamma Q_{rs}^\gamma = \delta_{\gamma\eta} (\delta_{qr} P_{ps}^\gamma - \delta_{q,r+(d_\gamma/2)} P_{p+(d_\gamma/2),s}^\gamma), \quad (p \neq q), \quad (2.23b)$$

$$P_{pq}^\gamma Q_{rs}^\gamma = \delta_{\gamma\eta} (\delta_{qr} Q_{ps}^\gamma + \delta_{q,r+(d_\gamma/2)} Q_{p+(d_\gamma/2),s}^\gamma), \quad (p \sim q), \quad (2.23c)$$

$$Q_{pq}^\gamma P_{rs}^\gamma = \delta_{\gamma\eta} (\delta_{qr} Q_{ps}^\gamma - \delta_{q,r+(d_\gamma/2)} Q_{p+(d_\gamma/2),s}^\gamma), \quad (p \neq q), \quad (2.23d)$$

and finally if D^γ is of type III

$$P_{pq}^\gamma P_{rs}^\gamma = \delta_{\gamma\eta} \delta_{qr} P_{ps}^\gamma, \quad (p \sim q), \quad (2.24a)$$

$$Q_{pq}^\gamma Q_{rs}^\gamma = \delta_{\gamma\eta} \delta_{qr} P_{ps}^\gamma, \quad (p \neq q), \quad (2.24b)$$

$$P_{pq}^\gamma Q_{rs}^\gamma = \delta_{\gamma\eta} \delta_{qr} Q_{ps}^\gamma, \quad (p \sim q), \quad (2.24c)$$

$$Q_{pq}^\gamma P_{rs}^\gamma = \delta_{\gamma\eta} \delta_{qr} Q_{ps}^\gamma, \quad (p \neq q). \quad (2.24d)$$

From the orthogonality relations and Eq. (2.16) it follows that the operators P_{rs}^γ and Q_{rs}^γ act as follows on the basis elements $\psi_k^{\gamma,\tau}$. If D^γ is of type I

$$P_{pq}^\gamma \psi_k^{\gamma,\tau} = \frac{1}{2} \delta_{\gamma\eta} \delta_{qk} \psi_p^{\gamma,\tau}, \quad (2.25a)$$

$$Q_{pq}^\gamma \psi_k^{\gamma,\tau} = \frac{1}{2} \delta_{\gamma\eta} \delta_{qk} \psi_p^{\gamma,\tau}, \quad (2.25b)$$

if D^γ is of type II

$$P_{pq}^\gamma \psi_k^{\gamma,\tau} = \delta_{\gamma\eta} (\delta_{qk} \psi_p^{\gamma,\tau} + \delta_{q,k+(d_\gamma/2)} \psi_{p+(d_\gamma/2)}^{\gamma,\tau}), \quad (p \sim q), \quad (2.26a)$$

$$Q_{pq}^\gamma \psi_k^{\gamma,\tau} = \delta_{\gamma\eta} (\delta_{qk} \psi_p^{\gamma,\tau} - \delta_{q,k+(d_\gamma/2)} \psi_{p+(d_\gamma/2)}^{\gamma,\tau}), \quad (p \neq q), \quad (2.26b)$$

and if D^γ is of type III

$$P_{pq}^\gamma \psi_k^{\gamma,\tau} = \delta_{\gamma\eta} \delta_{qk} \psi_p^{\gamma,\tau}, \quad (p \sim q), \quad (2.27a)$$

$$Q_{pq}^\gamma \psi_k^{\gamma,\tau} = \delta_{\gamma\eta} \delta_{qk} \psi_p^{\gamma,\tau}, \quad (p \neq q). \quad (2.27b)$$

3. EQUATIONS FOR THE CLEBSCH-GORDAN COEFFICIENTS

Let us consider the matrix elements of P_{pq}^γ and Q_{pq}^γ with respect to the basis $\{\phi_{ij}^{\alpha\beta}\}$. From Eq. (2.15) it follows

$$(\phi_{mn}^{\alpha\beta}, P_{pq}^\gamma \phi_{ij}^{\alpha\beta}) = \frac{d_\gamma}{|G|} \sum_{g \in G_0} D_{pq}^{\gamma*}(g) D_{mi}^\alpha(g) D_{nj}^\beta(g) \quad (3.1a)$$

and

$$(\phi_{mn}^{\alpha\beta}, Q_{pq}^\gamma \phi_{ij}^{\alpha\beta}) = \frac{d_\gamma}{|G|} \sum_{g \in G_0} D_{pq}^{\gamma*}(g) D_{mi}^\alpha(g) D_{nj}^\beta(g). \quad (3.1b)$$

On the other hand, it follows from Eq. (2.18) that

$$(\phi_{mn}^{\alpha\beta}, P_{pq}^\gamma \phi_{ij}^{\alpha\beta}) = \sum_{\eta,\tau,k} \sum_{\eta',\tau',k'} \begin{pmatrix} \alpha & \beta & | & \eta & \tau \\ m & n & | & k & \tau \end{pmatrix} \times \begin{pmatrix} \alpha & \beta & | & \eta' & \tau' \\ i & j & | & k' & \tau' \end{pmatrix}^* (\psi_k^{\eta,\tau}, P_{pq}^\gamma \psi_{k'}^{\eta',\tau'}) \quad (3.2a)$$

and

$$(\phi_{mn}^{\alpha\beta}, Q_{pq}^\gamma \phi_{ij}^{\alpha\beta}) = \sum_{\eta,\tau,k} \sum_{\eta',\tau',k'} \begin{pmatrix} \alpha & \beta & | & \eta & \tau \\ m & n & | & k & \tau \end{pmatrix} \times \begin{pmatrix} \alpha & \beta & | & \eta' & \tau' \\ i & j & | & k' & \tau' \end{pmatrix}^* (\psi_k^{\eta,\tau}, Q_{pq}^\gamma \psi_{k'}^{\eta',\tau'}) \quad (3.2b)$$

$$\times \begin{pmatrix} \alpha & \beta \\ i & j \end{pmatrix} \begin{vmatrix} \eta' & \tau' \\ k' & \end{vmatrix} (\psi_k^{\eta',\tau}, Q_{pq}^\gamma \psi_k^{\eta',\tau'}) \quad (3.2b)$$

