Detuning effects of FM mode-locking in atmospheric CO₂ lasers

W. J. WITTEMAN, A. H. M. OLBERTZ

Department of Applied Physics, Twente University of Technology, Enschede, The Netherlands

Received 22 October 1979; revised 2 January 1980

We describe a study of the detuning of the intracavity FM modulation of the homogeneously broadened laser applied to CO₂ systems. The analysis is done by working in the frequency domain and by describing the field in terms of discrete amplitudes and phases. The coupled mode equations are solved for modulation frequencies near the axial mode-separation frequency. The asymmetry of the mode-locking behaviour due to the dispersive behaviour of the active medium is verified experimentally. We have measured both the pulse and the phase shift of the pulse with respect to the modulation signal as a function of detuning. The experimental results show clearly the asymmetry with detuning, in agreement with theory.

1. Introduction

The problem of mode-locking and pulse forming by means of intracavity AM and FM modulation has been treated in the past by several authors [1–8]. For homogeneously broadened systems one may say that the problem has been approached from two different sides. One way is to work completely in the time domain and to assume a Gaussian optical pulse inside the resonator [4, 5]. The pulse parameters are then obtained from the requirement that after a complete round-trip through the resonator the pulse is unchanged. The other approach is to work in the frequency domain and to describe the field in terms of discrete mode amplitudes and phases [1–3].

In the present paper we analyse the detuning effects of FM mode-locked CO₂ lasers by working in the frequency domain. We derive as a function of the detuned modulation frequency spectrum, pulse phase shift with respect to the modulation, the gain, and the shift of the centre modulation frequency of the pulse. The theoretical results are analogous to those obtained in the time domain approach [4]. The difference is that we start from the discrete set of oscillating modes and show that the amplitude and phase distribution of these modes satisfy a Gaussian form. Experimentally we measured for the case of a CO₂ TEA laser the dependence of the pulsewidth on the frequency detuning for two different modulation signals. The dependence of the phase shift of the pulse with respect to the periodic modulation on the detuning is measured for 1 and 2 atmosphere laser mixtures. The experimental results show clearly a strong asymmetry in accordance with the analysis. Small detuning effects turn out to have a large bearing on the pulse parameters.

2. Derivation of the basic equation

In the following we shall work in the frequency domain, in order to obtain an analytical solution for a stationary phase-locked, internally FM modulated, laser system.

A fruitful analysis is to start with the self-consistency equations of Lamb [10], which describe the effect of an arbitrary polarization upon the electric fields. In the case of mode-locking the polarization includes contributions of both the active medium and the modulation. We shall follow the notation of Harris and McDuff [1] in order to arrive at an equation for the complex field amplitudes using the axial modes as a basic set.

The modulation is an electro-optic device that produces a periodic phase modulation. For our system we used a CdTe crystal of 40 mm length and an oscillator with a maximum output voltage of 2000 V. The phase modulation of such a modulator can be described by $\Delta\chi'(z, t)$, the in-phase component of the susceptibility:

$$\Delta\chi'(z, t) = \Delta\chi'(z) \cos \omega_m t \quad (1)$$

where ω_m is the modulation frequency.

In the case where the modulator is located near the out-coupling mirror at $z = d$ and its length is much smaller than the distance d between the mirrors we finally find the basic equation for the electric field:

$$E_{n+1} e^{i\phi_{n+1}} + E_{n-1} e^{i\phi_{n-1}} = \frac{1}{\delta} E_n e^{i\phi_n} \left[-\left(\frac{2d}{c} \dot{\phi}_n - \frac{2d}{c} n\Delta\omega + \psi_n \right) - iG_n + iL \right] \quad (2)$$

where ϕ_n is the phase of the n th mode, L is the single pass power loss, and

$$\psi_n = \frac{\omega_n \chi_n' d}{c} \quad (3)$$

is the phase retardation of the n th mode caused by the active medium

$$G_n = -\frac{\omega_n \chi_n'' d}{c} \quad (4)$$

being the single pass power gain of the n th mode through the medium;

$$n\Delta\omega = \Omega_n - \omega_n \quad (5)$$

being the difference between the n th empty cavity mode and the real mode; and

$$\delta = \frac{\omega_n}{c} \int_0^a \Delta\chi' \sin k_n z \sin k_{n \pm 1} z \, dz \quad (6)$$

being the modulation parameter.

