POLYNOMIAL SOLUTIONS TO H_∞ PROBLEMS

GJERRIT MEINSMA*

Department of Applied Mathematics, University of Twente, PO Box 217, 7500 AE Enschede, The Netherlands

SUMMARY

The paper presents a polynomial solution to the standard H_∞ -optimal control problem. Based on two polynomial J -spectral factorization problems, a parameterization of all suboptimal compensators is obtained. A bound on the McMillan degree of suboptimal compensators is derived and an algorithm is formulated that may be used to solve polynomial J -spectral factorization problems.

KEY WORDS H_∞ -optimization Polynomial methods J -spectral factorization McMillan degree

1. INTRODUCTION

We present a polynomial solution to the standard H_∞ -optimal control problem. The basic result is that, given a solution to two polynomial J -spectral factorization problems, all compensators may be generated that stabilize the plant and make the closed-loop transfer matrix satisfy a given ∞ -norm bound, provided any such compensators exist. The result was derived in preliminary form in 1989 by Kwakernaak (see Reference 19). The derivation presented in this paper is based on what may be called J -lossless theory. It is a modified and corrected version of that given in Reference 27 and it is linked to work by Ball and Cohen,² Ball and Helton,⁴ Helton¹⁴ and most of all to work by Green.¹³ The essential difference from other approaches is that in our approach nonproper plants can be handled. This, for instance, makes it possible to recast mixed sensitivity problems with nonproper shaping filters directly as standard H_∞ problems. In order to make the known state-space results applicable to these 'nonproper problems', first a rational matrix has to be absorbed into the standard plant to make the standard plant proper. This, however, increases the McMillan degree of the standard plant, and thereby increases the McMillan degree of suboptimal compensators, a problem that does not occur if polynomial methods are used directly on the nonproper problem. This is shown in an example in Section 5. The example gives as we hope a good explanation of why polynomial methods are useful.

The solution to the standard H_∞ -optimal control problem hinges on the solution to J -spectral factorization problems. J -spectral factorization problems are often solved using solutions of one or two indefinite Riccati equations (see References 12 and 1). In Section 4 it is shown how polynomial J -spectral factorization problems may be solved by means of a factor extraction procedure. This polynomial algorithm has computational as well as theoretical value.

To keep the paper readable, most of the proofs are listed in the Appendix.

* Present address: Department of Electrical and Computer Engineering, The University of Newcastle, University Drive, Callaghan, Newcastle, NSW 2308, Australia.

This paper was recommended for publication by editor M. G. Safonov

2. PRELIMINARIES

Throughout this paper we use the following notation.

$\mathbb{R}, \mathbb{C}$	real and complex numbers
$\mathbb{C}_-, \mathbb{C}_+, \mathbb{C}_0$	open left half complex plane, open right half complex plane, imaginary axis
$\mathbb{R}^{m \times n}, \mathbb{C}^{m \times n}$	real and complex valued $m \times n$ matrices
$\bar{s}$	complex conjugate of $s \in \mathbb{C}$
$\mathbb{R}[s], \mathbb{R}(s)$	polynomials in s and rational functions in s with real coefficients
$\mathbb{R}^{m \times n}[s], \mathbb{R}^{m \times n}(s)$	$m \times n$ matrices with entries in $\mathbb{R}[s], \mathbb{R}(s)$
$\mathcal{L}_2(a, b)$	$\{w: (a, b) \rightarrow \mathbb{C}^m \mid \int_a^b w^*(t)w(t) dt < \infty\}$
A^*, A^T	complex conjugate transpose and transpose of $A \in \mathbb{C}^{m \times n}$
H^-, H^*	$H^-(s) = H(-\bar{s})^*$ and $H^*(s) = (H(s))^*$ for $H \in \mathbb{C}^{m \times n}(s)$
$A > B, A \geq B$	$A(s) - B(s) > 0, A(s) - B(s) \geq 0$ for all $s \in \mathbb{C}_0$ (including infinity if $A, B \in \mathbb{C}^{n \times n}(s)$ are proper)
$\rho_i(R), \rho(R)$	i th row degree and sum of row degrees of a full row rank polynomial matrix R
$\gamma_i(R), \gamma(R)$	i th column degree and sum of column degrees of a full column rank polynomial matrix R
$\delta_r(R)$	minimum of $\rho(UR)$ over all polynomial unimodular matrices U of a full row rank polynomial matrix R
$\delta_c(R)$	minimum of $\gamma(RU)$ over all polynomial unimodular matrices U of a full column rank polynomial matrix R
$\delta_M(G)$	McMillan degree of rational G
$[z]$	number of components of a vector valued signal z
$\text{In}(A) = (\pi(A), \nu(A), \delta(A))$	The inertia of $A \in \mathbb{C}^{n \times n}$, that is, a triple of integers denoting the numbers of eigenvalues of A in $\mathbb{C}_+, \mathbb{C}_-$ and $\mathbb{C}_0$
$\ H\ _\infty$	supremum over all $s \in \mathbb{C}_0$ of the largest singular value of $H(s)$

Definition 2.1 (Kailath¹⁶)

The McMillan degree of a proper rational matrix G denoted by $\delta_M(G)$ is defined as

$$\delta_M(G) = \sum_i \deg \psi_i \quad (1)$$

where ψ_i are the denominator polynomials of the Smith–McMillan form of G :

$$G = U \begin{pmatrix} \frac{\varepsilon_1}{\psi_1} & & 0 \\ & \ddots & \\ & & \frac{\varepsilon_r}{\psi_r} \\ 0 & & & 0 \end{pmatrix} V \quad (2)$$

U and V are unimodular polynomial matrices, (ε_i, ψ_i) are coprime, $\varepsilon_i \mid \varepsilon_{i+1}$ and $\psi_{i+1} \mid \psi_i$. $\square$

This definition does not make sense for nonproper transfer matrices. Via a Möbius transformation this problem can be circumvented. In Rosenbrock²⁹ the McMillan degree of

a nonproper transfer matrix G is defined as the McMillan degree of H , where H is defined as $H(p) = G(\alpha p/(p-1))$ for some $\alpha \neq 0$. Note that H is proper if G does not have a pole at α . This somewhat tricky definition is not very appealing. The following result makes the McMillan degree easier to handle.

Lemma 2.2

$$\delta_M(G) = \delta_r(-N \ D) = \delta_c\left(\frac{\bar{D}}{\bar{N}}\right) \quad (3)$$

for any left and right coprime polynomial fractions of $G = D^{-1}N = \bar{N}\bar{D}^{-1}$. $\square$

A proof may be found in Reference 23. In Reference 32 Lemma 2.2 is formulated as an exercise. Lemma 2.2 shows that the differential equation determines the McMillan degree. In other words the McMillan degree can be determined of a system whose signals satisfy a set of differential equations; it does not depend on a partitioning of the signals into inputs and outputs. This is an interesting fact (see Reference 34).

A transfer matrix G is *stable* if all poles of G , except those at infinity, lie in $\mathbb{C}_-$. In particular polynomial matrices are considered stable. A rational matrix A is *para-Hermitian* if $A^\sim = A$. In the case where A is a para-Hermitian rational matrix with constant inertia on the imaginary axis, then $\text{In}(A)$ denotes the inertia of A on the imaginary axis. A rational matrix E is *inner* if it is stable and $E^\sim E = I$. A rational matrix $\bar{E}$ is *co-inner* if $\bar{E}^\top$ is inner. A rational matrix Q is a *J-spectral factor* of A if Q and Q^{-1} are stable and $A = Q^\sim J Q$. The matrix J is assumed to be a diagonal matrix of the form

$$\begin{pmatrix} I_q & 0 \\ 0 & -I_p \end{pmatrix} \quad (4)$$

A rational matrix $\bar{Q}$ is a *J-spectral cofactor* of A if $\bar{Q} J \bar{Q}^\sim = A$ and $\bar{Q}$ and $\bar{Q}^{-1}$ are stable. A square polynomial matrix R is (strictly) *Hurwitz* if the roots of $\det R$ lie in the (open) left half complex plane. The *zeros* of a rational matrix G are the roots of the nonzero numerator polynomials e_i in the Smith–McMillan form of G . A polynomial matrix R is *left prime* if it has full row rank everywhere on the complex plane.

Central to our solution to the standard H_∞ -optimal control problem is the following theorem. For a proof see References 26 and 24.

Theorem 2.3^{24,26}

Suppose $R \in \mathbb{R}^{m \times (q+p)}[s]$ is a polynomial matrix that has full row rank on the imaginary axis, and suppose $(w^\top \ z^\top)^\top$ is a partitioned vector valued time signal with $[w] = q$ and $[z] = p$. The following are equivalent if $p \leq m$.

1. There exists an $\varepsilon > 0$ such that every solution $v := (w^\top \ z^\top)^\top$ in $\mathcal{L}_2(-\infty, T)$ of the differential equation $R(d/dt)v(t) = 0$ satisfies

$$\int_{-\infty}^T w^*(t)w(t) - z^*(t)z(t) \, dt \geq \varepsilon \int_{-\infty}^T w^*(t)w(t) + z^*(t)z(t) \, dt \quad (5)$$

2. There exists a strictly Hurwitz solution Q to the J -spectral cofactorization problem

$$R \begin{pmatrix} I_q & 0 \\ 0 & -I_p \end{pmatrix} R^\sim = Q \begin{pmatrix} I_{m-p} & 0 \\ 0 & -I_p \end{pmatrix} Q^\sim \quad (6)$$

with $Q^{-1}R$ proper. Moreover, such a Q has the property that the inequality

$$Q^{-1}R \begin{pmatrix} I_q & 0 \\ 0 & -I_p \end{pmatrix} (RQ^{-1})^* \leq \begin{pmatrix} I_{m-p} & 0 \\ 0 & -I_p \end{pmatrix} \quad (7)$$

is satisfied in $\mathbb{C}_+$, or equivalently, such that the matrix $(Q_1 \ R_2)$ consisting of the left $m-p$ columns of Q and the right p columns of R is strictly Hurwitz. $\square$

Example 2.4

Suppose w and z are one-dimensional signals related through the differential equation

$$R \left(\frac{d}{dt} \right) \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} = 0, \quad R(s) = \begin{pmatrix} 1 & -\gamma \\ s-1 & 0 \end{pmatrix} \quad (8)$$

All solutions are of the form $z(t) = (1/\gamma)w(t) = ce^t$. Therefore there exists an $\varepsilon > 0$ such that (5) is satisfied for all solutions iff $|\gamma| > 1$.

Next we check when the conditions of the second item of Theorem 2.3 are met. A strictly Hurwitz solution Q of the equation

$$Q \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Q^{-1} = R \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R^{-1} = \begin{pmatrix} 1-\gamma^2 & -s-1 \\ s+1 & 1-s^2 \end{pmatrix} \quad (9)$$

with $Q^{-1}R$ proper, is

$$Q = \begin{pmatrix} 1 & -\gamma \\ s + \frac{\gamma^2+1}{\gamma^2-1} & \frac{-2\gamma}{\gamma^2-1} \end{pmatrix} \quad (10)$$

The matrix $(Q_1 \ R_2)$ is strictly Hurwitz iff $s + (\gamma^2+1)/(\gamma^2-1)$ is strictly Hurwitz, which is the case iff $|\gamma| > 1$. If $|\gamma| = 1$ then Q as given here is not defined. It follows from a result of Section 4 that for $|\gamma| = 1$ there do not exist strictly Hurwitz solutions Q such that $Q^{-1}R$ is proper. $\square$

3. THE STANDARD H_∞ -OPTIMAL CONTROL PROBLEM

We first briefly review the standard H_∞ -optimal control problem. As shown in Figure 1, the 'plant' G has an external input w and a control input u . The outputs of the plant are the control error z and the observed output y . The system is described in transfer matrix form by

Figure 1. The standard system

the input–output map

$$\begin{pmatrix} z \\ y \end{pmatrix} = G \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} \quad (11)$$

with G a rational transfer matrix. The system is controlled by a feedback compensator

$$u = Ky \quad (12)$$

The resulting closed-loop system has the closed-loop transfer matrix

$$H = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \quad (13)$$

The standard H_∞ -optimal control problem is the problem of minimizing the ∞ -norm $\|H\|_\infty$ of the closed-loop transfer matrix with respect to all compensators K that stabilize the closed-loop system. The standard H_∞ -suboptimization problem is the problem of generating stabilizing compensators K that make the closed-loop transfer matrix H satisfy $\|H\|_\infty < \lambda$ for a given bound λ . Such compensators are said to be *admissible*.