Using Eq. (2.25), (2.26), and (2.27) this becomes, if D^γ is of type I,

$$(\phi_{mn}^{\alpha\beta}, P_{pq}^\gamma \phi_{ij}^{\alpha\beta}) = \frac{1}{2} \sum_\tau \begin{pmatrix} \alpha & \beta \\ m & n \end{pmatrix} \begin{vmatrix} \gamma & \tau \\ p & \end{vmatrix} \begin{pmatrix} \alpha & \beta \\ i & j \end{pmatrix} \begin{vmatrix} \gamma & \tau \\ q & \end{vmatrix}^* \quad (3.3a)$$

and

$$(\phi_{mn}^{\alpha\beta}, Q_{pq}^\gamma \phi_{ij}^{\alpha\beta}) = \frac{1}{2} \sum_\tau \begin{pmatrix} \alpha & \beta \\ m & n \end{pmatrix} \begin{vmatrix} \gamma & \tau \\ p & \end{vmatrix} \begin{pmatrix} \alpha & \beta \\ i & j \end{pmatrix} \begin{vmatrix} \gamma & \tau \\ q & \end{vmatrix}; \quad (3.3b)$$

if D^γ is of type II,

$$\begin{aligned} (\phi_{mn}^{\alpha\beta}, P_{pq}^\gamma \phi_{ij}^{\alpha\beta}) &= \sum_\tau \begin{pmatrix} \alpha & \beta \\ m & n \end{pmatrix} \begin{vmatrix} \gamma & \tau \\ p & \end{vmatrix} \begin{pmatrix} \alpha & \beta \\ i & j \end{pmatrix} \begin{vmatrix} \gamma & \tau \\ q & \end{vmatrix}^* \\ &+ \sum_\tau \begin{pmatrix} \alpha & \beta \\ m & n \end{pmatrix} \begin{vmatrix} \gamma & \tau \\ p + \frac{d_\gamma}{2} & \end{vmatrix} \begin{pmatrix} \alpha & \beta \\ i & j \end{pmatrix} \begin{vmatrix} \gamma & \tau \\ q & \end{vmatrix}^* \\ &\times \begin{pmatrix} \alpha & \beta \\ i & j \end{pmatrix} \begin{vmatrix} \gamma & \tau \\ q + \frac{d_\gamma}{2} & \end{vmatrix}^* \quad (p \sim q), \end{aligned} \quad (3.4a)$$

and

$$\begin{aligned} (\phi_{mn}^{\alpha\beta}, Q_{pq}^\gamma \phi_{ij}^{\alpha\beta}) &= \sum_\tau \begin{pmatrix} \alpha & \beta \\ m & n \end{pmatrix} \begin{vmatrix} \gamma & \tau \\ p & \end{vmatrix} \begin{pmatrix} \alpha & \beta \\ i & j \end{pmatrix} \begin{vmatrix} \gamma & \tau \\ q & \end{vmatrix}^* \\ &- \sum_\tau \begin{pmatrix} \alpha & \beta \\ m & n \end{pmatrix} \begin{vmatrix} \gamma & \tau \\ p + \frac{d_\gamma}{2} & \end{vmatrix} \begin{pmatrix} \alpha & \beta \\ i & j \end{pmatrix} \begin{vmatrix} \gamma & \tau \\ q & \end{vmatrix}^* \\ &\times \begin{pmatrix} \alpha & \beta \\ i & j \end{pmatrix} \begin{vmatrix} \gamma & \tau \\ q + \frac{d_\gamma}{2} & \end{vmatrix}^* \quad (p \neq q), \end{aligned} \quad (3.4b)$$

and if D^γ is of type III,

$$(\phi_{mn}^{\alpha\beta}, P_{pq}^\gamma \phi_{ij}^{\alpha\beta}) = \sum_\tau \begin{pmatrix} \alpha & \beta \\ m & n \end{pmatrix} \begin{vmatrix} \gamma & \tau \\ p & \end{vmatrix} \begin{pmatrix} \alpha & \beta \\ i & j \end{pmatrix} \begin{vmatrix} \gamma & \tau \\ q & \end{vmatrix}^* \quad (p \sim q), \quad (3.5a)$$

and

$$(\phi_{mn}^{\alpha\beta}, Q_{pq}^\gamma \phi_{ij}^{\alpha\beta}) = \sum_\tau \begin{pmatrix} \alpha & \beta \\ m & n \end{pmatrix} \begin{vmatrix} \gamma & \tau \\ p & \end{vmatrix} \begin{pmatrix} \alpha & \beta \\ i & j \end{pmatrix} \begin{vmatrix} \gamma & \tau \\ q & \end{vmatrix}^* \quad (p \neq q) \quad (3.5b)$$

Let $A(\gamma, p, q)$ and $B(\gamma, p, q)$ be the matrices of P_{pq}^γ and Q_{pq}^γ respectively with respect to the basis $\{\phi_{ij}^{\alpha\beta}\}$

$$A(\gamma, p, q)_{mn,ij} = (\phi_{mn}^{\alpha\beta}, P_{pq}^\gamma \phi_{ij}^{\alpha\beta}), \quad (3.6a)$$

$$B(\gamma, p, q)_{mn,ij} = (\phi_{mn}^{\alpha\beta}, Q_{pq}^\gamma \phi_{ij}^{\alpha\beta}). \quad (3.6b)$$

Let $c(\gamma, \tau, k)$ be column matrix of dimension $d_\alpha d_\beta$ given by

$$c(\gamma, \tau, k)_{ij} = \begin{pmatrix} \alpha & \beta \\ i & j \end{pmatrix} \begin{vmatrix} \gamma & \tau \\ k & \end{vmatrix}. \quad (3.7)$$

Then Eqs. (3.3), (3.4), and (3.5) can be written as Eqs. (3.8), (3.9), and (3.10), respectively:

$$A(\gamma, p, q) = \frac{1}{2} \sum_\tau c(\gamma, \tau, p) \tilde{c}^*(\gamma, \tau, q), \quad (3.8a)$$

$$B(\gamma, p, q) = \frac{1}{2} \sum_\tau c(\gamma, \tau, p) \tilde{c}(\gamma, \tau, q), \quad (3.8b)$$

$$\begin{aligned} A(\gamma, p, q) &= \sum_\tau c(\gamma, \tau, p) \tilde{c}^*(\gamma, \tau, q) \\ &+ \sum_\tau c\left(\gamma, \tau, p + \frac{d_\gamma}{2}\right) \tilde{c}^*\left(\gamma, \tau, q + \frac{d_\gamma}{2}\right), \end{aligned} \quad (p \sim q), \quad (3.9a)$$

$$\begin{aligned} B(\gamma, p, q) &= \sum_\tau c(\gamma, \tau, p) \tilde{c}(\gamma, \tau, q) \\ &- \sum_\tau c\left(\gamma, \tau, p + \frac{d_\gamma}{2}\right) \tilde{c}\left(\gamma, \tau, q + \frac{d_\gamma}{2}\right), \end{aligned} \quad (p \neq q), \quad (3.9b)$$

$$A(\gamma, p, q) = \sum_\tau c(\gamma, \tau, p) \tilde{c}^*(\gamma, \tau, q), \quad (p \sim q), \quad (3.10a)$$

$$B(\gamma, p, q) = \sum_\tau c(\gamma, \tau, p) \tilde{c}(\gamma, \tau, q), \quad (p \neq q), \quad (3.10b)$$

Here $\tilde{c}$ denotes the row matrix which is the transpose of c .