In deriving Equation 2 we have omitted the integrals of the form

$$\int_0^a \Delta\chi'(z) \sin k_n z \cos k_{n \pm 1} z \, dz$$

because they are negligible for the position of the modulator near $z = d$.

3. Steady-state solution for the homogeneously broadened line

In the following we look for an analytical solution of Equation 2 for the case in which we are dealing with a homogeneously broadened line. Due to the interactions of the radiation field with the inverted medium the gain profile changes over its full frequency range. In general it turns out that the frequency width of the pulse is small as compared to the line-width. This is illustrated in Fig. 1, where the gain and the mode amplitudes are plotted against the frequency. This means that in the frequency region of interest we may, to a good approximation, expand the gain profile and the phase retardation of such a line profile to the second order.

This can be done in terms of the mode number n . Thus the gain and the phase retardation for the n th mode will be given by

$$G_n = G_0 + nG_1 + \frac{1}{2}n^2 G_2 \quad (7)$$

$$\psi_n = \psi_0 + n\psi_1 + \frac{1}{2}n^2 \psi_2. \quad (8)$$

In looking for an analytical solution of Equation 2 it has been argued [11] that for small detuning the quadratic terms in Equation 2 take over the effects of the linear terms on the field distribution and the

Figure 1 Schematic presentation of the mode spectrum with respect to the optical line. The centre pulse frequency ($n = 0$) has a frequency shift η with respect to the line centre.

mode amplitudes assume the Gaussian form. Therefore we describe the fields of the laser modes by a general Gaussian distribution:

$$E_n e^{i\phi_n} = E_0 \exp(-\alpha n^2 + i\theta n) \quad (9)$$

where the constants α and θ may be complex; i.e. $\alpha = \alpha_1 + i\alpha_2$ and $\theta = \theta_1 + i\theta_2$. From this we obtain:

$$E_{n+1} e^{i\phi_{n+1}} + E_{n-1} e^{i\phi_{n-1}} = E_n e^{i\phi_n} [\exp(-2n\alpha - \alpha + i\theta) + \exp(2n\alpha - \alpha - i\theta)]. \quad (10)$$

The arguments of the expressions on the right-hand side of Equation 10 are small, as will be seen later on. Therefore we expand the last expression up to second order. We obtain:

$$E_{n+1} e^{i\phi_{n+1}} + E_{n-1} e^{i\phi_{n-1}} = E_n e^{i\phi_n} (2 - 2\alpha + \alpha^2 + 4n\alpha^2 - \theta^2 - 4n\alpha\theta i). \quad (11)$$

By comparing Equations 2 and 11 and making use of Equations 7 and 8 we equate terms of the same order of n for the real and imaginary parts. Then we obtain the following six equations:

$$2 - 2\alpha_1 + \alpha_1^2 - \alpha_2^2 - \theta_1^2 + \theta_2^2 = -\frac{2d}{\delta c} \dot{\phi}_n - \frac{\psi_0}{\delta} \quad (12)$$

$$-2\alpha_2 - 2\theta_1\theta_2 + 2\alpha_1\alpha_2 = \frac{L}{\delta} - \frac{G_0}{\delta} \quad (13)$$

$$4\theta_1\alpha_2 + 4\alpha_1\theta_2 = -\frac{\psi_1}{\delta} + \frac{2d}{c\delta} \Delta\omega \quad (14)$$

$$-4\theta_1\alpha_1 + 4\alpha_2\theta_2 = -\frac{G_1}{\delta} \quad (15)$$

$$4\alpha_1^2 - 4\alpha_2^2 = -\frac{1}{2\delta} \psi_2 \quad (16)$$

$$8\alpha_1\alpha_2 = -\frac{1}{2\delta} G_2. \quad (17)$$

In solving this set of equations we have to consider two different situations, which are described by the sign of the modulation parameters δ . For a positive value of δ we are dealing with a pulse that passes the modulator when its phase perturbation is at or close to its maximum value. The other situation, given by a negative value of δ , corresponds to the passage of the pulse at or close to the minimum of the phase perturbation. The pulse forming at these two different situations is sometimes called the positive and negative mode, respectively [4]. The two solutions give different values for α_1 and α_2 . It turns out that the frequency shift η of the centre pulse frequency changes sign by changing the sign of δ . The values of the pulse width and pulse-band width as a function of the detuning frequency do not depend on the sign of δ . The two solutions are similar.

We shall consider here only a positive value of δ .