We write the plant G as a fraction of coprime polynomial matrices.

$$G = (D_1 \ D_2)^{-1}(N_1 \ N_2) \quad (14)$$

The partitioning of the numerator and the denominator polynomial matrix is compatible with the partitioning of the input and output signals. If we write the compensator K as $K = X^{-1}Y$, with X and Y left coprime polynomial matrices, we have that the closed-loop system is described by the differential equations

$$\begin{pmatrix} -N_1 & D_1 & D_2 & -N_2 \\ 0 & 0 & -Y & X \end{pmatrix} \begin{pmatrix} w \\ z \\ y \\ u \end{pmatrix} = 0 \quad (15)$$

We do not make a distinction between a time signal and its Laplace transform. Throughout we assume the following.

Assumption 3.1

1. $(-N_1 \ D_1)$ has full row rank on $\mathbb{C}_0$.
2. $(D_2 \ -N_2)$ has full column rank on $\mathbb{C}_0$.

□

3.1. Stability of the standard system

The standard system as in Figure 2 is *stable* if the transfer matrix from inputs (w, v_1, v_2) to outputs (z, y, u) is stable.

We assume that the plant and compensator are given in transfer matrix form. Writing K and G as left coprime polynomial matrix fractions $K = X^{-1}Y$ and $G = D^{-1}N = (D_1 \ D_2)^{-1}(N_1 \ N_2)$, we may describe the closed-loop system of Figure 2 by the differential equations

$$\underbrace{\begin{pmatrix} D_1 & D_2 & -N_2 \\ 0 & -Y & X \end{pmatrix}}_R \begin{pmatrix} z \\ y \\ u \end{pmatrix} = \underbrace{\begin{pmatrix} N_1 & D_2 & N_2 \\ 0 & 0 & 0 \end{pmatrix}}_P \begin{pmatrix} w \\ v_1 \\ v_2 \end{pmatrix} \quad (16)$$

Because the fractions of G and K are left coprime, also the pair (R, P) is left coprime and, hence, the zeros of R are the closed-loop poles and the closed-loop system is internally stable

Figure 2. The standard system; setup for internal stability

if and only if R is strictly Hurwitz. The zeros of D_1 and $(D_1 \ D_2 \ -N_2)$ are also zeros of R . These zeros do not depend on the choice of compensator and are called the *fixed* closed-loop poles. A compensator is said to be *stabilizing* if it places all the closed-loop poles in the open left half complex plane. There exist stabilizing compensators if and only if the fixed closed-loop poles all lie in the open left half complex plane. This may be seen as follows. By unimodular transformation from the left, R may be brought in the form

$$R = \begin{pmatrix} D_{11} & D_{12} & -N_{12} \\ 0 & A'D'_{22} & -A'N'_{22} \\ 0 & -Y & X \end{pmatrix} \quad (17)$$

with D_{11} square and $(D'_{22} - N'_{22})$ left prime and A' square nonsingular. The zeros of R that are always there independent of the choice of compensator are the zeros of D_{11} and A' , which in turn are precisely the zeros of D_1 and $(D_1 \ D_2 \ -N_2)$. The remaining zeros of R are the zeros of

$$\begin{pmatrix} D'_{22} & -N'_{22} \\ -Y & X \end{pmatrix} \quad (18)$$

These zeros may be placed at will by a suitable choice of compensator and are called the *assignable* closed-loop poles. Note that $G_{22} = (D'_{22})^{-1}N'_{22}$ and, hence, if any stabilizing compensator exists then K stabilizes iff it stabilizes G_{22} (see Reference 8).

Note that a stable closed-loop system is not necessarily *well-posed*, that is, the map from inputs (w, v_1, v_2) to outputs (z, u, y) is not necessarily proper. We explain in Section 5 why we want the closed-loop system to be stable but not necessarily well-posed. Note also that we do not require the compensator K to be proper; see Section 5.

3.2. Solution to the standard H_∞ -suboptimization problem

In this subsection the standard H_∞ -suboptimal control problem is solved. For simplicity we take $\lambda = 1$. At the end of this subsection we give a summary of our findings in the form of an algorithm.

Recall that the closed-loop system is described by the differential equation

$$\begin{pmatrix} -N_1 & D_1 & D_2 & -N_2 \\ 0 & 0 & -Y & X \end{pmatrix} \begin{pmatrix} w \\ z \\ y \\ u \end{pmatrix} = 0 \quad (19)$$

Written in this form clearly shows that the set

$$\left\{ (w, z, y, u) \mid (-N_1 \ D_1) \begin{pmatrix} w \\ z \end{pmatrix} = 0, \ y = 0, \ u = 0 \right\} \quad (20)$$

is a linear subset of the solutions of (19) that does not depend on the compensator. Therefore if the standard H_∞ problem has a solution K such that $\|H\|_\infty < 1$ then certainly on this compensator independent subset (20) the inequality

$$\int_{-\infty}^T w^*(t)w(t) - z^*(t)z(t) \, dt \geq \varepsilon \int_{-\infty}^T w^*(t)w(t) + z^*(t)z(t) \, dt \quad (21)$$

holds for some $\varepsilon > 0$. By Assumption 3.1 the matrix $(-N_1 \ D_1)$ has full row rank on $\mathbb{C}_0$, which makes Theorem 2.3 applicable. From Theorem 2.3 we may conclude that admissible compensators exist such that $\|H\|_\infty < 1$ only if (22) has a strictly Hurwitz solution Q with $Q^{-1}(-N_1 \ D_1)$ proper and $(Q_1 \ D_1)$ strictly Hurwitz. Combined with Redheffer's Lemma²⁸ we have Lemma 3.2.

Lemma 3.2

If there exist stabilizing compensators such that $\|H\|_\infty < 1$ then there exist strictly Hurwitz solutions Q of

$$Q \begin{pmatrix} I_{[y]} & 0 \\ 0 & -I_{[z]} \end{pmatrix} Q^{-1} = (-N_1 \ D_1) \begin{pmatrix} I_{[w]} & 0 \\ 0 & -I_{[z]} \end{pmatrix} \begin{pmatrix} -N_1^* \\ D_1^* \end{pmatrix} \quad (22)$$

with $Q^{-1}(-N_1 \ D_1)$ proper. Let Q be such a J -spectral cofactor with $Q^{-1}(-N_1 \ D_1)$ proper, and partition $Q = (Q_1 \ Q_2)$, with Q_1 the left $[y]$ columns of Q and Q_2 the right $[z]$ columns of Q . Consider the system as depicted in Figure 3. The following holds:

- E is well defined and there exist compensators K stabilizing G such that $\|H\|_\infty < 1$ only if E is co-inner. In that case the set of compensators that stabilize G such that $\|H\|_\infty < 1$ ($\|H\|_\infty \leq 1$) coincides with the set of compensators that stabilize G' such that the closed-loop transfer matrix H' from w' to z' satisfies $\|H'\|_\infty < 1$ ($\|H'\|_\infty \leq 1$).
- $\delta_M(G) = \delta_r(-N_1 \ D_1 \ D_2 \ -N_2) = \delta_r(Q \ D_2 \ -N_2) \leq \delta_M(G')$. Generically, equality holds. $\square$

$$E = (D_1 \ -Q_1)^{-1} (N_1 \ Q_2)$$

$$G' = (Q_2 \ D_2)^{-1} (-Q_1 \ N_2)$$

Figure 3. The first associated standard system

A proof is listed in the Appendix. It should be clear that G is precisely the interconnection of E and G' . Note that we do not require G' to be well defined. $(Q_2 \ D_2)$ may well be singular. This is no real problem; it means that (w', u) is not a suitable set of inputs to the system represented wrongly in transfer matrix form by G' . The proof does not rely on nonsingularity of $(Q_2 \ D_2)$. Note that always $EE^* = I$, and that E is co-inner iff $(D_1 - Q_1)$ is strictly Hurwitz.

According to Lemma 3.2 we may as well concentrate on the standard H_∞ suboptimization problem with plant G' . This we do next. The closed-loop system is described by

$$\begin{pmatrix} Q_1 & Q_2 & D_2 & -N_2 \\ 0 & 0 & -Y & X \end{pmatrix} \begin{pmatrix} w' \\ z' \\ y \\ e \end{pmatrix} = 0 \quad (23)$$

In the terminology of Reference 34 this is an AR-representation of the system. It is also possible to construct an MA-representation (see Reference 34) of this closed-loop system. To this end, define a right coprime pair (Δ, Λ) such that $\Delta\Lambda^{-1} = Q^{-1}(D_2 - N_2)$. If we partition Δ and Λ as

$$\Delta = \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} \quad (24)$$

compatibly with the partitioning of $Q = (Q_1 \ Q_2)$ and $(D_2 - N_2)$, we may obtain a right coprime fraction of G' :

$$G' = (Q_2 \ D_2)^{-1}(-Q_1 \ N_2) = \begin{pmatrix} \Delta_2 \\ -\Lambda_1 \end{pmatrix} \begin{pmatrix} \Delta_1 \\ -\Lambda_2 \end{pmatrix}^{-1} \quad (25)$$

With this right fraction of G' we may rewrite

$$\begin{pmatrix} z' \\ y \end{pmatrix} = G' \begin{pmatrix} w' \\ u \end{pmatrix} \quad (26)$$

as

$$\begin{pmatrix} w' \\ z' \\ y \\ u \end{pmatrix} = \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ -\Lambda_1 \\ -\Lambda_2 \end{pmatrix} l_{G'} \quad (27)$$

Similarly, with a right coprime fraction representation of the compensator $K = \bar{Y}\bar{X}^{-1}$, the relation $y = Ku$ may be rewritten as

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} l_K \quad (28)$$

This, together with (27), gives the following description of the closed-loop system:

$$\begin{pmatrix} w' \\ z' \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \Delta_1 & 0 \\ \Delta_2 & 0 \\ \Lambda_1 & \bar{X} \\ \Lambda_2 & \bar{Y} \end{pmatrix} \begin{pmatrix} l_{G'} \\ l_K \end{pmatrix} \quad (29)$$

Equation (29) looks very much like a transposed version of equation (15). In fact by transposing the problem we may invoke Lemma 3.2 again. This may be seen as follows. The closed-loop transfer matrix of the closed-loop system with plant G'^T and compensator K^T is

H'^T . Moreover, K^T stabilizes G'^T iff K stabilizes G' . Hence we may as well solve the problem with plant

$$G'^T = (\Delta_1^T \quad -\Lambda_2^T)^{-1} (\Delta_2^T \quad -\Lambda_1^T) \quad (30)$$

The signals in the closed-loop system with plant G'^T and compensator K^T satisfy the differential equations

$$\begin{pmatrix} -\Delta_2^T & \Delta_1^T & -\Lambda_2^T & \Lambda_1^T \\ 0 & 0 & -\bar{Y}^T & \bar{X}^T \end{pmatrix} \begin{pmatrix} w \\ z \\ y \\ u \end{pmatrix} = 0 \quad (31)$$

for some signals w, z, y and u . This closed-loop system satisfies assumptions similar to those we made for the original problem, namely that $(-\Delta_2^T \quad \Delta_1^T)$ has full row rank on $\mathbb{C}_0$, and that $(-\Lambda_2^T \quad \Lambda_1^T)$ has full column rank on $\mathbb{C}_0$. This is because Λ is strictly Hurwitz and $\Delta = Q^{-1}(D_2 - N_2)\Lambda$ is tall and has full column rank on $\mathbb{C}_0$. Hence, we may invoke Lemma 3.2 again. In a transposed version Lemma 3.2 in this case reads:

Lemma 3.3

If there exist stabilizing compensators such that $\|H'\|_\infty < 1$ then there exist strictly Hurwitz solutions Γ of

$$\Gamma^{-1} \begin{pmatrix} I_{[y]} & 0 \\ 0 & -I_{[u]} \end{pmatrix} \Gamma = \Delta^{-1} \begin{pmatrix} I_{[y]} & 0 \\ 0 & -I_{[z]} \end{pmatrix} \Delta \quad (32)$$

with $\Delta\Gamma^{-1}$ proper. Let Γ be such a J -spectral factor with $\Delta\Gamma^{-1}$ proper, and partition

$$\Gamma = \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} \quad (33)$$

with Γ_1 the upper $[y]$ rows of Γ and Γ_2 the lower $[u]$ rows of Γ . Consider the system as depicted in Figure 4. The following holds:

- $\bar{E}$ is well defined and there exist compensators stabilizing G' such that $\|H'\|_\infty < 1$ only if $\bar{E}$ is inner. In that case the set of compensators that stabilize G' such that $\|H'\|_\infty < 1$ ($\|H'\|_\infty \leq 1$) coincides with the set of compensators that stabilize G'' such that the closed-loop transfer matrix U from w'' to z'' satisfies $\|U\|_\infty < 1$ ($\|U\|_\infty \leq 1$).
- $\delta_M(G) = \delta_r(Q \quad D_2 \quad -N_2) \geq \delta_c(\hat{\Lambda}) = \delta_c(\hat{\Lambda}) \geq \delta_M(G'')$. Generically, equality holds. $\square$

Thus the problem is reduced to finding compensators K that stabilize the standard system with plant G'' and make the closed-loop transfer matrix U from w'' to z'' satisfy $\|U\|_\infty < 1$. In the remainder of this section we show that U is a free parameter in the sense that for every stable U there exist $\bar{X}$ and $\bar{Y}$ such that the closed-loop system with plant G'' and compensator $\bar{X}\bar{Y}^{-1}$ is stable and has closed-loop transfer matrix U .

The closed-loop system with plant G'' is described by

$$\begin{pmatrix} w'' \\ z'' \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \Gamma_1 & 0 \\ \Gamma_2 & 0 \\ \Lambda_1 & \bar{X} \\ \Lambda_2 & \bar{Y} \end{pmatrix} \begin{pmatrix} I_{G''} \\ I_K \end{pmatrix} \quad (34)$$

Figure 4. The second associated standard system

$$\bar{E} = \begin{pmatrix} \Delta_2 \\ \Gamma_1 \end{pmatrix} \begin{pmatrix} \Delta_1 \\ -\Gamma_2 \end{pmatrix}^{-1}$$

$$G'' = \begin{pmatrix} \Gamma_2 \\ -\Lambda_1 \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ -\Lambda_2 \end{pmatrix}^{-1}$$

Since Γ and Λ are square, $l_{G'}$ may be eliminated, resulting in

$$\Lambda \Gamma^{-1} \begin{pmatrix} w'' \\ z'' \end{pmatrix} + \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} l_K = 0 \quad (35)$$

Without loss of generality we may write

$$\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \Lambda \Gamma^{-1} \begin{pmatrix} A \\ B \end{pmatrix} \quad (36)$$

With this, (35) reduces to

$$\begin{pmatrix} w'' \\ z'' \end{pmatrix} = - \begin{pmatrix} A \\ B \end{pmatrix} l_K \quad (37)$$

Hence, $U = BA^{-1}$, implying nonsingularity of A . As $K = \bar{Y}\bar{X}^{-1}$ does not depend on multiplication on the right in (36), $K = \bar{Y}\bar{X}^{-1}$ is equivalently described by

$$\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \Lambda \Gamma^{-1} \begin{pmatrix} I \\ U \end{pmatrix} \quad (38)$$

That such compensators are stabilizing follows from

$$\begin{pmatrix} I & 0 \\ 0 & \Gamma \Lambda^{-1} \end{pmatrix} \begin{pmatrix} \Gamma_1 & 0 \\ \Lambda_1 & \bar{X} \\ \Lambda_2 & \bar{Y} \end{pmatrix} \begin{pmatrix} \Gamma^{-1} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ I & 0 & I \\ 0 & I & U \end{pmatrix} \quad (39)$$

The term on the right is nonsingular in $\mathbb{C}_+ \cup \mathbb{C}_0$ for stable U . Hence, so is the term on the left. This implies that the square matrix

$$\begin{pmatrix} \Gamma_1 & 0 \\ \Lambda_1 & \bar{X} \\ \Lambda_2 & \bar{Y} \end{pmatrix} \quad (40)$$

is nonsingular in $\mathbb{C}_+ \cup \mathbb{C}_0$ because the other two factors on the left of (39), and their inverses, are stable. To investigate internal stability, examine the extended closed-loop system with

Figure 5. A standard system; setup for internal stability

extra internal signals v_1 and v_2 as in Figure 5. It is clear that this system is described by

$$\begin{pmatrix} z'' \\ y \\ u \\ w'' \\ v_1 \\ -v_2 \end{pmatrix} = \begin{pmatrix} \Gamma_2 & 0 \\ 0 & \bar{X} \\ 0 & \bar{Y} \\ \Gamma_1 & 0 \\ \Lambda_1 & \bar{X} \\ \Lambda_2 & \bar{Y} \end{pmatrix} \begin{pmatrix} I_{G''} \\ I_K \end{pmatrix} \quad (41)$$

Hence, the transfer matrix from (w'', v_1, v_2) to (z'', y, u) is stable because $\bar{X}$ and $\bar{Y}$ are stable and (40) is nonsingular in $\mathbb{C}_0 \cup \mathbb{C}_+$.

Summarizing, reintroducing λ , we have the following algorithm.

Algorithm 3.4

Given: A left coprime polynomial matrix fraction description of the plant $G = (D_1 \ D_2)^{-1}(N_1 \ N_2)$. Assumptions: $(-N_1 \ D_1)$ has full row rank on $\mathbb{C}_0$ and $(D_2 \ -N_2)$ has full column rank on $\mathbb{C}_0$.

(a) Choose $\lambda \in \mathbb{R}$.

(b) Compute, if possible, a J -spectral cofactor Q such that

$$Q \begin{pmatrix} I_{[y]} & 0 \\ 0 & -I_{[z]} \end{pmatrix} Q^{-1} = (-N_1 \ D_1) \begin{pmatrix} I_{[w]} & 0 \\ 0 & -\lambda^2 I_{[z]} \end{pmatrix} \begin{pmatrix} -N_1 \\ D_1 \end{pmatrix} \quad (42)$$

with $Q^{-1}(-N_1 \ D_1)$ proper. If this solution exists and $(Q_1 \ D_1)$ is strictly Hurwitz, with Q_1 the left $[y]$ columns of Q , then proceed to (c). Otherwise, no admissible compensator exists; λ need be increased and (b) repeated.

(c) Find right coprime polynomial matrices Δ and Λ such that $\Delta \Lambda^{-1} = Q^{-1}(D_2 \ -N_2)$.

(d) Compute, if possible, a J -spectral factor Γ such that

$$\Gamma^{-1} \begin{pmatrix} I_{[y]} & 0 \\ 0 & -I_{[u]} \end{pmatrix} \Gamma = \Delta^{-1} \begin{pmatrix} I_{[y]} & 0 \\ 0 & -I_{[z]} \end{pmatrix} \Delta \quad (43)$$

with $\Delta \Gamma^{-1}$ proper. If this solution exists and

$$\begin{pmatrix} \Delta_1 \\ \Gamma_2 \end{pmatrix} \quad (44)$$

is strictly Hurwitz, with Γ_2 the lower $[u]$ rows of Γ and Δ_1 the upper $[y]$ rows of Δ , then proceed to (e). Otherwise, no admissible compensator exists; λ need be increased and (b)–(d) repeated.

- (e) These exist stabilizing compensators such that $\|H\|_\infty < \lambda$. All compensators $K = \bar{Y}\bar{X}^{-1}$ that stabilize and make $\|H\|_\infty \leq \lambda$ are generated by

$$\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \Lambda \Gamma^{-1} \begin{pmatrix} I \\ U \end{pmatrix}, \quad U \text{ stable and } \|U\|_\infty \leq 1 \quad (45)$$

In the next section we give an algorithm that may be used to solve polynomial J -spectral factorization problems. The algorithm produces solutions Q and Γ with $Q^{-1}(-N_1 \ D_1)$ and $\Delta\Gamma^{-1}$ proper, iff such solutions exist. With such an algorithm at hand we are thus able to delimit the minimally achievable ∞ -norm λ_{opt} , by varying λ . It should be noted, however, that at $\lambda = \lambda_{\text{opt}}$ the algorithm usually fails to generate stabilizing compensators.

4. POLYNOMIAL J -SPECTRAL FACTORIZATION

In this section we treat the polynomial J -spectral factorization problem. Most proofs are listed in the appendix.

The basic theorem originates from Jacobovič.

Theorem 4.1 (Jacobovič¹⁵)

A square nonsingular para-Hermitian polynomial matrix A that does not have zeros on the imaginary axis may be factored as

$$\Gamma^* J \Gamma = A \quad (46)$$

where Γ is strictly Hurwitz and J is a unique signature matrix of the form

$$J = \begin{pmatrix} I_q & 0 \\ 0 & -I_p \end{pmatrix} \quad (47) \quad \square$$

The algorithm to be formulated later in this section provides an alternative proof of the above theorem, and in fact it proves a more general version of Theorem 4.1. In the case where $J = \pm I$, it is well known that the spectral factor Γ is unique up to premultiplication by a constant unitary matrix. This does not hold in case $J \neq \pm I$.

Lemma 4.2

Under the conditions of Theorem 4.1 a solution Γ is unique up to multiplication from the left by a polynomial unimodular J -unitary matrix U , that is,

$$U^* J U = J \quad (48)$$

The matrix U is necessarily a constant matrix only if $J = \pm I$. $\square$

Proof 4.3

(see Green *et al.*¹²). Suppose that Γ and $\bar{\Gamma}$ are both J -spectral factors of A . Then $U = \Gamma^{-1}\bar{\Gamma}$ and its inverse are stable and, hence, U^* and its inverse have all their poles in $\mathbb{C}_+$. Then $U^* J U = J$ implies that U can have only poles and zeros at infinity, i.e., U must be a polynomial unimodular matrix. On the imaginary axis we have $\text{tr}((U(s))^* J U(s)) = \text{tr}(J)$, and, hence, if $J = \pm I$ we have that $\|U(i\omega)\|_2^2 = \pm \text{tr}(U^*(i\omega) J U(i\omega)) = \text{tr}(I)$ is bounded on the imaginary axis, which implies that the polynomial matrix U must be constant. If $J \neq \pm I$ then polynomial

(nonconstant) J -unitary matrices U exist. For instance,

$$U = \begin{pmatrix} s+1 & s \\ -s & -s+1 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (49) \quad \square$$

The following example illustrates some points.

Example 4.4

Suppose A is given as

$$A = P^{-1} \begin{pmatrix} I_2 & 0 \\ 0 & -\lambda^2 \end{pmatrix} P, \quad P = \begin{pmatrix} 1 & 1 \\ 1 & -s \\ -1 & 0 \end{pmatrix} \quad (50)$$

so that

$$A = \begin{pmatrix} 2 - \lambda^2 & 1 - s \\ 1 + s & 1 - s^2 \end{pmatrix} \quad (51)$$

From $\det A = (1 - s^2)(1 - \lambda^2)$ it follows that A admits a $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ -spectral factorization only for $|\lambda| > 1$. Depending on λ we have a $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ -spectral factor Γ given by

$$\begin{pmatrix} 0 & \frac{\sqrt{\lambda^2 - 1}(s+1)}{\sqrt{\lambda^2 - 2}} \\ \sqrt{\lambda^2 - 2} & \frac{-1+s}{\sqrt{\lambda^2 - 2}} \end{pmatrix}, \quad \begin{pmatrix} 1+s & 1-s^2/2 \\ 1+s & -s^2/2 \end{pmatrix}, \quad \begin{pmatrix} \sqrt{2-\lambda^2} & \frac{1-s}{\sqrt{2-\lambda^2}} \\ 0 & \frac{\sqrt{\lambda^2 - 1}(s+1)}{\sqrt{2-\lambda^2}} \end{pmatrix} \quad (52)$$

for $\lambda > \sqrt{2}$, $\lambda = \sqrt{2}$, and $\sqrt{2} > \lambda > 1$, respectively. For $\lambda \neq \sqrt{2}$ the column degrees of Γ equal those of P . For $\lambda = \sqrt{2}$ there does not exist a Γ that has the same column degrees as P . This is because all J -spectral factors of A are of the form $U\Gamma$ and, hence, in this example, for $\lambda = \sqrt{2}$ the first column of $U\Gamma$ is (nonzero) divisible by $s+1$, whereas the first column of P is constant.