Now we will show that a solution of these equations is indeed a unitary matrix of Clebsch–Gordan coefficients.

The orthogonality relations of the Clebsch–Gordan coefficients take the form

$$\sum_{\gamma, \tau, k} c(\gamma, \tau, k) \tilde{c}^*(\gamma, \tau, k) = 1. \quad (3.11)$$

But from Eqs. (3.8a), (3.9a) and (3.10a) it follows that $\sum_{\gamma, \tau, k} c(\gamma, \tau, k) \tilde{c}^*(\gamma, \tau, k)$ is the same for each solution, and therefore this sum must be equal to the unit matrix, since we know that a unitary matrix of Clebsch–Gordan coefficients exists. So each solution indeed gives a unitary matrix. To show that this unitary matrix is a matrix of Clebsch–Gordan coefficients we have to prove Eq. (2.16) if Eq. (2.15) holds and if the basis $\{\psi_k^{\eta',\tau}\}$ is defined by Eq. (2.17) where the coefficients are some solution of Eq. (3.8) (for the D^γ of type I), (3.9) (for the D^γ of type II), and (3.10) (for the D^γ of type III).

From Eqs. (2.17) and (2.15) it follows that

$$P_{pq}^\gamma \psi_k^{\eta',\tau} = \sum_{i,j,m,n} \begin{pmatrix} \alpha & \beta \\ i & j \end{pmatrix} \begin{vmatrix} \eta & \tau \\ k & \end{vmatrix} A(\gamma, p, q)_{mn,ij} \phi_{ij}^{\alpha\beta}. \quad (3.12)$$

If we substitute for $A(\gamma, p, q)_{mn,ij}$ the right-hand side of Eq. (3.3a), (3.4a), or (3.5a), take into account the orthogonality relations for the coefficients and use Eq. (2.17) again, then we obtain Eqs. (2.25a), (2.26a), and (2.27a), respectively. Equations (2.25b), (2.26b) and (2.27b) are derived similarly. From Eqs. (2.25a), (2.26a), and (2.27a) it follows that

$$P_{pq}^\gamma \psi_k^{\eta',\tau} = \alpha \psi_k^{\eta',\tau}, \quad (3.13)$$

where $\alpha = \frac{1}{2}$ if D^γ is of type I and $\alpha = 1$ if D^γ is of type II or

of type III. If we apply $T(g)$ on both sides of this equation for $g \in G_0$, it follows from Eq. (2.21a) that

$$\sum_k D_{kp}^\gamma(g) P_{kq}^\gamma \psi_q^{\gamma,\tau} = \alpha T(g) \psi_p^{\gamma,\tau}, \quad (3.14)$$

and using Eq. (3.13) we obtain Eq. (2.16) for $g \in G_0$. From Eqs. (2.25b), (2.26b), and (2.27b) it follows that

$$Q_{pq}^\gamma \psi_q^{\gamma,\tau} = \alpha \psi_p^{\gamma,\tau}. \quad (3.15)$$

Applying $T(g)$ on both sides of this Eq. for $g \in G \setminus G_0$ we again obtain Eq. (3.14), and thus Eq. (2.16), by using Eq. (2.21d).

4. SOLUTION OF THE EQUATIONS

In this section we will give a method for solving Eqs. (3.8), (3.9), and (3.10). Let us start with Eq. (3.8). Since here D^γ is of type I it follows from the Eqs. (2.22) and (3.6) that

$$A(\gamma, p, q) A(\gamma, r, s) = \frac{1}{2} \delta_{qr} A(\gamma, p, s), \quad (4.1a)$$

$$B(\gamma, p, q) B^*(\gamma, r, s) = \frac{1}{2} \delta_{qr} A(\gamma, p, s), \quad (4.1b)$$

$$A(\gamma, p, q) B(\gamma, r, s) = \frac{1}{2} \delta_{qr} B(\gamma, p, s), \quad (4.1c)$$

$$B(\gamma, p, q) A^*(\gamma, r, s) = \frac{1}{2} \delta_{qr} B(\gamma, p, s). \quad (4.1d)$$

Moreover, since $P_{pq}^{\gamma\dagger} = P_{qp}^\gamma$ and $Q_{pq}^{\gamma\dagger} = Q_{qp}^\gamma$ it follows from Eq. (3.6) that

$$\tilde{A}^*(\gamma, p, q) = A(\gamma, q, p), \quad (4.2a)$$

$$\tilde{B}(\gamma, p, q) = B(\gamma, q, p). \quad (4.2b)$$

Since we know that the solutions of Eq. (3.8) obey the orthogonality relations, it follows that these solutions also satisfy

$$A(\gamma, p, q) c(\gamma, \tau, k) = \frac{1}{2} \delta_{qk} c(\gamma, \tau, p), \quad (4.3a)$$

$$B(\gamma, p, q) c^*(\gamma, \tau, k) = \frac{1}{2} \delta_{qk} c(\gamma, \tau, p), \quad (4.3b)$$

and, with $p = q = k$,

$$A(\gamma, k, k) c(\gamma, \tau, k) = \frac{1}{2} c(\gamma, \tau, k), \quad (4.4a)$$

$$B(\gamma, k, k) c^*(\gamma, \tau, k) = \frac{1}{2} c(\gamma, \tau, k). \quad (4.4b)$$