From Equations 16 and 17 we find for the α s:

$$\alpha_1 = \frac{1}{4\sqrt{\delta}} [-\psi_2 + (\psi_2^2 + G_2^2)^{1/2}]^{1/2} \quad (18)$$

and

$$\alpha_2 = \mp \frac{1}{4\sqrt{\delta}} [\psi_2 + (\psi_2^2 + G_2^2)^{1/2}]^{1/2}. \quad (19)$$

The sign of α_2 depends on the sign of G_2 . In practical situations with a mode centre frequency close to the line centre where G_2 is negative we have to use the positive sign in Equation 19, otherwise the negative sign.

For values of $\alpha_2 \neq 0$ we are dealing with a frequency chirp in the light pulse, i.e. in the time domain there is a linear frequency shift during the pulse.

Using Equations 18 and 19 we obtain from Equations 14 and 15:

$$\theta_1 = \frac{1}{2\sqrt{\delta}(\psi_2^2 + G_2^2)^{1/2}} \left\{ \mp \frac{2d}{c} \Delta\omega_t [\psi_2 + (\psi_2^2 + G_2^2)^{1/2}]^{1/2} + G_1 [\psi_2 + (\psi_2^2 + G_2^2)^{1/2}]^{1/2} \right\} \quad (20)$$

and

$$\theta_2 = \frac{1}{2\sqrt{\delta}(\psi_2^2 + G_2^2)^{1/2}} \left\{ \frac{2d}{c} \Delta\omega_t [-\psi_2 + (\psi_2^2 + G_2^2)^{1/2}]^{1/2} \pm G_1 [\psi_2 + (\psi_2^2 + G_2^2)^{1/2}]^{1/2} \right\} \quad (21)$$

where the upper and lower signs refer to, respectively, the positive and negative values of G_2 , and

$$\Delta\omega_t = \Delta\omega - \frac{c}{2d} \psi_1. \quad (22)$$

For mode-locking at the ideal situation we have $\Delta\omega_t = 0$. The mode with the highest intensity must be found for $n = 0$, so that according to Equation 9 θ_2 must be equal to zero. This condition gives us the relation between $\Delta\omega_t$ and the position of the pulse centre frequency with respect to the line profile. Setting $\theta_2 = 0$ we obtain from Equation 21:

$$\frac{2d}{c} \Delta\omega_t = \mp G_1 \frac{[\psi_2 + (\psi_2^2 + G_2^2)^{1/2}]^{1/2}}{[-\psi_2 + (\psi_2^2 + G_2^2)^{1/2}]^{1/2}}. \quad (23)$$

From this equation it is readily seen that for $G_2 = 0$ at the lower frequency side of the profile, where ψ_2 is positive, the detuning $\Delta\omega_t$ goes to infinity. This means that by detuning the system for positive values of $\Delta\omega_t$ the centre frequency of the pulse will never be smaller than the frequency for which G_2 at the lower frequency side of the profile becomes zero. For negative values of $\Delta\omega_t$ the centre pulse frequency moves onto the higher frequency part of the profile. It is also seen from Equation 23 that for $\Delta\omega_t = 0$ the value of ψ_2 must be zero, i.e. the pulse centre frequency coincides with the line centre. The frequency shift from the ideal situation is not given by $\Delta\omega_t$, because by detuning the system the pulse centre frequency moves away from the line centre, so that the gain at the line centre has to increase in order to balance the cavity losses and accordingly ψ_1 will increase. The detuning from the ideal situation $\Delta\omega_{\text{det}}$ is then given by:

$$\Delta\omega_{\text{det}} = \Delta\omega_t + \Delta \frac{c}{2d} \psi_1 \quad (24)$$

where $\Delta[c/(2d)] \psi_1$ is the change of $(c/2d)\psi_1$ with respect to the ideal situation. It turns out that as $\Delta\omega_{\text{det}}$ decreases from the zero value the centre frequency of the pulse shifts continuously to higher frequencies on the high frequency side of the line profile. Furthermore there is only a small frequency range for negative detuning as compared to positive detuning, provided there is sufficient gain in the medium.