◇

Often the matrix A to be factored is given as $A = P^{-1}JP$ for some tall or not strictly Hurwitz P . As the example shows, it is apparently not always possible to find Γ such that $P\Gamma^{-1}$ is proper, or, equivalently if P is column reduced, such that Γ is column reduced with the same column degrees as P . If $P\Gamma^{-1}$ is proper, we have

$$\delta_c \begin{pmatrix} P \\ \Gamma \end{pmatrix} = \delta_c(P) = \delta_c(\Gamma) \quad (53)$$

Generically, it is possible to find Γ such that $P\Gamma^{-1}$ is proper. As we have seen in the previous section, this is an important property of P .

We next present an algorithm to compute J -spectral factors numerically. It is based on Callier's method for ordinary polynomial spectral factorization by symmetric factor extraction.⁵ For details we refer to Reference 19. An algorithm based on diagonalization, is described in Reference 30. It can handle J -spectral factorization problems where the matrix to be factored may be singular.

By $\underline{m}$ we mean $\{1, 2, \dots, m\}$, the set of positive integers from 1 up to and including m .

Algorithm 4.5

Given $A = P^* J' P$, with P tall column reduced with m columns and A nonsingular on C_0 , the algorithm determines J and produces a J -spectral factor Γ of A .

- (a) $n := \frac{1}{2} \deg \det A$. Compute all n zeros $\zeta_j \in \mathbb{C}_-$ of $\det A$. Set the virtual column degrees d_j : $d_j := \gamma_j(P)$ for $j \in \underline{m}$. Set $i := 0$ and $A_1 := A$.
- (b) $i := i + 1$. Compute a constant null vector $e = (e_1, \dots, e_m)^T$ such that $A_i(\zeta_i)e = 0$.
- (c) Select a pivot index k from the maximal active index set

$$\mathcal{M}_i = \{j \in \underline{m} \mid e_j \neq 0 \text{ and } d_j \geq d_l \text{ for all } l \in \underline{m} \text{ for which } e_l \neq 0\} \quad (54)$$

- (d) Compute the polynomial matrix $A_{i+1} = (T_i^*)^{-1} A_i T_i^{-1}$, where T_i is defined as

$$T_i(s) = \begin{bmatrix} 1 & & -\frac{e_1}{e_k} & & \\ & \ddots & \vdots & & \\ & & 1 & -\frac{e_{k-1}}{e_k} & \\ & & & s - \zeta_i & \\ & & & -\frac{e_{k+1}}{e_k} & 1 \\ & & & \vdots & \\ & & & -\frac{e_m}{e_k} & 1 \end{bmatrix} \quad (55)$$

- (e) $d_k := d_k - 1$ (update of the virtual column degrees of $PT_1^{-1} \dots T_i^{-1}$).
- (f) If $i < n$ then go to (b).
- (g) A_{n+1} is unimodular. Compute unimodular W such that

$$W^* J W = A_{n+1} \quad (56)$$

by whatever method (see for instance References 15 and 5).

- (h) $\Gamma = W T_n \dots T_1$ is a J -spectral factor of A . □

Γ generated this way may turn out to have complex valued coefficients. In the case where A itself has only real valued coefficients, the extractions may be rearranged such that Γ is also real.^{19,20} For completeness we briefly discuss it here.

Suppose A is real and suppose that the algorithm is at a point that the next zero to be extracted ζ_i is complex valued. Without loss of generality we may assume $\zeta_{i+1} = \bar{\zeta}_i$. The following combines in this case the i th and i plus first extraction step. There are two cases. Suppose e satisfies $A(\zeta_i)e = 0$. Write e in Cartesian form as $e = p + jq$. Define the matrix $C = (p, q)$, where p, q is the vector whose entries are the elements p_j of p for which $j \in \mathcal{M}_i$. The vector q, q is defined similarly. If the rank of C is one, then steps (d) and (e) need to be replaced by:

- (d) Write in Cartesian form: $e = p + jq = (p_1, \dots, p_m)^T + j(q_1, \dots, q_m)^T$ and $\zeta_i = \sigma + j\omega$.

Compute the polynomial matrix $A_{i+2} = (T_i)^{-1} A_i T_i^{-1}$, where T_i is defined as

$$T_i(s) = \begin{pmatrix} I & -a_1 - b_1 s & 0 \\ 0 & (s - \xi_i)(s - \bar{\xi}_i) & 0 \\ 0 & -a_2 - b_2 s & I \end{pmatrix} \quad (57)$$

$\uparrow$
 k th column

with a_1, a_2, b_1 and b_2 determined by

$$k\text{th row} \rightarrow \begin{pmatrix} a_1 & b_1 \\ 1 & 0 \\ a_2 & b_2 \end{pmatrix} = (p \quad q) \begin{pmatrix} p_k & q_k \\ \sigma p_k - \omega q_k & \sigma q_k + \omega p_k \end{pmatrix}^{-1} \quad (58)$$

(e) $d_k := d_k - 2$ and $i := i + 1$, $T_{i+1} = I$.

Note that the determinant of the 2 by 2 matrix on the right in equation (58) is $\omega(p_k^2 + q_k^2)$, which is nonzero because $k \in \mathcal{M}_i$ and ξ_i is not real valued.

The quadratic factor takes a different form if C has rank two. In this case $\mathcal{M}_i$ contains at least two elements. From $\mathcal{M}_i$ two pivot elements k and l have to be selected such that

$$\begin{pmatrix} p_k & q_k \\ p_l & q_l \end{pmatrix} \quad (59)$$

is nonsingular. Steps (c), (d) and (e) have to be replaced by:

(c) Select two pivot indices k and l from $\mathcal{M}_i$ such that (59) is nonsingular.

(d) Write in Cartesian form: $e = p + jq = (p_1, \dots, p_m)^T + j(q_1, \dots, q_m)^T$ and $\xi_i = \sigma + j\omega$.

Compute the polynomial matrix $A_{i+2} = (T_i)^{-1} A_i T_i^{-1}$, where T_i is defined as

$$T_i(s) = \begin{pmatrix} I & -a_1 & 0 & -b_1 & 0 \\ 0 & s - \alpha & 0 & -\beta & 0 \\ 0 & -a_2 & I & -b_2 & 0 \\ 0 & -\gamma & 0 & s - \delta & 0 \\ 0 & -a_3 & 0 & -b_3 & I \end{pmatrix} \quad (60)$$

$\uparrow$
 k th column

$\uparrow$
 l th column

with $a_1, a_2, a_3, b_1, b_2, b_3, \alpha, \beta, \gamma$ and δ determined by

$$\begin{aligned} k\text{th row} &\rightarrow \begin{pmatrix} a_1 & b_1 \\ 1 & 0 \\ a_2 & b_2 \end{pmatrix} \\ l\text{th row} &\rightarrow \begin{pmatrix} 0 & 1 \\ a_3 & b_3 \end{pmatrix} \end{aligned} = (p \quad q) \begin{pmatrix} p_k & q_k \\ p_l & q_l \end{pmatrix}^{-1} \quad (61)$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \sigma I_2 - \omega \begin{pmatrix} p_k & q_k \\ p_l & q_l \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_k & q_k \\ p_l & q_l \end{pmatrix}^{-1} \quad (62)$$

(e) $d_k := d_k - 1$, $d_l := d_l - 1$ and $i := i + 1$, $T_{i+1} = I$.

We make clear, by means of a lemma, why we call d_j the *virtual column degrees* of $PT_1^{-1} \dots T_i^{-1}$.

Lemma 4.6

In the notation of Algorithm 4.5, either in the real or in the complex version,

$$PT_1^{-1} \dots T_i^{-1} \begin{pmatrix} s^{-d_1} & & \\ & \ddots & \\ & & s^{-d_m} \end{pmatrix} \quad (63)$$

is proper and has full column rank at infinity for all $i \in \{0, \dots, n\}$. $\square$

From Lemma 4.6 we see that if on exit of the algorithm all virtual column degrees are zero, the matrix $PT_1^{-1} \dots T_i^{-1}$ itself is proper and has full column rank at ∞ . We can in fact prove a more general result.

Lemma 4.7

In the notation of Algorithm 4.5, there exist J -spectral factors Γ such that $P\Gamma^{-1}$ is proper if and only if the algorithm terminates with $d_j = 0$ for all $j \in m$. In this case A_{n+1} is constant, and Γ is a J -spectral factor of A with $P\Gamma^{-1}$ proper iff $\bar{\Gamma} = WT_n \dots T_1$ with W a constant matrix satisfying $\tilde{W}JW = A_{n-1}$. In particular, all solutions Γ with $P\Gamma^{-1}$ proper are unique up to multiplication from the left by a constant J -unitary matrix. $\square$

Example 4.8

We take the same matrix as in Example 4.4:

$$A = P^*JP = \begin{pmatrix} 2 - \lambda^2 & 1 - s \\ 1 + s & 1 - s^2 \end{pmatrix} \quad (64)$$

For $1 < \lambda \neq \sqrt{2}$ we go through the steps of the algorithm.

- (a) $n = 1$, $\xi_1 = -1$, $A_1 = A$, $m = 1$, and $d_1 = 0$, $d_2 = 1$.
- (b) $i = 1$,

$$A(\xi_1) = A(-1) = \begin{pmatrix} 2 - \lambda^2 & 2 \\ 0 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 2 \\ \lambda^2 - 2 \end{pmatrix} \quad (65)$$

- (c) As k is to be chosen from the set $\{2\}$, we have $k = 2$.
- (d) $A_2 = (T_1^*)^{-1}A_1T_1^{-1}$, with

$$T_1 = \begin{pmatrix} 1 & -\frac{2}{\lambda^2 - 2} \\ 0 & s + 1 \end{pmatrix} \quad (66)$$

so that

$$T_1^{-1} = \begin{pmatrix} 1 & \frac{2}{\lambda^2 - 2} & \frac{1}{s + 1} \\ 0 & \frac{1}{s + 1} \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 2 - \lambda^2 & -1 \\ -1 & 1 \end{pmatrix} \quad (67)$$

- (e) $d_2 := 0$.
- (f) $i = n = 1$ and all d_j are zero. Hence, by Lemma 4.7, A_{n+1} is constant.

(g)

$$A_2 = \begin{pmatrix} 2 - \lambda^2 & -1 \\ -1 & 1 \end{pmatrix} \quad (68)$$

$$= \begin{pmatrix} -1 & \sqrt{\lambda^2 - 1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ \sqrt{\lambda^2 - 1} & 0 \end{pmatrix} = W^* J W \quad (69)$$

$$(h) \quad \Gamma = W T_1 = \begin{pmatrix} -1 & 1 \\ \sqrt{\lambda^2 - 1} & 0 \end{pmatrix} \begin{pmatrix} 1 & -\frac{2}{\lambda^2 - 2} \\ 0 & s + 1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{\lambda^2}{\lambda^2 - 2} + s \\ \sqrt{\lambda^2 - 1} & \frac{-2}{\lambda^2 - 2} \sqrt{\lambda^2 - 1} \end{pmatrix} \quad (70)$$

If $\lambda = \sqrt{2}$ the pivot element can only be $k = 1$ because then $e = (2 \ 0)^T$. This gives rise to a discontinuity of the J -spectral factor as a function of λ at $\lambda = \sqrt{2}$. Lemma 4.7 shows that at $\lambda = \sqrt{2}$ no J -spectral factor Γ of A exists with $P\Gamma^{-1}$ proper. $\square$

5. EXAMPLE AND REMARKS

The standard H_∞ problem is 'standard' because many other H_∞ control problems may be recast as a standard H_∞ problem. In this section we examine one of these problems in more detail: the mixed sensitivity problem. The example of the mixed sensitivity problem that follows clarifies why we do not bother about properness and well-posedness much.