Suppose we have found m_γ orthonormal solutions $c(\gamma, \tau, k)$ ($\tau = 1, 2, \dots, m_\gamma$) of Eq. (4.4) for some fixed value of k , then it follows from Eq. (4.3a) that for $p \neq k$ the vectors $c(\gamma, \tau, p)$ are given by

$$c(\gamma, \tau, p) = 2A(\gamma, p, k) c(\gamma, \tau, k) \quad (4.5)$$

We will show that Eq. (3.8) is then satisfied. Since $2P_{kk}^\gamma$ is a projection operator which projects on a subspace of dimension m_γ [see Eq. (2.25a)] the matrix $A(\gamma, k, k)$ has exactly m_γ independent eigenvectors with eigenvalue $\frac{1}{2}$ and the other independent eigenvectors have all eigenvalue 0. Since we supposed we found m_γ orthogonal eigenvectors $c(\gamma, \tau, k)$ ($\tau = 1, 2, \dots, m_\gamma$) with eigenvalue $\frac{1}{2}$, this set may be extended with eigenvectors with eigenvalue 0 to an orthonormal basis of eigenvectors of $A(\gamma, k, k)$. The relation

$$A(\gamma, k, k) = \frac{1}{2} \sum_\tau c(\gamma, \tau, k) \tilde{c}^*(\gamma, \tau, k) \quad (4.6a)$$

is now proved by verifying that both sides give the same result if they are applied to this orthonormal set of eigenvectors. In the same way the relation

$$B(\gamma, k, k) = \frac{1}{2} \sum_\tau c(\gamma, \tau, k) \tilde{c}(\gamma, \tau, k) \quad (4.6b)$$

is proved, using the complex conjugates of the orthonormal set of eigenvectors. Equation (3.8) may now be derived immediately from Eqs. (4.5), (4.6), (4.1), and (4.2).

So the problem of solving Eq. (3.8) is reduced to the problem of finding m_γ orthonormal vectors $c(\gamma, \tau, k)$ ($\tau = 1, 2, \dots, m_\gamma$) which satisfy Eq. (4.4).

Let ψ be a nonzero column of $A(\gamma, k, k)$, say the m th column:

$$\psi_{ij} = A(\gamma, k, k)_{ij, mn}. \quad (4.7)$$

Note that $\tilde{\psi}^* \psi = \frac{1}{2} A(\gamma, k, k)_{mn, mn}$, which means that a column of $A(\gamma, k, k)$ is nonzero if and only if the diagonal term of this column is nonzero. Let ϕ be the m th column of $B(\gamma, k, k)$:

$$\phi_{ij} = B(\gamma, k, k)_{ij, mn}. \quad (4.8)$$

It follows immediately from Eq. (4.1) that

$$A(\gamma, k, k) \psi = \frac{1}{2} \psi, \quad (4.9a)$$

$$A(\gamma, k, k) \phi = \frac{1}{2} \phi, \quad (4.9b)$$

$$B(\gamma, k, k) \psi^* = \frac{1}{2} \phi, \quad (4.9c)$$

$$B(\gamma, k, k) \phi^* = \frac{1}{2} \psi. \quad (4.9d)$$

If $A(\gamma, k, k)_{mn, mn}$ is not equal to $-B(\gamma, k, k)_{mn, mn}$ then $\psi + \phi$ is nonzero and satisfies Eq. (4.4). So after normalizing this vector we may define it to be $c(\gamma, 1, k)$:

$$c(\gamma, 1, k) = (\psi + \phi) \{ A(\gamma, k, k)_{mn, mn} + \frac{1}{2} B^*(\gamma, k, k)_{mn, mn} + \frac{1}{2} B(\gamma, k, k)_{mn, mn} \}^{-1/2}. \quad (4.10)$$

If it happens that $A(\gamma, k, k)_{mn, mn} = -B(\gamma, k, k)_{mn, mn}$ then the vector $i(\psi - \phi)$ is nonzero and satisfies Eq. (4.4); so we may then take $c(\gamma, 1, k)$ equal to this vector after normalizing it:

$$c(\gamma, 1, k) = \frac{i(\psi - \phi)}{\{ 2A(\gamma, k, k)_{mn, mn} \}^{1/2}}. \quad (4.11)$$

If $m_\gamma \geq 2$ we define $A'(\gamma, k, k)$ and $B'(\gamma, k, k)$ by

$$A'(\gamma, k, k) = A(\gamma, k, k) - \frac{1}{2} c(\gamma, 1, k) \tilde{c}^*(\gamma, 1, k) \quad (4.12a)$$

$$B'(\gamma, k, k) = B(\gamma, k, k) - \frac{1}{2} c(\gamma, 1, k) \tilde{c}(\gamma, 1, k) \quad (4.12b)$$

It is easy to check that the Eqs. (4.1) and (4.2) (with p, q, r , and s all equal to k) hold as well for $A'(\gamma, k, k)$ and $B'(\gamma, k, k)$. Therefore, we obtain in the same way as above a normalized vector $c(\gamma, 2, k)$ which satisfies

$$A'(\gamma, k, k) c(\gamma, 2, k) = \frac{1}{2} c(\gamma, 2, k), \quad (4.13a)$$

$$B'(\gamma, k, k) c^*(\gamma, 2, k) = \frac{1}{2} c(\gamma, 2, k). \quad (4.13b)$$

Now since

$$A(\gamma, k, k) A'(\gamma, k, k) = \frac{1}{2} A'(\gamma, k, k) \quad (4.14)$$

and

$$B(\gamma, k, k) B'^*(\gamma, k, k) = \frac{1}{2} A'(\gamma, k, k), \quad (4.15)$$

we obtain from Eq. (4.13a)

$$A(\gamma, k, k) c(\gamma, 2, k) = \frac{1}{2} c(\gamma, 2, k), \quad (4.16a)$$

and from Eq. (4.13b)

$$B(\gamma, k, k) c^*(\gamma, 2, k) = \frac{1}{2} c(\gamma, 2, k). \quad (4.16b)$$

If we apply both sides of Eq. (4.12a) to the vector $c(\gamma, 2, k)$ we find $\tilde{c}^*(\gamma, 1, k) c(\gamma, 2, k) = 0$; so $c(\gamma, 1, k)$ and $c(\gamma, 2, k)$ are ortho-

ogonal. Thus, we have shown that the vector $c(\gamma, 2, k)$ satisfies all requirements. It is obvious that if $m_\gamma \geq 3$ we may proceed in this way until m_γ orthonormal vectors satisfying Eq. (4.4) are obtained.