The gain of the system for the pulse can be obtained from Equation 13. Substituting Equations 18–20 and $\theta_2 = 0$ we find:

$$G_0 = L \mp [\frac{1}{4}\delta\psi_2 + \frac{1}{4}\delta(\psi_2^2 + G_2^2)^{1/2}]^{1/2} + \frac{G_2}{8}. \quad (25)$$

The pulse form can be obtained by summing the field over all modes. Thus the total field is given by

$$E_{\text{tot}} = \sum_n E_n \exp [i(\Omega_0 + n\omega_m + \dot{\phi})t + i\phi_n]. \quad (26)$$

Making use of Equation 9 we get:

$$E_{\text{tot}} = E_0 [\exp i(\Omega_0 + \dot{\phi})t] \sum_n \exp (-\alpha n^2 + i\theta n + in\omega_m t). \quad (27)$$

In order to calculate the radiation intensity we have to take $E_{\text{tot}}E_{\text{tot}}^*$. The summation over the modes is straightforward. For simplicity we increase the time by θ_1/ω_m . In other words we substitute

$$t' = t + \theta_1/\omega_m \quad (28)$$

and omit the prime in the final result. We then find:

$$E_{\text{tot}}E_{\text{tot}}^* = C^2 \left| \exp \left(-\frac{\omega_m^2 t'^2}{2\alpha} \right) \right| \quad (29)$$

where

$$C = \left| \sum_n \exp \left\{ -\alpha \left(n - i \frac{\omega_m t'}{2\alpha} \right)^2 \right\} \right|. \quad (30)$$

Because the time duration of the pulse is small it turns out that C can be considered as time-independent.

The pulse-width, defined as the time between half intensity points, is according to Equation 29 given by:

$$\tau_p = \frac{2}{\omega_m} (\ln 2)^{1/2} [(2\alpha_1^2 + 2\alpha_2^2)/\alpha_1]^{1/2}. \quad (31)$$

Using Equations 18 and 19 we obtain

$$\tau_p = \frac{2}{\omega_m} (\ln 2)^{1/2} \delta^{-1/4} \frac{(\psi_2^2 + G_2^2)^{1/4}}{[-\psi_2 + (\psi_2^2 + G_2^2)^{1/2}]^{1/4}}. \quad (32)$$

From Equation 28 we find that the pulse passes through the modulator at an instant $\Delta t = \theta_1/\omega_m$ away from the peak of the modulation cycle. So we find that

$$\Delta\phi_{\text{pulse}} = -\theta_1 \quad (33)$$

is the phase difference between the modulator and the pulse. (For positive values of $\Delta\phi_{\text{pulse}}$ the pulse is lagging.)

The frequency band-width $\Delta\nu_{\text{pulse}}$ of the laser pulse is defined as the frequency between half-power points of the mode spectrum. From Equations 9 and 18 we get:

$$\Delta\nu_{\text{pulse}} = \frac{2\nu_m (2 \ln 2)^{1/2} \delta^{1/4}}{[-\psi_2 + (\psi_2^2 + G_2^2)^{1/2}]^{1/4}}, \quad (34)$$

where $\nu_m = \omega_m/2\pi$.

4. Lorentzian line profile

The line profile being known, the pulse parameters τ_p , $\Delta\nu_{\text{pulse}}$, $\Delta\phi_{\text{pulse}}$, G_0 and η , being the frequency difference between the pulse centre frequency and line centre frequency, can be calculated as a function

Figure 2 The ratio τ_{pulse} to the round-trip time T of the optical pulse is plotted as a function of the relative frequency detuning for $L = 0.1$, $\nu_m/\Delta\nu_N = 0.04$ and two different values of δ .

of $\Delta\omega_{\text{det}}$ by means of Equations 23–25, 31, 33 and 34. In order to do these calculations we have to know the line-profile parameters as a function of η . The pressure-broadened line profile of an atmospheric CO_2 laser can be described by a Lorentzian line profile. For this case the single pass intensity gain for the n th mode with frequency $\omega_n + \phi$ can be expressed by:

$$G_n(\omega_n + \phi) = \bar{G} \left[1 + 4 \left(\frac{\eta + n\omega_m}{\Delta\omega_N} \right)^2 \right]^{-1}. \quad (35)$$

The phase retardation of the n th mode for a round-trip through the medium for this medium given by

$$\psi_n = \frac{2(\eta + n\omega_m)}{\Delta\omega_N} \bar{G} \left[1 + 4 \left(\frac{\eta + n\omega_m}{\Delta\omega_N} \right)^2 \right]^{-1}. \quad (36)$$

Comparing Equation 7 with Equation 35 we find:

$$G_0 = \bar{G}(1 + p^2)^{-1} \quad (37)$$

$$G_1 = -2\bar{G}pq(1 + p^2)^{-2} \quad (38)$$

$$G_2 = 2\bar{G}q^2(3p^2 - 1)(1 + p^2)^{-3}, \quad (39)$$

where

$$\frac{2\eta}{\Delta\omega_N} = p \quad \text{and} \quad \frac{2\omega_m}{\Delta\omega_N} = q$$

