ON A PATH INTEGRAL WITH A TOPOLOGICAL CONSTRAINT

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We discuss a new method to evaluate a path integral with a topological constraint involving a point singularity in a plane. The path integration is performed explicitly in the universal covering space. Our method is an alternative to an earlier method of Inomata.

In various problems in theoretical physics one has to evaluate a path integral with a constraint which is of a topological nature. This arises, for example, in the theory of polymers or in the analysis of the Aharonov-Bohm effect (see ref. [1] and the literature quoted therein).

In order to elucidate some of these problems Inomata and Singh [2] considered path integrals over trajectories in a plane (with polar coordinates r and θ , $0 < r < \infty$, $0 < \theta < 2\pi$) where the constraint could be expressed as

$$\int_0^T \dot{\theta} dt - \phi = 0, \quad (1)$$

where ϕ is a real constant. The trajectories are followed during the time interval $(0, T)$. They introduce the constraint through a Dirac δ -function with the left-hand side of (1) as the argument. Next the δ -function is written as a Fourier integral which enables them to write the propagator

$$K_\phi(r'', T|r', 0) \equiv K_\phi(r'', \theta'', T|r', \theta', 0)$$

for a quantum mechanical free particle of mass m obeying the constraint (1) as

$$K_\phi(r'', T|r', 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda e^{i\lambda\phi} K_\lambda(r'', T|r', 0), \quad (2)$$

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where

$$K_\lambda(\mathbf{r}'', T|\mathbf{r}', 0) = \int \exp[(i/\hbar)S_\lambda(\mathbf{r}(t))] d[\mathbf{r}(t)] \quad (3)$$

denotes the propagator associated with the action S_λ given by

$$S_\lambda = \frac{1}{2}m \int_0^T \dot{\mathbf{r}}^2 dt - \lambda \hbar \int_0^T \dot{\theta} dt. \quad (4)$$

Note that (3) represents an unconstrained path integral, the effect of the constraint being taken care of through the introduction of an extra term in the action (4) namely

$$\lambda \hbar \int_0^T \dot{\theta} dt.$$

In refs. [2] and [3] the essential step in the evaluation of K_λ and hence K is the observation that for an infinitesimal time interval ϵ the change $\Delta\theta$ in the angular coordinates can be expressed as a cosine term

$$-a\epsilon \Delta\theta \approx \cos(\Delta\theta + a\epsilon) - \cos(\Delta\theta) + \frac{1}{2}a^2\epsilon^2 + O(\epsilon^3). \quad (5)$$

This enables one to take all contributions up to order ϵ in the infinitesimal propagator into account. Once the terms involving $\Delta\theta$ in the infinitesimal propagator are modified using (5) the integration over intermediate angular coordinates from 0 to 2π in the polygonal scheme finally enables the authors of refs. [2] and [3] to obtain the exact propagator associated with the problem.

The present note is devoted to an alternative way to calculate such path integrals which might be easier to generalize to more complicated geometries and constraints. We first note that the introduction of a singularity at the origin makes the space multiple connected. The various trajectories connecting $\mathbf{r}'$ and $\mathbf{r}''$ can be classified into several distinct homotopy classes each characterized by the number of turns (winding number) around the singular point. The constrained propagator (2), therefore, can be expressed as a sum

$$K = \sum_{n=-\infty}^{\infty} K_\phi^{(n)}(\mathbf{r}'', T|\mathbf{r}', 0), \quad (6)$$

where

$$K_\phi^{(n)}(\mathbf{r}'', T|\mathbf{r}', 0) = K_\phi^{(n)}(r'', \theta'', T|r', \theta', 0)$$

represents the path integral over the set of trajectories characterized by the winding number n . Secondly, one performs the path integration by going over to the universal covering space which essentially means that one lets the angular coordinate θ take all values between $-\infty$ and $+\infty$. Further, since we want to evaluate K^n , we must demand

$$\int_0^T \dot{\theta} dt = \theta'' - \theta' + 2\pi n. \quad (7)$$

With these observations we proceed to evaluate the propagator for a free particle obeying the constraint (1).