Let us now consider Eq. (3.9). Note that we only have to consider Eq. (3.9a) for $p, q \leq d_\gamma/2$ and Eq. (3.9b) for $p \leq d_\gamma/2$ and $q > d_\gamma/2$. Let us define the vectors $d(\gamma, \tau, p)$ for $\tau = 1, 2, \dots, 2m_\gamma$ and $p = 1, 2, \dots, d_\gamma/2$ by

$$d(\gamma, \tau, p) = c(\gamma, \tau, p) \quad 1 \leq \tau \leq m_\gamma, \quad (4.17a)$$

$$d(\gamma, \tau, p) = c\left(\gamma, \tau - m_\gamma p + \frac{d_\gamma}{2}\right) \quad m_\gamma < \tau \leq 2m_\gamma, \quad (4.17b)$$

This definition only implies a different labeling of the vectors $c(\gamma, \tau, p)$. Equation (3.9a), for $p, q \leq d_\gamma/2$ now becomes

$$A(\gamma, p, q) = \sum_{\tau=1}^{2m_\gamma} d(\gamma, \tau, p) \tilde{d}^*(\gamma, \tau, q), \quad (4.18a)$$

and Eq. (3.9b), for $p \leq d_\gamma/2$ and $q > d_\gamma/2$ becomes

$$B(\gamma, p, q) = \sum_{\tau=1}^{m_\gamma} d(\gamma, \tau, p) \tilde{d}\left(\gamma, \tau + m_\gamma q - \frac{d_\gamma}{2}\right) - \sum_{\tau=1}^{m_\gamma} d(\gamma, \tau + m_\gamma p) \tilde{d}\left(\gamma, \tau, q - \frac{d_\gamma}{2}\right),$$

which may be written as

$$C(\gamma, p, q) = \sum_{\tau=1}^{2m_\gamma} \epsilon(\tau) d(\gamma, \tau, p) \tilde{d}(\gamma, \tau + m_\gamma q), \quad (4.18b)$$

where $p, q \leq d_\gamma/2$, $C(\gamma, p, q)$ is defined by

$$C(\gamma, p, q) = B\left(\gamma, p, q + \frac{d_\gamma}{2}\right), \quad (4.19)$$

$\epsilon(\tau)$ is defined by

$$\epsilon(\tau) = \begin{cases} 1 & \text{if } 1 \leq \tau \leq m_\gamma \\ -1 & \text{if } m_\gamma < \tau \leq 2m_\gamma \end{cases} \quad (4.20)$$

and $\tau + m_\gamma$ should be read as $\tau + m_\gamma \pmod{2m_\gamma}$. So Eq. (3.9) has been rewritten as Eq. (4.18), with the definitions of Eqs. (4.17), (4.19), and (4.20), and the parameters p, q now run from 1 to $d_\gamma/2$.

From Eqs. (2.23), (3.6), and (4.19) and the fact that

$$Q_{pq}^\gamma = -Q_{p+(d_\gamma/2), q+(d_\gamma/2)}^\gamma$$

and

$$P_{pq}^\gamma = P_{p+(d_\gamma/2), q+(d_\gamma/2)}^\gamma$$

if D^γ is of type II, it follows that for $p, q, r, s \leq d_\gamma/2$

$$A(\gamma, p, q)A(\gamma, r, s) = \delta_{qr}A(\gamma, p, s), \quad (4.21a)$$

$$C(\gamma, p, q)C^*(\gamma, r, s) = -\delta_{qr}A(\gamma, p, s), \quad (4.21b)$$

$$A(\gamma, p, q)C(\gamma, r, s) = \delta_{qr}C(\gamma, p, s), \quad (4.21c)$$

$$C(\gamma, p, q)A^*(\gamma, r, s) = \delta_{qr}C(\gamma, p, s). \quad (4.21d)$$

Moreover, since $P_{pq}^{\gamma\dagger} = P_{qp}^\gamma$ and $Q_{pq}^{\gamma\dagger} = Q_{qp}^\gamma$ it follows from Eqs. (3.6) and (4.19) that

$$\tilde{A}^*(\gamma, p, q) = A(\gamma, q, p), \quad (4.22a)$$

$$\tilde{C}(\gamma, p, q) = -C(\gamma, q, p). \quad (4.22b)$$

Since we know that the solutions of Eq. (4.18) satisfy the

orthogonality relations, it follows that these solutions also satisfy

$$A(\gamma, p, q)d(\gamma, \tau, k) = \delta_{qk}d(\gamma, \tau, p), \quad (4.23a)$$

$$C(\gamma, p, q)d^*(\gamma, \tau, k) = \delta_{qk}\epsilon(\tau + m_\gamma)d(\gamma, \tau + m_\gamma, p), \quad (4.23b)$$

and, with $p = q = k$,

$$A(\gamma, k, k)d(\gamma, \tau, k) = d(\gamma, \tau, k), \quad (4.24a)$$

$$C(\gamma, k, k)d^*(\gamma, \tau, k) = \epsilon(\tau + m_\gamma)d(\gamma, \tau + m_\gamma, k). \quad (4.24b)$$

Suppose we found $2m_\gamma$ orthonormal solutions $d(\gamma, \tau, k)$ ($\tau = 1, 2, \dots, 2m_\gamma$) of Eq. (4.24) for some fixed value of k ; it then follows from Eq. (4.23a) that for $p \neq k$ the vectors $d(\gamma, \tau, p)$ are given by

$$d(\gamma, \tau, p) = A(\gamma, p, k)d(\gamma, \tau, k). \quad (4.25)$$

These vectors satisfy Eq. (4.18) as may be verified immediately, noting that P_{kk}^γ is a projection operator which projects on a subspace of dimension $2m_\gamma$ [see Eq. (2.26a)] and reasoning along the same lines as above, where we derived Eq. (3.8) from Eqs. (4.4) and (4.5).

So the problem of solving Eq. (4.18) is reduced to the problem of finding $2m_\gamma$ orthonormal vectors $d(\gamma, \tau, k)$ ($\tau = 1, 2, \dots, 2m_\gamma$) which satisfy Eq. (4.24). This can be done as follows. Let ψ be a nonzero column of $A(\gamma, k, k)$, say the m th column:

$$\psi_{ij} = A(\gamma, k, k)_{ij, mn}, \quad (4.26)$$

and let ϕ be the m th column of $C(\gamma, k, k)$:

$$\phi_{ij} = C(\gamma, k, k)_{ij, mn}. \quad (4.27)$$

Then the following properties may be derived immediately from Eqs. (4.21) and (4.22):

$$A(\gamma, k, k)\psi = \psi, \quad (4.28a)$$

$$A(\gamma, k, k)\phi = \phi, \quad (4.28b)$$

$$C(\gamma, k, k)\psi^* = \phi, \quad (4.28c)$$

$$C(\gamma, k, k)\phi^* = -\psi, \quad (4.28d)$$

$$\tilde{\psi}^*\psi = \tilde{\phi}^*\phi = A(\gamma, k, k)_{mn, mn}, \quad (4.28e)$$