Similarly we obtain by comparing Equations 8 and 38:

$$\psi_0 = \bar{G}p(1 + p^2)^{-1} \quad (40)$$

$$\psi_1 = \bar{G}q(1 - p^2)(1 + p^2)^{-2} \quad (41)$$

$$\psi_2 = 2\bar{G}pq^2(p^2 - 3)(1 + p^2)^{-3}. \quad (42)$$

Figure 3 The ratio of the pulse band-width to the line width is plotted as a function of the detuning frequency for $L = 0.3$, $\nu_m/\Delta\nu_m = 0.04$ and for 1 and 2 atmosphere CO₂ laser mixtures.

By means of Expressions 37–42 the pulse parameters are calculated for our experimental parameters $L = 0.3$, $\nu_m/\Delta\nu_m = 0.04$ and different values of the modulation parameter δ as a function of the frequency detuning from the ideal situation. The results are plotted in Figs. 2–6. It is found that the centre mode frequency shift η is practically independent of the modulation depth parameter δ . The results for η are plotted in Fig. 6 for $\delta = 0.2$ up to $\delta = 1$. It should be noted that for the ideal situation, $\Delta\omega_{\text{det}} = 0$, where $p = 0$, the pulse parameters are easily derived. Substituting Equation 39 for $p = 0$ into Equations 32 and 34 we find for ideal mode-locking:

$$\tau_p = \frac{2}{\pi} (\ln 2)^{1/2} \left(\frac{\bar{G}}{2\delta} \right)^{1/4} \frac{1}{(\nu_m \Delta\nu_N)^{1/2}} \quad (43)$$

Figure 4 The phase difference between the modulator and the pulse is plotted as a function of the detuning frequency for $L = 0.3$, $\nu_m/\Delta\nu_m = 0.02$ and for 1 and 2 atmosphere CO₂ laser mixtures.

Figure 5 The single-pass power gain at the line centre as a function of the frequency detuning for $L = 0.3$, $\nu_m/\Delta\nu_N \approx 0.04$ and for 1 and 2 atmosphere CO_2 laser mixtures.

Figure 6 The ratio of the centre pulse frequency shift to the line width is plotted versus the relative frequency detuning for $L = 0.3$, $\nu_m/\Delta\nu_N \approx 0.04$ and for 1 and 2 atmosphere CO_2 mixtures. It turns out that this shift is practically independent of the modulation depth parameter.

and

$$\Delta\nu_{\text{pulse}} = (2 \ln 2)^{1/2} \left(\frac{2\delta}{G} \right)^{1/4} (\nu_m \Delta\nu_N)^{1/2}. \quad (44)$$

These values are the same as those found [4] for a time domain solution of an optical pulse in a resonator.

5. Experimental procedure and observations

The experimental system consists basically of a CO₂ TEA laser and a CdTe crystal for electro-optic phase modulation. The TEA laser (1 : 2 : 4 mixture) employs two brass uniform field electrodes of 30 cm length and a separation distance of 5 mm, shaped according to the paper of Chang [12], with $k = 0.01$. The discharge is fed from a Blumlein circuit with a total capacity of 6 nF and a charge voltage of 20 kV. The cavity is formed by one total reflecting mirror with a radius of curvature of 2.5 m and a germanium flat with a reflection coefficient of 70%.

The mode-locking occurs by means of active mode-locking with an electro-optic device in order to produce the required periodic phase modulation. The modulator consists of a CdTe crystal with both ends AR coated and sizes of $40 \times 5 \times 5$. The modulator is driven by an oscillator that delivers a maximum voltage of 2000 V to the crystal. The long term frequency stability of the oscillator was within 10 kHz. The modulation frequency at 120.42 MHz was controlled by a quartz crystal. The detuning of the system was performed by just changing the distance between the mirrors while the modulator frequency remained fixed at this stabilized frequency. The centre mirror separation distance for ideal mode-locking is the distance for which the observed beat frequency of the modes of the system with the modulation power switched off is equal to the modulator frequency.