It is well known [4,5] that while performing the path integration in two-dimensional polar coordinates one must add an extra

$$\int_0^T \frac{\hbar^2}{8m} \frac{1}{r^2(t)} dt$$

to the action functional. This term arises essentially from the scalar curvature of the space. Thus the effective action functional for a free particle in two-dimensional polar coordinates can be written as

$$S = \int_0^T \left[\frac{1}{2} m \left(\frac{dr}{dt} \right)^2 + \frac{\hbar^2}{8m} \frac{1}{r^2} + \frac{1}{2} m r^2 \left(\frac{d\theta}{dt} \right)^2 \right] dt, \quad (8)$$

and consequently the propagator $K_\phi^{(n)}$ can be expressed as

$$K_\phi^{(n)} = \delta(\phi - \theta'' + \theta' - 2\pi n) \int \exp \left(\frac{i}{\hbar} S \right) d[r(t)], \quad (9)$$

where the integral is over trajectories of winding number n .

Next, the path integration over the angular coordinates $\theta(t)$ is identical to an integration over the paths of a free particle with a time-dependent moment of inertia $mr^2(t)$. An expression for the propagator for similar systems has been evaluated by Khandekar and Lawande [6]. Following a similar line of steps the path integration over angular coordinates can be done and the result can be expressed as

$$K_\phi^{(n)} = \delta(\theta'' - \theta' + 2\pi n - \phi) \times \int \exp \left\{ \frac{i}{\hbar} \int_0^T \left[\frac{1}{2} m \left(\frac{dr}{dt} \right)^2 + \frac{\hbar^2}{8m} \frac{1}{r^2} \right] dt \right\} \left(\frac{M}{2\pi i \hbar r' r''} \right)^{1/2} \exp \left(\frac{iM}{2\hbar} (\theta'' - \theta' + 2\pi n)^2 \right) d[r(t)]. \quad (10)$$

Here the quantity

$$M = m \left(\int_0^T \frac{1}{r^2} dt \right)^{-1} \quad (11)$$

is still a functional of the "radial" path, and the radial path integration has still to be performed.

To perform the radial path integration we make use of the identity

$$\left(\frac{M}{i\hbar} \right)^{1/2} \exp \left(\frac{iM}{2\hbar} \phi^2 \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left(-\frac{i\hbar}{2M} \xi^2 + i\xi\phi \right) d\xi, \quad (12)$$

and rewrite (10) as

$$K_\phi^{(n)} = \delta(\theta'' - \theta' + 2\pi n - \phi) \int_{-\infty}^{\infty} \exp[i\xi(\theta'' - \theta' + 2\pi n)] K_\xi d\xi. \quad (13)$$

Here K_ξ is the path integral

$$K_\xi = \frac{1}{2\pi \sqrt{r' r''}} \int \exp \left(\frac{i}{\hbar} S_\xi \right) d[\gamma(t)] \quad (14)$$

associated with the "radial action"

$$S_\xi = \frac{1}{2} m \int_0^T \left(\frac{dr}{dt} \right)^2 dt - \frac{\hbar^2}{2m} \left(\xi^2 - \frac{1}{4} \right) \int_0^T \frac{1}{r^2} dt. \quad (15)$$

But this is exactly equal to the action of a fictitious one dimensional particle of mass m which moves on the half line $0 < r < \infty$ under the influence of the potential

$$\frac{\hbar^2}{2m} \left(\xi^2 - \frac{1}{4} \right) \frac{1}{r^2}.$$

For this system the propagator has been evaluated by Khandekar and Lawande [7]. Using their result we obtain

$$K_{\phi}^{(n)} = \delta(\theta'' - \theta' + 2\pi n - \phi) \times \frac{m}{2\pi i \hbar T} \exp\left(i(r'^2 + r''^2) \frac{m}{2\hbar T}\right) \int_{-\infty}^{\infty} \exp[i\xi(\theta'' - \theta' + 2\pi n)] I_{|\xi|}(mr'r''/i\hbar T) d\xi, \quad (16)$$

where I is the modified Bessel function. The propagator expression obtained in (16) coincides with the corresponding expressions of refs. [2] and [3], which shows that this method does indeed yield the correct results.

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