$$\tilde{\psi}^*\phi = 0. \quad (4.28f)$$

Therefore, we may define

$$d(\gamma, 1, k) = \frac{\psi}{\{A(\gamma, k, k)_{mn, mn}\}^{1/2}} \quad (4.29a)$$

and

$$d(\gamma, m_\gamma + 1, k) = \frac{-\phi}{\{A(\gamma, k, k)_{mn, mn}\}^{1/2}}; \quad (4.29b)$$

these vectors are orthonormal and satisfy Eq. (4.24). If $m_\gamma \geq 2$ we define $A'(\gamma, k, k)$ and $C'(\gamma, k, k)$ by

$$A'(\gamma, k, k) = A(\gamma, k, k) - d(\gamma, 1, k)\tilde{d}^*(\gamma, 1, k) - d(\gamma, m_\gamma + 1, k)\tilde{d}^*(\gamma, m_\gamma + 1, k), \quad (4.30a)$$

$$C'(\gamma, k, k) = C(\gamma, k, k) - d(\gamma, 1, k)\tilde{d}(\gamma, m_\gamma + 1, k) + d(\gamma, m_\gamma + 1, k)\tilde{d}(\gamma, 1, k). \quad (4.30b)$$

It is easy to check that Eqs. (4.21) and (4.22) (with p, q, r and s

allequal to k) hold as well for $A'(\gamma, k, k)$ and $C'(\gamma, k, k)$. Therefore, we obtain in the same way as above two orthonormal vectors $d(\gamma, 2, k)$ and $d(\gamma, m_\gamma + 2, k)$ which satisfy

$$A'(\gamma, k, k)d(\gamma, 2, k) = d(\gamma, 2, k), \quad (4.31a)$$

$$A'(\gamma, k, k)d(\gamma, m_\gamma + 2, k) = d(\gamma, m_\gamma + 2, k), \quad (4.31b)$$

$$C'(\gamma, k, k)d^*(\gamma, 2, k) = -d(\gamma, m_\gamma + 2, k), \quad (4.31c)$$

$$C'(\gamma, k, k)d^*(\gamma, m_\gamma + 2, k) = d(\gamma, 2, k). \quad (4.31d)$$

Now since

$$A(\gamma, k, k)A'(\gamma, k, k) = A'(\gamma, k, k) \quad (4.32a)$$

and

$$C(\gamma, k, k)C'^*(\gamma, k, k) = -A'(\gamma, k, k), \quad (4.32b)$$

we obtain from Eq. (4.31)

$$A(\gamma, k, k)d(\gamma, 2, k) = d(\gamma, 2, k), \quad (4.33a)$$

$$A(\gamma, k, k)d(\gamma, m_\gamma + 2, k) = d(\gamma, m_\gamma + 2, k), \quad (4.33b)$$

$$C(\gamma, k, k)d^*(\gamma, m_\gamma + 2, k) = d(\gamma, 2, k), \quad (4.33c)$$

$$C(\gamma, k, k)d^*(\gamma, 2, k) = d(\gamma, m_\gamma + 2, k). \quad (4.33d)$$

Since $A'(\gamma, k, k)d(\gamma, 2, k) = A(\gamma, k, k)d(\gamma, 2, k)$ Eq. (4.30a) gives $\{d(\gamma, 1, k)\tilde{d}^*(\gamma, 1, k) - d(\gamma, m_\gamma + 1, k)\tilde{d}^*(\gamma, m_\gamma + 1, k)\}d(\gamma, 2, k) = 0$. By multiplying on the left with $\tilde{d}^*(\gamma, 1, k)$ we find, since $\tilde{d}^*(\gamma, 1, k)d(\gamma, 1, k) = 1$ and $\tilde{d}^*(\gamma, 1, k)d(\gamma, m_\gamma + 1, k) = 0$, that $\tilde{d}^*(\gamma, 1, k)d(\gamma, 2, k) = 0$; so $d(\gamma, 1, k)$ and $d(\gamma, 2, k)$ are orthogonal. In the same way we may show that $d(\gamma, 1, k)$, $d(\gamma, 2, k)$, $d(\gamma, m_\gamma + 1, k)$ and $d(\gamma, m_\gamma + 2, k)$ are orthonormal vectors. Thus, we indeed obtained correct solutions $d(\gamma, 2, k)$ and $d(\gamma, m_\gamma + 2, k)$. It is obvious that if $m_\gamma \geq 3$ we may proceed in this way until $2m_\gamma$ orthonormal vectors satisfying Eq. (4.24) are obtained. The solutions $c(\gamma, \tau, p)$ of Eq. (3.9) are then given by Eqs. (4.25) and (4.17).

Let us finally consider Eq. (3.10). Since here D^γ is of type III, it follows from Eqs. (2.24) and (3.6) that

$$A(\gamma, p, q)A(\gamma, r, s) = \delta_{qr}A(\gamma, p, s) \quad (p \sim q), \quad (4.34a)$$

$$B(\gamma, p, q)B^*(\gamma, r, s) = \delta_{qr}A(\gamma, p, s) \quad (p \not\sim q), \quad (4.34b)$$

$$A(\gamma, p, q)B(\gamma, r, s) = \delta_{qr}B(\gamma, p, s) \quad (p \sim q), \quad (4.34c)$$

$$B(\gamma, p, q)A^*(\gamma, r, s) = \delta_{qr}B(\gamma, p, s) \quad (p \not\sim q). \quad (4.34d)$$