Although the above analyses have been done for steady-state conditions and the experiments were done under pulsed-discharge conditions, we notice that the discharge and inversion duration time is long enough to obtain a quasi-stationary condition. The observed pulse train consisted of about 20–30 pulses with a duration of about 250 ns. The pulse width at the beginning of the train and at the end of the train were practically equal. By detuning the laser system for fixed parameters of electrical input energy and modulation depth the pulse form disappears abruptly for a small negative detuning. However, for positive detuning the pulse form does not disappear, but the height of the pulses decreases only slightly. In contrast to AM the output does not disappear for positive detuning of the system. In Fig. 2 we have plotted the pulse duration as a function of the detuning for two different values of δ . In order to be able to measure the pulse durations with our detection system (a hot-hole detector with a rise time of 200 ns) we had to use small values for the modulation parameter δ . From the figure we see that the detuning is asymmetrical and that there is an excellent agreement between the calculated and measured values for τ_{pulse} .

The phase shift of the pulse with respect to the modulation signal was obtained by putting both the unchanged modulation signal and the pulse on a dual-beam oscilloscope. The phase shift in which we are interested is then simply obtained by subtracting from the observed phase shift on the oscilloscope its value for the ideal situation with no detuning. The observed values for gas mixtures of respectively 1 and 2 atmosphere are plotted in Fig. 4. Again we find a very good agreement between predictions and observations.

6. Discussion

It is seen that the pulse parameters change asymmetrically with the detuning frequency and that the negative detuning range is small as compared to the positive detuning, which seems to be unlimited provided there is sufficient gain. The asymmetry of the pulse behaviour comes from the asymmetry of the dispersion of the active medium.

The condition to be fulfilled for pulse forming by means of mode-locking is to find a way of obtaining a constant axial frequency separation equal to the modulation frequency. In any laser system the

axial mode frequency spacing is not constant for all modes due to the dispersion of the active medium. Further, in the case of pulse forming the gain is not constant for all modes due to the gain profile, so that in order to maintain over a period a certain mode amplitude distribution there must be a redistribution of the amplitudes by passing through the modulator. Both the constancy of mode frequency and redistribution of the mode amplitudes can be achieved by a non-linear phase spacing between the axial modes, given by α_2 . By this mechanism the modes with higher and lower frequencies will pass the modulator effectively at different times.

In the case of detuning the 'natural' mode spacing is not equal to the modulation frequency. The pulse will now not pass the modulator at the extremes of the phase variation of the modulator, i.e. $\Delta\phi_{\text{pulse}} \neq 0$. The modulator will then give a Doppler shift to the mode spectrum and therefore also to the axial mode separation. The phase difference between modulator and pulse would yield a further change in frequency each time the pulse passes the modulator. This, however, is not the case, because the position of the mode spectrum with respect to the line profile will be such that after a passage through the medium the mode spectrum has suffered an opposite frequency change as compared with that caused by the modulator. The modes closer to the line centre suffer more gain than those further away from the line centre. This means that the pulse centre frequency, i.e. the frequency of the mode having largest amplitude, has changed effectively towards the line centre by passing through the medium. Since the frequency shift of the modulator is more or less abrupt and the 'correction' in frequency occurs continuously by passing through the medium, the phase difference between adjacent modes equal to $\int_0^T \omega_m dt = 2\pi$ can be reached during the modulation period.

References

1. S. E. HARRIS and O. P. McDUFF, *IEEE J. Quant. Elect.* QE1 (1965) 245–62.
2. O. P. McDUFF and S. E. HARRIS, *ibid* QE-3 (1967) 101–111.
3. H. HAKEN and M. PAUTHIER, *ibid* QE-4 (1968) 454–59.
4. D. J. KUIZENGA and A. E. SIEGMAN, *ibid* QE-6 (1970) 694–708.
5. A. E. SIEGMAN and D. J. KUIZENGA, *ibid* QE-6 (1970) 803–808.
6. D. J. KUIZENGA and A. E. SIEGMAN, *ibid* QE-6 (1970) 709–15.
7. H. A. HAUS, *ibid* QE-11 (1975) 323–30.
8. A. E. SIEGMAN and D. J. KUIZENGA, *Opto-Electronics* 6 (1974) 43–66.
9. W. J. WITTEMAN and A. H. M. OLBERTZ, *IEEE J. Quant. Elect.* QE-13 (1977) 381–387.
10. W. E. LAMB, Jr., *Phys. Rev.* 134 (1964) A1429–A1450.
11. T. J. NELSON, *IEEE J. Quant. Elect.* QE-8 (1972) 29–33.
12. T. Y. CHANG, *Rev. Scient. Instr.* 44 (1973) 405–407.