Example 5.1

We consider a mixed sensitivity problem as shown in Figure 6. The plant P is given and the compensator K is to be determined such that it makes the closed-loop system 'behave well'. The idea is that 'behaving well' may adequately be translated in terms of ∞ -norm bounds, that is, if the filters V , W_1 and W_2 are designed 'correctly' then stabilizing compensators that make the ∞ -norm of the transfer matrix H from w to (z_1, z_2) small, make the closed-loop system behave well. This is in a few words the mixed sensitivity problem. How to translate 'behaving well' in terms of these shaping filters is a problem by its own and we are not going to dwell on it here. For details, see Reference 19. Properness of K and well-posedness of the closed-loop system ($I + P(\infty)K(\infty)$ nonsingular) is usually essential for a closed-loop system to behave well. In other words, correctly designed shaping filters have the property that nearly optimal

Figure 6. The mixed sensitivity problem

compensators K are proper and that the closed-loop system is well-posed. In many cases properness of K is not enough; K should be strictly proper, or better, it should be small outside the closed-loop bandwidth. In terms of shaping filters this means that W_2 has to be chosen nonproper.

Suppose that the plane is given as

$$P(s) = \frac{1}{s} \quad (71)$$

It may be argued that

$$W_1(s) = 1, \quad V(s) = \frac{s+1}{s}, \quad W_2(s) = c(1+rs) \quad (72)$$

are correctly designed shaping filters if $0 \leq r \leq 1$ and $c > 0$. With these filters, the smaller a stabilizing compensator makes

$$\|H\|_\infty = \left\| \frac{W_1(I+PK)^{-1}V}{W_2K(I+PK)^{-1}V} \right\|_\infty = \left\| \frac{\frac{s+1}{s+K}}{c(1+rs)K \frac{s+1}{s+K}} \right\|_\infty \quad (73)$$

the better it makes the closed-loop system behave. Expression (73) is finite only if $K(s)$ is bounded at infinity and, hence, admissible compensators are always proper and even strictly proper if $r \neq 0$. It shows that admissible compensators always make the closed-loop system well-posed in this example ($1 + P(\infty)K(\infty) = 1$ is nonsingular).

The generalized plant G in the corresponding standard system is

$$G = \left(\begin{array}{c|c} W_1V & W_1P \\ \hline 0 & W_2 \\ \hline -V & -P \end{array} \right) \quad (74)$$

In our example W_2 is nonproper, which is typical for mixed sensitivity problems. As a result, G is nonproper too. It shows that in this example the standard system *never* is well-posed for admissible compensators.* This is the reason for not insisting on well-posedness in the standard system. The underlying control system will always be well-posed for admissible compensators if the filters are correctly chosen.

Though G often is nonproper, the problem may still be solved using state-space techniques. An obvious solution to this problem is to bring in an extra stable factor F^{-1} . If K_{tmp} is an admissible compensator for the standard system with plant

$$G_{\text{tmp}} = G \begin{pmatrix} I & 0 \\ 0 & F^{-1} \end{pmatrix} = \left(\begin{array}{c|c} W_1V & W_1PF^{-1} \\ \hline 0 & W_2F^{-1} \\ \hline -V & -PF^{-1} \end{array} \right) \quad (75)$$

then $K = F^{-1}K_{\text{tmp}}$ is an admissible compensator for the original problem, and vice versa (see Krause¹⁸). Often $F = W_2$ will do. Unfortunately there is one drawback to this method: the McMillan degree of the compensator is higher than necessary, unless factors cancel. Without cancellation (which, if at all possible, is numerically unattractive,) we have for suboptimal

*The transfer matrix from v_2 to z is $W_2(I+KP)^{-1}$, which behaves as W_2 for high frequencies since for admissible compensators $I+KP$ is biproper. See Figure 2.

compensators computed this way

$$\delta_M(K) = \delta_M(G_{\text{tmp}}) + \delta_M(F) = \delta_M(P) + \delta_M(W_1) + \delta_M(W_2) + \delta_M(F) \quad (76)$$

whereas for suboptimal compensators computed polynomially we have

$$\delta_M(K) = \delta_M(G) = \delta_M(P) + \delta_M(W_1) + \delta_M(W_2) \quad (77)$$

In (76) and (77) we use the assumption that V and P have the same denominators and that V is proper (see Kwakernaak¹⁹). The two equations (76) and (77) do not hold in general as there may be cancellation of common factors. If K is computed polynomially we have

$$\delta_M(K) \leq \delta_M(G) \leq \delta_M(P) + \delta_M(W_1) + \delta_M(W_2) \quad (78)$$

This may be seen as follows. With left coprime fractions $V = D^{-1}M$, $P = D^{-1}N$, $W_1 = B_1^{-1}A_1$ and $W_2 = B_2^{-1}A_2$ the open-loop system is determined by the differential equation

$$\begin{pmatrix} M & 0 & 0 & D & N \\ 0 & B_1 & 0 & A_1 & 0 \\ 0 & 0 & B_2 & 0 & -A_2 \end{pmatrix} \begin{pmatrix} w \\ z_1 \\ z_2 \\ y \\ u \end{pmatrix} = 0 \quad (79)$$

so

$$\delta_M(G) \leq \delta_c(M \ N \ D) + \delta_c(B_1 \ A_1) + \delta_c(B_2 \ A_2) = \delta_M(P) + \delta_M(W_1) + \delta_M(W_2) \quad (80)$$

Again we assume here that $V = D^{-1}M$ is proper. In Remark 5.3 we show that K may be chosen such that $\delta_M(K) \leq \delta_M(G)$. To make (76) plausible we mention only that if no cancellation takes place, then $\delta_M(F^{-1}K_{\text{tmp}}) = \delta_M(F^{-1}) + \delta_M(K) = \delta_M(G_{\text{tmp}}) + \delta_M(F)$, because F^{-1} and K_{tmp} are both proper. Furthermore, if no cancellation takes place, $\delta_M(G_{\text{tmp}}) \geq \delta_M(W_1 P F^{-1}) = \delta_M(W_1) + \delta_M(P) + \delta_M(F)$ because W_1 , P and F^{-1} are usually all proper.

The polynomial solution to the problem defined by (71)–(74) goes as follows. A left coprime fraction of G is

$$G = \left(\begin{array}{c|c} (s+1)/s & 1/s \\ 0 & c(1+rs) \\ \hline -(s+1)/s & -1/s \end{array} \right) \quad (81)$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & c(1+rs) \\ -(s+1) & -1 \end{pmatrix} = (D_1 \ D_2)^{-1} (N_1 \ N_2) \quad (82)$$

The matrix $(-N_1 \ D_1)$ is square and strictly Hurwitz, so a J -spectral cofactor Q is

$$Q = (-N_1 \ \lambda D_1) = \begin{pmatrix} 0 & \lambda & 0 \\ 0 & 0 & \lambda \\ s+1 & 0 & 0 \end{pmatrix} \quad (83)$$

The matrices Δ and Λ follow from a left-to-right conversion.

$$Q^{-1}(D_2 - N_2) = \begin{pmatrix} \frac{s}{s+1} & \frac{1}{s+1} \\ \frac{1}{\lambda} & 0 \\ 0 & -c \frac{1+rs}{\lambda} \end{pmatrix} \quad (84)$$

$$= \begin{pmatrix} \lambda & -\lambda s \\ 1 & -1-s \\ -c(1+rs) & 0 \end{pmatrix} \begin{pmatrix} \lambda & -\lambda(1+s) \\ \lambda & (s+1)\lambda \end{pmatrix}^{-1} = \Delta \Lambda^{-1} \quad (85)$$

The next step in the algorithm is the computation of Γ . The $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ -spectral factor Γ need satisfy

$$\Gamma^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma = \Delta^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Delta \quad (86)$$

$$= \begin{pmatrix} \lambda^2 - 1 - c^2(1 - r^2 s^2) & -(\lambda^2 - 1)s + 1 \\ (\lambda^2 - 1)s + 1 & -(\lambda^2 - 1)s^2 - 1 \end{pmatrix} \quad (87)$$

For simplicity we take $r = 0$, in which case $\det(\Delta^{-1} J \Delta) = c^2 - \lambda^2 + c^2(-1 + \lambda^2)s^2$. From this we see that a $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ -spectral factor Γ exists only if $\lambda > \max(1, c)$. The stable zero of (87) then is

$$\zeta_1 = \frac{1}{c} \sqrt{\frac{\lambda^2 - c^2}{\lambda^2 - 1}} \quad (88)$$

The extraction algorithm may be applied and the result is that for $\max(1, c) \leq \lambda \neq \sqrt{1 + c^2}$, a solution with the correct degree structure is

$$\Gamma = \begin{pmatrix} \sqrt{\lambda^2 - 1} & -\sqrt{\lambda^2 - 1} \left(s + \frac{1}{c} \sqrt{\frac{\lambda^2 - c^2}{\lambda^2 - 1}} - \frac{1}{c} \frac{\sqrt{(\lambda^2 - 1)(\lambda^2 - c^2) + c}}{\lambda^2 - 1 - c^2} \right) \\ c & \frac{\sqrt{(\lambda^2 - 1)(\lambda^2 - c^2) + c}}{\lambda^2 - 1 - c^2} \end{pmatrix} \quad (89)$$

The zero of

$$\begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} = \begin{pmatrix} \lambda & -\lambda s \\ c & \frac{\sqrt{(\lambda^2 - 1)(\lambda^2 - c^2) + c}}{\lambda^2 - 1 - c^2} \end{pmatrix} \quad (90)$$

lies in the left half plane iff $\lambda > \sqrt{1 + c^2}$ and, hence, $\lambda_{\text{opt}} = \sqrt{1 + c^2}$. For $\lambda > \lambda_{\text{opt}}$ all suboptimal compensators $K = \bar{Y} \bar{X}^{-1}$ are given by

$$\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \Lambda \Gamma^{-1} \begin{pmatrix} 1 \\ U \end{pmatrix} \quad (91)$$

with U stable and $\|U\|_{\infty} \leq 1$. The central compensator is

$$K = \frac{\sqrt{(\lambda^2 - 1)(\lambda^2 - c^2) + c} - c - c(1 + s)(\lambda^2 - 1 - c^2)}{\sqrt{(\lambda^2 - 1)(\lambda^2 - c^2) + c} + c + c(1 + s)(\lambda^2 - 1 - c^2)} \quad (92)$$

This central compensator satisfies $\delta_M(K) \leq \delta_M(G) = 1$, which holds in general as we show in Remark 5.3. Note that the central compensator is not unique. For $\lambda > \sqrt{1+c^2}$ another solution Γ with the correct degree structure is

$$\Gamma = \begin{pmatrix} \sqrt{\lambda^2 - 1 - c^2} & \frac{-(\lambda^2 - 1)s + 1}{\sqrt{\lambda^2 - 1 - c^2}} \\ 0 & \frac{\sqrt{c^2(\lambda^2 - 1)s + \sqrt{\lambda^2 - c^2}}}{\sqrt{\lambda^2 - 1 - c^2}} \end{pmatrix} \quad (93)$$

In this case the central compensator is $K = 1$, independent of λ and c . Exceptionally, in this case $\delta_M(K)$ is strictly less than $\delta_M(G)$.

Remark 5.2 (Stable common factors)

Algorithm 3.4 assumes that a left coprime fraction of $G = D^{-1}N$ is available. Possible stable common factors (strictly Hurwitz common factors) of (D, N) do not affect the stability tests and the existence of J -spectral factors in Algorithm 3.4, however. In other words, the algorithm works as long as $D^{-1}N = G$ is a polynomial fraction of G that has stable common factors only. This we can use to advantage in connection with the mixed sensitivity problem.