Moreover, since $P_{pq}^\dagger = P_{qp}^\gamma$ and $Q_{pq}^\dagger = Q_{qp}^\gamma$ it follows from Eq. (3.6) that

$$\tilde{A}^*(\gamma, p, q) = A(\gamma, q, p), \quad (4.35a)$$

$$\tilde{B}(\gamma, p, q) = B(\gamma, q, p). \quad (4.35b)$$

Since we know that the solutions of Eq. (3.10) satisfy the orthogonality relations, it follows that these solutions also satisfy

$$A(\gamma, p, q)c(\gamma, \tau, k) = \delta_{qk}c(\gamma, \tau, p) \quad (p \sim q), \quad (4.36a)$$

$$B(\gamma, p, q)c^*(\gamma, \tau, k) = \delta_{qk}c(\gamma, \tau, p) \quad (p \not\sim q), \quad (4.36b)$$

and, with $p = q = k$

$$A(\gamma, k, k)c(\gamma, \tau, k) = c(\gamma, \tau, k). \quad (4.37)$$

Suppose we have found m_γ orthonormal solutions $c(\gamma, \tau, k)$ ($\tau = 1, 2, \dots, m_\gamma$) of Eq. (4.37) for some fixed value of k ; it then follows from Eq. (4.36) that for $p \neq k$ the vectors $c(\gamma, \tau, p)$ are given by

$$c(\gamma, \tau, p) = A(\gamma, p, k)c(\gamma, \tau, k) \quad (4.38a)$$

if $p \sim k$, and by

$$c(\gamma, \tau, p) = B(\gamma, p, k)c^*(\gamma, \tau, k) \quad (4.38b)$$

if $p \not\sim k$. As in the preceding cases we can show that Eq. (3.10) is then satisfied. So the problem of solving Eq. (3.10) is reduced to the problem of finding m_γ orthonormal vectors $c(\gamma, \tau, k)$ ($\tau = 1, 2, \dots, m_\gamma$) which satisfy Eq. (4.37).

But this problem is exactly the same as the corresponding problem for unitary representations of nonmagnetic groups,⁵ the solution being as follows: let ψ be a nonzero column of $A(\gamma, k, k)$, say the m th column. Then

$$c(\gamma, 1, k) = \frac{\psi}{\{A(\gamma, k, k)_{mn, mn}\}^{1/2}}. \quad (4.39)$$

Define (if $m_\gamma \geq 2$)

$$A'(\gamma, k, k) = A(\gamma, k, k) - c(\gamma, 1, k)\tilde{c}^*(\gamma, 1, k), \quad (4.40)$$

and take $c(\gamma, 2, k)$ equal to a normalized column of $A'(\gamma, k, k)$. In this way one may proceed until m_γ orthonormal solutions $c(\gamma, \tau, k)$ ($\tau = 1, 2, \dots, m_\gamma$) of Eq. (4.37) are obtained.

5. SUMMARY OF THE METHOD

The decomposition of the Kronecker product and the notation of the Clebsch–Gordan coefficients is given in Eq. (2.14) and the lines below. For each D^γ appearing in the decomposition the Clebsch–Gordan coefficients are calculated separately, the method depending on the type of D^γ .

Suppose D^γ is of type I. We may choose $k = 1$ and calculate the matrices $A(\gamma, p, k)$ and $B(\gamma, p, k)$ for $1 \leq p \leq d_\gamma$ from Eqs. (3.1) and (3.6). The Clebsch–Gordan coefficients are written as column matrices with Eq. (3.7). We choose a diagonal element $A(\gamma, k, k)_{mn, mn}$ which is not equal to zero. If $A(\gamma, k, k)_{mn, mn}$ is not equal $-B(\gamma, k, k)_{mn}$ then $c(\gamma, 1, k)$ is given by Eqs. (4.10), (4.7), and (4.8), and otherwise by Eqs. (4.11), (4.7), and (4.8). If $m_\gamma \geq 2$ then $A'(\gamma, k, k)$ and $B'(\gamma, k, k)$ are obtained with Eq. (4.12) and from these matrices $c(\gamma, 2, k)$ is obtained in the same way as $c(\gamma, 1, k)$ was obtained from $A(\gamma, k, k)$ and $B(\gamma, k, k)$. This procedure is continued until $c(\gamma, \tau, k)$ is known for $1 \leq \tau \leq m_\gamma$. The remaining Clebsch–Gordan coefficients are given by Eq. (4.5).

Suppose D^γ is of type II. We may choose $k = 1$ and calculate the matrices $A(\gamma, p, k)$ and $C(\gamma, p, k)$ for $1 \leq p \leq d_\gamma/2$ from Eqs. (3.1), (3.6), and (4.19). The Clebsch–Gordan coefficients are written as column matrices with Eqs. (3.7) and (4.17). We choose a diagonal element $A(\gamma, k, k)_{mn, mn}$ which is not equal to zero. Then $d(\gamma, 1, k)$ is given by Eqs. (4.29a) and (4.26); $d(\gamma, m_\gamma + 1, k)$ is given by Eqs. (4.29b) and (4.27). If $m_\gamma \geq 2$, then $A'(\gamma, k, k)$ and $C'(\gamma, k, k)$ are obtained with Eq. (4.30), and from these matrices we obtain $d(\gamma, 2, k)$ and $d(\gamma, m_\gamma + 2, k)$ in the same way as we obtained $d(\gamma, 1, k)$ and $d(\gamma, m_\gamma + 1, k)$ from $A(\gamma, k, k)$ and $C(\gamma, k, k)$. This procedure is continued until $d(\gamma, \tau, k)$ is known for $1 \leq \tau \leq 2m_\gamma$. The remaining Clebsch–Gordan coefficients are given by Eq. (4.25).

Suppose D^γ is of type III. We may choose $k = 1$ and calculate the matrices $A(\gamma, p, k)$ and $B(\gamma, p, k)$ for $1 \leq p \leq d_\gamma$ from Eqs. (3.1) and (3.6). The Clebsch–Gordan coefficients

are written as column matrices with Eq. (3.7). We choose a diagonal element $A(\gamma, k, k)_{mn, mn}$ which is not equal to zero. Then $c(\gamma, 1, k)$ is given by Eqs. (4.39) and (4.26). If $m_\gamma \geq 2$ then $A'(\gamma, k, k)$ is obtained with Eq. (4.40) and we obtain $c(\gamma, 2, k)$ from $A'(\gamma, k, k)$ in the same way as we obtained $c(\gamma, 1, k)$ from $A(\gamma, k, k)$. This procedure is continued until $c(\gamma, \tau, k)$ is known for $1 \leq \tau \leq m_\gamma$. The remaining Clebsch–Gordan coefficients are given by Eq. (4.38a) for $2 \leq p \leq d_\gamma/2$ and by Eq. (4.39b) for $d_\gamma/2 < p \leq d_\gamma$.

6. THE WIGNER ECKART THEOREM FOR FINITE MAGNETIC GROUPS

The Wigner–Eckart theorem has been generalized for UA representations of finite magnetic groups by Aviran and Zak.³ Here we will give a derivation of the Wigner–Eckart theorem which brings to light the close connection with the lemma of Schur.

Let T be a UA representation in a Hilbert space H and let $\{\phi_i^\beta\}$ ($i = 1, \dots, d_\beta$) and $\{\phi_j^\gamma\}$ ($j = 1, \dots, d_\gamma$) be orthonormal sets of elements of H which transform under T according to the irreducible UA representations D^β and D^γ of G :

$$T(g)\phi_i^\beta = \sum_j D_{ji}^\beta(g)\phi_j^\beta, \quad \forall g \in G, \quad (5.1)$$

and similarly for $\{\phi_j^\gamma\}$. Let O^α be a irreducible tensor operator transforming under T according to the irreducible UA representation D^α of G :

$$T(g)O_k^\alpha T(g)^{-1} = \sum_l D_{lk}^\alpha(g)O_l^\alpha, \quad \forall g \in G. \quad (5.2)$$

We assume that the sets $\{\phi_i^\alpha\}$, $\{\phi_j^\beta\}$ and $\{O_k^\alpha\}$ have been chosen in such a way that D^α , D^β and D^γ are the fixed representatives of their equivalence classes (which is always possible). The elements $O_i^\alpha \phi_j^\beta$ transform according to $D^\alpha \otimes D^\beta$:

$$T(g)O_i^\alpha \phi_j^\beta = \sum_{kl} (D^\alpha \otimes D^\beta)_{kl, ij} O_k^\alpha \phi_l^\beta, \quad \forall g \in G. \quad (5.3)$$

If $D^\alpha \otimes D^\beta$ is equivalent with the direct sum $\sum_\eta m_\eta D^\eta$ then the elements $\psi_p^{\eta, \tau} (m_\eta \neq 0; \tau = 1, 2, \dots, m_\eta, p = 1, \dots, d_\eta)$, defined by

$$\psi_p^{\eta, \tau} = \sum_{ij} \left(\begin{matrix} \alpha & \beta \\ i & j \end{matrix} \middle| \begin{matrix} \eta & \tau \\ p & \end{matrix} \right) O_i^\alpha \phi_j^\beta, \quad (5.4)$$

where the $\left(\begin{matrix} \alpha & \beta \\ i & j \end{matrix} \middle| \begin{matrix} \eta & \tau \\ p & \end{matrix} \right)$ are the Clebsch–Gordan coefficients, transform according to

$$T(g)\psi_p^{\eta, \tau} = \sum_q D_{qp}^\eta(g)\psi_q^{\eta, \tau}, \quad \forall g \in G. \quad (5.5)$$

We are interested in the matrix elements $(\phi_k^\gamma, O_i^\beta \phi_j^\alpha)$. We have

$$(\phi_k^\gamma, O_i^\beta \phi_j^\alpha) = \sum_{\eta, \tau, p} \left(\begin{matrix} \alpha & \beta \\ i & j \end{matrix} \middle| \begin{matrix} \eta & \tau \\ p & \end{matrix} \right)^* (\phi_k^\gamma, \psi_p^{\eta, \tau}). \quad (5.6)$$

Define the matrix U by $U_{kp} = (\phi_k^\gamma, \psi_p^{\eta, \tau})$. One verifies immediately that

$$D^\gamma(g)U^g = UD^\eta(g) \quad \forall g \in G. \quad (5.7)$$

Therefore, if $\gamma \neq \eta$, U is equal to the zero matrix¹⁷; if $\gamma = \eta$, then from the generalized lemma of Schur it follows that U

has the form of Eq. (2.9). From Eq. (5.6) it follows that the Wigner–Eckart theorem now reads as follows: If D^γ is of type I, then

$$(\phi_k^\gamma, O_i^\beta \phi_j^\alpha) = \sum_\tau \left(\begin{matrix} \alpha & \beta \\ i & j \end{matrix} \middle| \begin{matrix} \gamma & \tau \\ k & \end{matrix} \right)^* \lambda_\gamma(\tau), \quad (5.8)$$

where $\lambda_\gamma(\tau)$ are real numbers which do not depend on i, j and k ; if D^γ is of type II, then

$$\begin{aligned} (\phi_k^\gamma, O_i^\beta \phi_j^\alpha) &= \sum_\tau \left(\begin{matrix} \alpha & \beta \\ i & j \end{matrix} \middle| \begin{matrix} \gamma & \tau \\ k & \end{matrix} \right)^* \lambda_\gamma(\tau) \\ &+ \sum_\tau \left(\begin{matrix} \alpha & \beta \\ i & j \end{matrix} \middle| \begin{matrix} \gamma & \tau \\ k + \frac{d_\gamma}{2} & \end{matrix} \right)^* \mu_\gamma(\tau), \end{aligned} \quad (5.9a)$$

if $k \leq d_\gamma/2$, and

$$\begin{aligned} (\phi_k^\gamma, O_i^\beta \phi_j^\alpha) &= \sum_\tau \left(\begin{matrix} \alpha & \beta \\ i & j \end{matrix} \middle| \begin{matrix} \gamma & \tau \\ k & \end{matrix} \right)^* \lambda_\gamma(\tau) \\ &- \sum_\tau \left(\begin{matrix} \alpha & \beta \\ i & j \end{matrix} \middle| \begin{matrix} \gamma & \tau \\ k - \frac{d_\gamma}{2} & \end{matrix} \right)^* \mu_\gamma(\tau), \end{aligned} \quad (5.9b)$$

if $d_\gamma/2 < k \leq d_\gamma$, where $\lambda_\gamma(\tau)$ and $\mu_\gamma(\tau)$ are complex numbers which do not depend on i, j and k ; if D^γ is of type III, then

$$(\phi_k^\gamma, O_i^\beta \phi_j^\alpha) = \sum_\tau \left(\begin{matrix} \alpha & \beta \\ i & j \end{matrix} \middle| \begin{matrix} \gamma & \tau \\ k & \end{matrix} \right)^* \lambda_\gamma(\tau), \quad (5.10a)$$

if $k \leq d_\gamma/2$, and

$$(\phi_k^\gamma, O_i^\beta \phi_j^\alpha) = \sum_\tau \left(\begin{matrix} \alpha & \beta \\ i & j \end{matrix} \middle| \begin{matrix} \gamma & \tau \\ k & \end{matrix} \right)^* \lambda_\gamma(\tau), \quad (5.10b)$$

if $d_\gamma/2 < k \leq d_\gamma$, where $\lambda_\gamma(\tau)$ are complex numbers which do not depend on i, j and k . Since the generalized lemma of Schur does not hold in general if the irreducible UA representations are not in standard form, the Wigner–Eckart theorem as formulated above will also not be valid in general for irreducible UA representations which are not in standard form.

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