Consider the mixed sensitivity problem as in Figure 6, and let $W_1 = B_1^{-1}A_1$, $W_2 = B_2^{-1}A_2$ and $V = D^{-1}M$ be polynomial left coprime fractions of the various filters. Without loss of generality we assume M , B_1 and B_2 to be strictly Hurwitz (if, say, M is not strictly Hurwitz then replace M by a spectral cofactor M_{co} of $MM^* = M_{co}M_{co}^*$). With polynomial left coprime fractions $D^{-1}N = P$ and $X^{-1}Y = K$ of plant and compensator respectively, we get a polynomial fraction of the standard plant

$$G = (D_1 \ D_2)^{-1} \begin{pmatrix} N_1 & N_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & D \\ B_1 & 0 & A_1 \\ 0 & B_2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -M & -N \\ 0 & 0 \\ 0 & A_2 \end{pmatrix} \quad (94)$$

The factorization is not necessarily coprime but possible common factors are stable. In the terminology of Section 3.1 there are no unstable fixed closed-loop poles and the assignable closed-loop poles are the zeros of

$$\begin{pmatrix} D & N \\ -Y & X \end{pmatrix} \quad (95)$$

That is, K stabilizes the standard system iff it stabilizes the system as depicted in Figure 6, a fact we silently assumed in Example 5.1. Note that with fraction (94) the first few steps of Algorithm 3.4 may be performed symbolically: Step (b) of Algorithm 3.4 is always satisfied with

$$Q = \begin{pmatrix} -N_1 & \lambda D_1 \end{pmatrix} = \begin{pmatrix} M & \lambda B_1 \\ & \lambda B_2 \end{pmatrix} \quad (96)$$

and Δ and Λ in Step (c) of Algorithm 3.4 follow from

$$\Delta\Lambda^{-1} = Q^{-1}(D_2 \quad -N_2) = \begin{pmatrix} M^{-1}D & M^{-1}N \\ \frac{1}{\lambda} B_1^{-1}A_1 & 0 \\ 0 & -\frac{1}{\lambda} B_2^{-1}A_2 \end{pmatrix} = \begin{pmatrix} V^{-1} & V^{-1}P \\ \frac{1}{\lambda} W_1 & 0 \\ 0 & -\frac{1}{\lambda} W_2 \end{pmatrix} \quad (97) \quad \square$$

Remark 5.3 (McMillan degrees)

If the compensator $K = \bar{Y}\bar{X}^{-1}$ is given as in algorithm 3.4:

$$\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \Lambda\Gamma^{-1} \begin{pmatrix} I \\ U \end{pmatrix} \quad (98)$$

then for any constant U we have

$$\delta_M(K) \leq \delta_M(G) \quad (99)$$

because with a right coprime polynomial fraction BA^{-1} of

$$\Lambda\Gamma^{-1} \begin{pmatrix} I \\ U \end{pmatrix} = BA^{-1} \quad (100)$$

we may see that

$$\delta_M(K) \leq \delta_c(B) \leq \delta_M(BA^{-1}) = \delta_M\left(\Lambda\Gamma^{-1} \begin{pmatrix} I \\ U \end{pmatrix}\right) \leq \delta_M(\Lambda\Gamma^{-1}) \leq \delta_M(G') \leq \delta_M(G) \quad (101)$$

if U is constant. Hence, the McMillan degree of a central compensator (that is, a compensator for $U=0$) does not exceed the McMillan degree of the plant. $\square$

Remark 5.4 (Dual solution)

The present algorithm is based on a left coprime fraction of the plant G . A similar algorithm may be derived starting with a *right* coprime fraction of the plant. A simple proof uses the fact that K is admissible for the standard system with plant G iff K^T is admissible for the standard system with plant G^T .

Algorithm 5.5

Given: A right coprime polynomial matrix fraction description of the plant

$$G = \begin{pmatrix} \bar{N}_1 \\ \bar{N}_2 \end{pmatrix} \begin{pmatrix} \bar{D}_1 \\ \bar{D}_2 \end{pmatrix}^{-1} \quad (102)$$

Assumptions:

$$\begin{pmatrix} -\bar{N}_1 \\ \bar{D}_1 \end{pmatrix}, \quad \begin{pmatrix} \bar{D}_2 \\ -\bar{N}_2 \end{pmatrix} \quad (103)$$

have full column rank and full row rank on $\mathbb{C}_0$, respectively.

(a) Choose $\lambda \in \mathbb{R}$.

(b) Compute, if possible, a J -spectral factor $\bar{Q}$ such that

$$\bar{Q}^* \begin{pmatrix} I_{[u]} & 0 \\ 0 & -I_{[w]} \end{pmatrix} \bar{Q} = (-\bar{N}_1^* \quad \bar{D}_1^*) \begin{pmatrix} I_{[z]} & 0 \\ 0 & -\lambda^2 I_{[w]} \end{pmatrix} \begin{pmatrix} -\bar{N}_1 \\ \bar{D}_1 \end{pmatrix} \quad (104)$$

with

$$\begin{pmatrix} -\bar{N}_1 \\ \bar{D}_1 \end{pmatrix} \bar{Q}^{-1} \quad (105)$$

proper. If this solution $\bar{Q}$ exists and

$$\begin{pmatrix} \bar{Q}_1 \\ \bar{D}_1 \end{pmatrix} \quad (106)$$

is strictly Hurwitz, with $\bar{Q}_1$ the top $[u]$ rows of $\bar{Q}$, then proceed to (c). Otherwise, no admissible compensator exists; λ need be increased and (b) repeated.

(c) Find left coprime polynomial matrices $\bar{\Delta}$ and $\bar{\Lambda}$ such that

$$\bar{\Lambda}^{-1} \bar{\Delta} = \begin{pmatrix} \bar{D}_2 \\ -\bar{N}_2 \end{pmatrix} \bar{Q}^{-1} \quad (107)$$

(d) Compute, if possible, a J -spectral cofactor $\bar{\Gamma}$ such that

$$\bar{\Gamma} \begin{pmatrix} I_{[u]} & 0 \\ 0 & -I_{[y]} \end{pmatrix} \bar{\Gamma}^* = \bar{\Delta} \begin{pmatrix} I_{[u]} & 0 \\ 0 & -I_{[w]} \end{pmatrix} \bar{\Delta}^* \quad (108)$$

with $\Delta \Gamma^{-1}$ proper. If this solution exists and $(\bar{\Delta}_1 \quad \bar{\Gamma}_2)$ is strictly Hurwitz, with $\bar{\Gamma}_2$ the right $[y]$ columns of $\bar{\Gamma}$ and $\bar{\Delta}_1$ the left $[u]$ columns of $\bar{\Delta}$, then proceed to (e). Otherwise, no admissible compensator exists; λ need be increased and (b)–(d) repeated.

(e) There exist stabilizing compensators such that $\|H\|_\infty < \lambda$. All compensators $K = X^{-1}Y$ that stabilize and make $\|H\|_\infty \leq \lambda$ are generated by

$$(X \quad Y) = (I \quad \bar{U}) \bar{\Gamma}^{-1} \bar{\Lambda}, \quad \bar{U} \text{ stable and } \|\bar{U}\|_\infty \leq 1 \quad (109) \quad \square$$

We refer to the above solution as the *dual* solution and to the solution given in Section 3 as the *primal* solution. There is a connection between the primal and dual solution. To make things more compact we define two rational matrices.

$$\Pi = \begin{pmatrix} D_2^* \\ -N_2^* \end{pmatrix} (N_1 N_1^* - \lambda^2 D_1 D_1^*)^{-1} \begin{pmatrix} D_2 & -N_2 \end{pmatrix} \quad (110)$$

$$\bar{\Pi} = \begin{pmatrix} \bar{D}_2^* \\ -\bar{N}_2^* \end{pmatrix} (\bar{N}_1 \bar{N}_1^* - \lambda^2 \bar{D}_1 \bar{D}_1^*)^{-1} \begin{pmatrix} \bar{D}_2 & -\bar{N}_2 \end{pmatrix} \quad (111)$$

Note that Π and $\bar{\Pi}$ have constant inertia on the imaginary axis. $\Gamma \Lambda^{-1}$ is a rational J -spectral factor of Π and $\bar{\Lambda}^{-1} \bar{\Gamma}$ a rational J -spectral cofactor of $\bar{\Pi}$. The connection between the primal and dual solution is easy to formulate in terms of Π and $\bar{\Pi}$:

$$\Pi^{-1} = - \begin{pmatrix} 0 & I_{[y]} \\ -I_{[u]} & 0 \end{pmatrix} \bar{\Pi} \begin{pmatrix} 0 & -I_{[u]} \\ I_{[y]} & 0 \end{pmatrix} \quad (112)$$

In Reference 27 this formula was the starting point for the development of an algorithm that combines the dual and primal solution procedure. At the moment this has not yet led to an attractive algorithm. $\square$

6. CONCLUDING REMARKS

In this paper we have shown how the standard H_∞ -suboptimal control problem may be solved polynomially, based on what may be called J -lossless theory. A mixed sensitivity problem was solved, showing the usefulness of polynomial methods.

A symmetric factor extraction algorithm for polynomial J -spectral factorization was formulated. The symmetric factor extraction algorithm has the important feature that it is conceptually simple and that it clearly shows the degree structure. Other algorithms, like Riccati-based algorithms and algorithms using 'Jordan chains' (see Reference 11), have yet to be investigated, tested and compared.

In Reference 25 the solution procedure as presented in this paper is extended to the optimal case.

7. APPENDIX

Proof 7.1 (Lemma 3.2)

We split the proof into two parts.

Lemma 7.2

With $G = (D_1 \ D_2)^{-1}(N_1 \ N_2)$ and $K = \bar{Y}\bar{X}^{-1}$ a left and right polynomial matrix fraction of plant and compensator, the closed-loop transfer matrix H satisfies $H = R^{-1}P$ with P and R defined as

$$(P \ R) = T(-N_1 \ D_1) \quad (113)$$

Here, T is a left prime polynomial matrix of maximal rank such that $T(D_2\bar{X} - N_2\bar{Y}) = 0$. Moreover, zeros of R are closed-loop poles and with P' and R' defined as

$$(P' \ R') = TQ \quad (114)$$

H' satisfies $H' = -R'^{-1}P'$. □

Proof 7.3

The signals in the closed-loop system with plant G satisfy the differential equations

$$\begin{pmatrix} -N_1 & D_1 & D_2 & -N_2 \\ 0 & 0 & -Y & X \end{pmatrix} \begin{pmatrix} w \\ z \\ y \\ u \end{pmatrix} = 0 \quad (115)$$

From this expression y and u may be eliminated to yield the closed-loop transfer matrix H . With $K = \bar{Y}\bar{X}^{-1}$ a polynomial right coprime fraction of the compensator, we have

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} I \quad (116)$$

and, hence, the closed-loop system is equivalently described by

$$(D_1 \ D_2\bar{X} - N_2\bar{Y}) \begin{pmatrix} z \\ I \end{pmatrix} = N_1 w \quad (117)$$

in combination with (116). Note that the zeros of the square matrix $(D_1 \ D_2\bar{X} - N_2\bar{Y})$ are the closed-loop poles because there exist $\bar{A}$ and $\bar{B}$ such that

$$\begin{pmatrix} D_1 & D_2 & -N_2 \\ 0 & -Y & X \end{pmatrix} \underbrace{\begin{pmatrix} I & 0 & 0 \\ 0 & \bar{X} & \bar{A} \\ 0 & \bar{Y} & \bar{B} \end{pmatrix}}_{\text{unimodular}} = \begin{pmatrix} D_1 & D_2\bar{X} - N_2\bar{Y} & ? \\ 0 & 0 & I \end{pmatrix} \quad (118)$$

Here, '?' denotes some polynomial matrix we do not bother about. Next, define a left prime polynomial matrix T of maximal row rank such that

$$T(D_2\bar{X} - N_2\bar{Y}) = 0 \quad (119)$$

Complete T to a unimodular matrix:

$$W = \begin{pmatrix} T \\ V \end{pmatrix}$$

Multiplying (117) from the left by W yields

$$\begin{pmatrix} TD_1 & 0 \\ VD_1 & ? \end{pmatrix} \begin{pmatrix} z \\ l \end{pmatrix} = \begin{pmatrix} TN_1 \\ ? \end{pmatrix} w \quad (120)$$

The external closed-loop behaviour therefore is determined by

$$T(-N_1 \ D_1) \begin{pmatrix} w \\ z \end{pmatrix} = 0 \quad (121)$$

Note that the zeros of TD_1 are part of the closed-loop poles. In particular, TD_1 is nonsingular and, hence, $H = (TD_1)^{-1}(TN_1)$.

The same matrix T may also be used to eliminate y and u in the standard system with plant G' , giving in a similar way the closed-loop behaviour

$$T(Q_1 \ Q_2) \begin{pmatrix} w' \\ z' \end{pmatrix} = 0 \quad (122)$$

If TQ_2 is nonsingular (and it is if K is admissible as we see later on) then $H' = -(TQ_2)^{-1}(TQ_1)$. If we now define $P' = TQ_1$ and $R' = TQ_2$ then $H' = -R'^{-1}P'$, and with R and P defined as

$$(P \ R) = (P' \ R')Q^{-1}(-N_1 \ D_1) \quad (123)$$

we have $H = R^{-1}P$ and a relation between H and H' depending only on $(-N_1 \ D_1)$. $\square$

We are now in a position to prove the first part of the lemma. By Theorem 2.3 we know that there exists a Q such that $Q^{-1}(-N_1 \ D_1)$ is proper if the problem has a solution with $\|H\|_\infty < 1$. Define M as

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = Q^{-1}(-N_1 \ D_1), \text{ with } M_{22} \text{ square.} \quad (124)$$

Obviously M satisfies

$$M \begin{pmatrix} I_{[w]} & 0 \\ 0 & -I_{[z]} \end{pmatrix} M^\sim = \begin{pmatrix} I_{[y]} & 0 \\ 0 & -I_{[z]} \end{pmatrix} \quad (125)$$

from which immediately follows using (123) that $RR^\sim - PP^\sim = R'R'^\sim - P'P'^\sim$ and, hence, that $\|R^{-1}P\|_\infty \leq 1 \Leftrightarrow \|R'^{-1}P'\|_\infty \leq 1$. Equation (125) implies that

$$M^\sim \begin{pmatrix} I_{[y]} & 0 \\ 0 & -I_{[z]} \end{pmatrix} M \leq \begin{pmatrix} I_{[w]} & 0 \\ 0 & -I_{[z]} \end{pmatrix} \quad (126)$$

If we now examine the lower right block of (126) we see that

$$M_{12}^\sim M_{12} - M_{22}^\sim M_{22} \leq -I_{[z]} \quad (127)$$

which implies nonsingularity of M_{22} . Properness of both M_{12} and M_{22} implies in addition

$$\|M_{22}M_{22}^{-1}\|_\infty < 1 \quad (128)$$

Now R may be written as

$$R = (P'M_{12} + R'M_{22}) = R'(H'M_{12}M_{22}^{-1} + I)M_{22} \quad (129)$$

From a small gain argument it follows that R does not have zeros in $\mathbb{C}_+ \cup \mathbb{C}_0$ iff $R'M_{22}$ does not have (see Vidyasagar,³³ pp. 274–275), which in turn implies that both R' and M_{22} are not allowed to have zeros in $\mathbb{C}_+ \cup \mathbb{C}_0$, because both R' and M_{22} are stable.

Observe that

$$Q^{-1}(Q_1 \ D_1) = \begin{pmatrix} I_{[y]} & M_{12} \\ 0 & M_{22} \end{pmatrix} \quad (130)$$

So $(Q_1 \ D_1)$ is nonsingular and strictly Hurwitz iff M_{22} has all its zeros in $\mathbb{C}_-$. Obviously $EE^* = I$ and, therefore, E is co-inner iff $(Q_1 \ D_1)$ is strictly Hurwitz. Because M is proper and has full column rank it follows that $\|H\|_\infty < 1$ iff $\|H'\|_\infty < 1$.

The proof of the first part is complete if we can prove that for compensators that achieve $\|H\|_\infty \leq 1$, we have that they stabilize G' iff they stabilize G . First note that the signals in the standard system with plant G' with extra disturbances v_1 and v_2 similar as in Figure 2, satisfy the differential equation

$$\begin{pmatrix} Q_2 & D_2 & -N_2 \\ 0 & -Y & X \end{pmatrix} \begin{pmatrix} z' \\ y \\ u \end{pmatrix} = \begin{pmatrix} Q_1 & D_2 & N_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v' \\ v_1 \\ v_2 \end{pmatrix} \quad (131)$$

Because Q is strictly Hurwitz, the above two matrices do not have unstable common factors, so, all unstable zeros of

$$\begin{pmatrix} Q_2 & D_2 & -N_2 \\ 0 & -Y & X \end{pmatrix} \quad (132)$$

are closed-loop poles and, hence, the closed-loop system is stable iff (132), or equivalently,

$$(Q_2 \ D_2 \bar{X} - N_2 \bar{Y}) \quad (133)$$

is strictly Hurwitz. Now examine

$$\begin{pmatrix} P' & R' \\ I & 0 \end{pmatrix} Q^{-1} (Q_2 \ D_2 \bar{X} - N_2 \bar{Y}) = \begin{pmatrix} R' & 0 \\ ? & \bar{A} \end{pmatrix} \quad (134)$$

$$\begin{pmatrix} P' & R' \\ I & 0 \end{pmatrix} Q^{-1} (D_1 \ D_2 \bar{X} - N_2 \bar{Y}) = \begin{pmatrix} R' & 0 \\ ? & \bar{A} \end{pmatrix} \quad (135)$$

This defines $\bar{A}$. In the above equations '?' represents matrices of which we only need to know that they are stable. Since

$$\begin{pmatrix} P' & R' \\ I & 0 \end{pmatrix} Q^{-1} \quad (136)$$

and its inverse are stable, we see from (134) and (135) that K stabilizes G with $\|H\|_\infty \leq 1$ iff it stabilizes G' with $\|H'\|_\infty \leq 1$. This completes the proof of the first part of Lemma 3.2. $\square$

Proof 7.4 (Second part of Lemma 3.2)

If $Q^{-1}(-N_1 \ D_1)$ is proper and has full column rank at infinity, then $\rho_i(UQ) = \rho_i(U(-N_1 \ D_1))$ for every polynomial matrix U and, hence, minimizing

$$\rho(U(-N_1 \ D_1 \ D_2 \ -N_2)) \quad (137)$$

over all unimodular U is the same as minimizing

$$\rho(U(Q \ D_2 \ -N_2)) \quad (138)$$

over all unimodular U , i.e.,

$$\delta_t(-N_1 \ D_1 \ D_2 \ -N_2) = \delta_t(Q \ D_2 \ -N_2) \quad (139)$$

This concludes the proof of Lemma 3.2. $\square$

Proof 7.5 (Lemma 4.6)

By induction. First consider the case where all factors T_i are first order. Obviously for $i=0$ we have that

$$PT_1^{-1} \dots T_i^{-1} \begin{pmatrix} s^{-d_1} & & \\ & \ddots & \\ & & s^{-d_n} \end{pmatrix} \quad (140)$$

is proper and has full column rank at infinity ($d_j = \gamma_j(P)$). Suppose without loss of generality that at

the l th extraction step the pivot index is $k = 1$. After performing step (e) ($d_k := d_k - 1$), we have

$$\begin{aligned} PT_1^{-1} \dots T_l^{-1} \begin{pmatrix} s^{-d_1} & & \\ & \ddots & \\ & & s^{-d_n} \end{pmatrix} \\ = PT_1^{-1} \dots T_{l-1}^{-1} \begin{pmatrix} s^{-(d_1+1)} & & \\ & \ddots & \\ & & s^{-d_n} \end{pmatrix} \begin{pmatrix} s^{d_1+1} & & \\ & \ddots & \\ & & s^{d_n} \end{pmatrix} T_l^{-1} \begin{pmatrix} s^{-d_1} & & \\ & \ddots & \\ & & s^{-d_n} \end{pmatrix} \end{aligned} \quad (141)$$

By induction hypothesis, (141) is proper and has full column rank at infinity because

$$\begin{pmatrix} s^{d_1+1} & & \\ & \ddots & \\ & & s^{d_n} \end{pmatrix} T_l^{-1} \begin{pmatrix} s^{-d_1} & & \\ & \ddots & \\ & & s^{-d_n} \end{pmatrix} = \begin{pmatrix} \frac{s}{s - \xi_l} & & \\ * & & \\ * & 1 & \\ * & & \ddots & \\ * & & & 1 \end{pmatrix} \quad (142)$$

is biproper, as the j th element ($j > 1$) of its first column is

$$\frac{e_j s^{d_j - d_1}}{e_1 s - \xi_l} \quad (143)$$

which is either zero ($e_j = 0$) or nonzero proper (if $e_j \neq 0$, we know that $d_j - d_k \leq 1$ because $k = 1$ is an element of the maximal active index set).

This proves the result in case all extracted factors are first order. The same line of reasoning may be used to show that the result is also valid in the case where quadratic factors of the form, as explained in Section 4, are extracted. $\square$

Proof 7.6 (Lemma 4.7)

First we show that J -spectral factors Γ of P^*JP such that $P\Gamma^{-1}$ is proper, are unique up to multiplication from the left by a constant J -unitary matrix. If Γ and $\bar{\Gamma}$ are two such J -spectral factors, then $U = \Gamma\bar{\Gamma}^{-1}$ is proper and by Lemma 4.2, U is polynomial and J -unitary, i.e., U is a constant J -unitary matrix.

($\Leftarrow$) Suppose $d_j = 0$ for all $j \in \underline{m}$. Then by Lemma 4.6

$$A_{n+1} = (T_n^{-1} \dots (T_1^{-1})^{-1} P^* J P T_1^{-1} \dots T_n^{-1}) \quad (144)$$

is proper and, hence, being polynomial, it is constant. Write A_{n+1} as $W^* J W = A_{n+1}$ with W constant. $\Gamma = WT_n \dots T_1$ is then a J -spectral factor of A . By Lemma 4.6, $P\Gamma^{-1}$ is proper.

($\Rightarrow$) Suppose there exists a J -spectral factor $\bar{\Gamma}$ of P^*JP such that $P\bar{\Gamma}^{-1}$ is proper. Using, Lemma 4.2 we know that

$$\bar{\Gamma} = UWT_n \dots T_1 \quad (145)$$

where W and T_i are given by the algorithm and U is some J -unitary polynomial matrix. Since (without loss of generality) P is assumed to be column reduced, we have after the l th extraction step that

$$\bar{\Gamma} T_1^{-1} \dots T_l^{-1} \begin{pmatrix} s^{-d_1} & & \\ & \ddots & \\ & & s^{-d_n} \end{pmatrix} \quad (146)$$

is proper and has full column rank at infinity. Suppose one of the virtual column degrees for the first time drops below zero, at, say, the l plus first time a root is extracted, that is, suppose $A_{l+1}(\xi_{l+1})e = 0$ for some vector e and that $d_j = 0$ for all j such that $e_j \neq 0$. This implies that

$$\bar{\Gamma} T_1^{-1} \dots T_l^{-1} e = \bar{\Gamma} T_1^{-1} \dots T_l^{-1} \begin{pmatrix} s^{-d_1} & & \\ & \ddots & \\ & & s^{-d_n} \end{pmatrix} e \quad (147)$$

is proper. Equation (147) is equal to $UWT_n \dots T_{l+1}e$ and, hence, being polynomial, it must be constant. Because $T_{l+1}(\xi_{l+1})e = 0$, (147) must in fact be zero. This leads to a contradiction because $UWT_n \dots T_1$ is strictly Hurwitz. Hence, on exit of the algorithm all virtual column degrees are nonnegative. If some

$d_j > 0$, then $\delta_c(\bar{\Gamma}) = \deg \det(\bar{\Gamma}) = n = \delta_c(P) - \sum d_j < \delta_c(P)$, which implies that $P\bar{\Gamma}^{-1}$ is nonproper. Therefore all virtual column degrees must be zero on exit of the algorithm. $\square$

Proof 7.7 (Remark 5.4, equation (112))

Given are a left and right coprime fraction of G :

$$G = (D_1 \ D_2)^{-1} (N_1 \ N_2) = \begin{pmatrix} \bar{N}_1 \\ \bar{N}_2 \end{pmatrix} \begin{pmatrix} \bar{D}_1 \\ \bar{D}_2 \end{pmatrix}^{-1} \quad (148)$$

Define polynomial matrices $A, \bar{A}$ and $B, \bar{B}$ as

$$\begin{aligned} A &= (-N_1 \ D_1), & B &= (D_2 \ -N_2) \\ \bar{A} &= \begin{pmatrix} -\bar{N}_1 \\ \bar{D}_1 \end{pmatrix}, & \bar{B} &= \begin{pmatrix} \bar{D}_2 \\ -\bar{N}_2 \end{pmatrix} \end{aligned} \quad (149)$$

And for convenience of notation define

$$J_{zw} = \begin{pmatrix} I_{[z]} & 0 \\ 0 & -I_{[w]} \end{pmatrix}, \quad J_{wz} = \begin{pmatrix} I_{[w]} & 0 \\ 0 & -I_{[z]} \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & I_w \\ -I_z & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & I_y \\ -I_u & 0 \end{pmatrix} \quad (150)$$

With these definitions the connection between the left and right fraction of G may be expressed as

$$AL_1\bar{A} = BL_2\bar{B} \quad (151)$$

Furthermore, we have for Π and $\bar{\Pi}$ in this notation

$$\Pi = \bar{B}^{\sim} (A J_{wz} A^{\sim})^{-1} B \quad \text{and} \quad \bar{\Pi} = \bar{B} (\bar{A}^{\sim} J_{zw} \bar{A})^{-1} \bar{B}^{\sim} \quad (152)$$

We prove that

$$\bar{\Pi} L_2^{\sim} \Pi L_2 \bar{\Pi} = -\bar{\Pi} \quad (153)$$

which is equivalent to equation (112).

$$\bar{\Pi} L_2^{\sim} \Pi L_2 \bar{\Pi} = \bar{B} (\bar{A}^{\sim} J_{zw} \bar{A})^{-1} \bar{B}^{\sim} L_2^{\sim} \bar{B}^{\sim} (A J_{wz} A^{\sim})^{-1} B L_2 \bar{B} (\bar{A}^{\sim} J_{zw} \bar{A})^{-1} \bar{B}^{\sim} \quad (154)$$

$$= \underbrace{\bar{B} (\bar{A}^{\sim} J_{zw} \bar{A})^{-1} \bar{A}^{\sim}}_C \underbrace{L_1^{\sim} A^{\sim} (A J_{wz} A^{\sim})^{-1} A L_1}_{E} \underbrace{\bar{A} (\bar{A}^{\sim} J_{zw} \bar{A})^{-1} \bar{B}^{\sim}}_{C^{\sim}} \quad (155)$$

This defines C and E . Note that $L_1^{\sim} J_{wz} L_1 = -J_{zw}$, and therefore,

$$\bar{\Pi} L_2^{\sim} \Pi L_2 \bar{\Pi} = C J_{wz} C^{\sim} + C(E - J_{wz})C^{\sim} = -\bar{\Pi} + C(E - J_{wz})C^{\sim} \quad (156)$$

Next we show that $C(E - J_{wz}) = 0$, which then completes the proof. Fix s and introduce two subspaces $V = \text{Im}(J_{wz} L_1 \bar{A})$ and $F = \text{Ker}(\bar{A}^{\sim} L_1^{\sim})$. The subspaces V and F are complementary for almost all s because $(\bar{A}^{\sim} L_1^{\sim})(J_{wz} L_1 \bar{A}) = -\bar{A}^{\sim} J_{zw} \bar{A}$ is nonsingular for almost all s . Let π_V be the projection along F onto V and define π_F similarly. Restricted to F we have

$$C|_F \pi_F (E - J_{wz}) = 0 \quad (157)$$

This is immediate because $C|_F = 0|_F$. Restricted to V we have

$$C|_V \pi_V (E - J_{wz}) = 0 \quad (158)$$

because $A J_{wz} (E - J_{wz}) = 0$ and $C|_V = -L_2^{-1} X A J_{wz}|_V$, with X a left inverse of B . That $C|_V = -L_2^{-1} X A J_{wz}|_V$ follows from

$$C(J_{wz} L_1 \bar{A}) = \bar{B} (\bar{A}^{\sim} J_{zw} \bar{A})^{-1} \bar{A}^{\sim} L_1^{\sim} (J_{wz} L_1 \bar{A}) = -\bar{B} \quad (159)$$

and

$$-L_2^{-1} X A J_{wz} (J_{wz} L_1 \bar{A}) = -L_2^{-1} X (A L_1 \bar{A}) = -L_2^{-1} X (B L_2 \bar{B}) = -\bar{B} \quad (160)$$

A left inverse X of B exists because $B = (D_2 \ -N_2)$ is tall and has full column rank by assumption. Summarizing we have that $C(E - J_{wz}) = 0$ for almost all $s \in \mathbb{C}_0$ and, hence, $C(E - J_{wz}) = 0$ as a rational matrix. $\square$

REFERENCES

1. Ball, J., I. Gohberg, and L. Rodman, 'Interpolation of Rational Matrix Functions', *Operator Theory: Adv. Appl.*, Vol. 45, Birkhäuser Verlag, Basle, 1990.
2. Ball, J., and N. Cohen, 'Sensitivity minimization in an $\mathcal{H}^\infty$ norm', *Int. J. Control*, **46**, 785–816 (1987).
3. Ball, J., and J. W. Helton, 'A Beurling–Lax theorem for the Lie group $U(m, n)$ which contains most classical interpolation theory', *J. Operator Theory*, **9**, 107–142 (1983).
4. Ball, J., and J. W. Helton, 'Shift invariant subspaces, passivity, reproducing kernels and $\mathcal{H}^\infty$ -optimization', *Operator Theory: Adv. Appl.*, Vol. 35, Birkhäuser, Verlag, Basle, 1988.
5. Callier, F. M., 'On polynomial matrix spectral factorization by symmetric extraction', *IEEE Trans. Automatic Control*, **AC-30**, 453–464 (1985).
6. Callier, F. M., and C. A. Desoer, *Multivariable Feedback Systems*, Springer-Verlag, New York, 1982.
7. Doyle, J. C., K. Glover, P. P. Khargonekar, and B. A. Francis, 'State space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems', *IEEE Trans. Automatic Control*, **AC-34**, 831–847 (1989).
8. Francis, B. A., *A Course in $\mathcal{H}_\infty$ Control Theory*, Springer Lecture Notes in Control and Information Sciences, Vol. 88, Springer-Verlag, Heidelberg, 1987.
9. Glover, K., and J. C. Doyle, 'A state space approach to $\mathcal{H}_\infty$ optimal control', in H. Nijmeijer and J. M. Schumacher (Eds), *Three Decades of Mathematical System Theory*, Lecture Notes in Control and Information Sciences, Vol. 135, Springer-Verlag, Heidelberg, 1989.
10. Glover, K., D. J. N. Limebeer, J. C. Doyle, E. M. Kasenally, and M. G. Safonov, 'A characterization of all solutions to the four-block general distance problem', *SIAM J. Control and Opt.*, **29**, 283–324 (1991).
11. Gohberg, I., P. Lancaster, and L. Rodman, *Matrix Polynomials*, Academic Press, New York, 1982.
12. Green, M., K. Glover, D. Limebeer, and J. C. Doyle, 'A J -spectral factorization approach to $\mathcal{H}_\infty$ control', *SIAM J. Control and Opt.*, **28**, 1350–1371 (1990).
13. Green, M., ' $\mathcal{H}_\infty$ controller synthesis by J -lossless coprime factorization', *SIAM J. Control and Opt.*, **30**, 522–547 (1992).
14. Helton, J. W., 'Operator theory, analytic functions matrices, and electrical engineering', Conference Board of the Mathematical Sciences, Regional Conference Series in Mathematics number 68, American Society, 1987.
15. Jakubovič, V. A., 'Factorization of symmetric matrix polynomials', *Dokl. Akad. Nauk SSSR*, **11**, 1261–1264 (1970).
16. Kailath, T., *Linear Systems*, Prentice Hall, Englewood Cliffs, NJ, 1980.
17. Kimura, H., Y. Lu, and R. Kawatani, 'On the structure of $\mathcal{H}^\infty$ control systems and related extensions', *IEEE Trans. Aut. Control*, **AC-36**, 653–667 (1991).
18. Krause, J. M., 'Comments on Grimble's comments on Stein's comments on rolloff of $\mathcal{H}^\infty$ optimal controllers', *IEEE Trans. Automatic Control*, **AC-37**, 702–702 (1992).
19. Kwakernaak, H., 'The polynomial approach to $\mathcal{H}_\infty$ -optimal regulation', in E. Mosca and L. Pandolfi (Eds), *$\mathcal{H}_\infty$ Control Theory*, Lecture Notes in Mathematics, Vol. 149, Springer-Verlag, Berlin, 1991.
20. Kwakernaak, H., 'MATLAB macros for polynomial $\mathcal{H}_\infty$ control system optimization', Memorandum 881, Department of Applied Mathematics, University of Twente, The Netherlands, 1990.
21. Kwakernaak, H., 'Polynomial solution of the one-block Nehari problem', in preparation, Department of Applied Mathematics, University of Twente, The Netherlands, 1991.
22. McFarlane, D. C., and K. Glover, *Robust Controller Design using Normalized Coprime Factor Descriptions*, Lecture Notes in Control and Information Sciences, Vol. 138, Springer-Verlag, Berlin, 1990.
23. Meinsma, G., 'Polynomial methods in $\mathcal{H}_\infty$ -optimal control', Memorandum 1008, Department of Applied Mathematics, University of Twente, The Netherlands, 1991.
24. Meinsma, G., 'Strict passivity in the frequency domain', preprint Second IFAC Workshop on System Structure and Control, Prague, 1992.
25. Meinsma, G., 'Optimal solutions to a two-block problem', *Int. J. Robust Nonlin. Control*, to appear.
26. Meinsma, G., and M. Green, 'On strict passivity and its application to interpolation and $\mathcal{H}_\infty$ control', *Proc. 31st CDC*, Tucson, Arizona, 1992, pp. 2802–2807.
27. Meinsma, G., and H. Kwakernaak, ' $\mathcal{H}_\infty$ -optimal control and behaviors', *Proc ECC-1991*, Grenoble, 1991.
28. Redheffer, R. M., 'On a certain linear fractional transformation', *J. Math. and Physics*, **39**, 269–286 (1960).
29. Rosenbrock, H. H., *State-Space and Multivariable Systems*, Wiley, New York, 1970.
30. Šebek, M., 'An algorithm for spectral factorization of polynomial matrices with any signature', Memorandum 912, Department of Applied Mathematics, University of Twente, The Netherlands, 1990.
31. Tsai, M. C., and I. Postlethwaite, 'On J -lossless coprime factorizations and $\mathcal{H}^\infty$ control', *Int. J. of Robust and Nonlinear Control*, **1**, 46–68 (1991).
32. Vardulakis, A. I. G., *Linear Multivariable Control*, Wiley, Chichester, 1991.
33. Vidyasagar, M., *Control System Synthesis: A Factorization Approach*, MIT Press, Cambridge, MA, 1986.
34. Willems, J. C., 'Paradigms and puzzles in the theory of dynamical systems', *IEEE Trans. Automatic Control*, **AC-36**, 259–294 (1991).