ROTATIONAL FRICTION COEFFICIENT OF A PERMEABLE CYLINDER IN A VISCOUS FLUID

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An exact expression is derived for the rotational friction coefficient of a cylinder of infinite length and constant permeability immersed in an incompressible viscous fluid. An asymptotic expression for the translational friction coefficient of a permeable cylinder moving in a sheet of viscous fluid embedded on both sides in a fluid of much lower viscosity is also given.

In this letter we consider a porous cylinder of infinite length and uniform permeability immersed in an incompressible fluid (viscosity η) and rotating with a constant angular velocity ω_0 around its axis. The problem is to calculate the rotational friction coefficient f_R, which is defined as the ratio T/ω_0, where T denotes the total torque of the forces per unit of length which the cylinder exerts on the fluid.

This problem is relevant to the rotational diffusion of a patch of cross-linked proteins in a cell membrane [1]. This rotational friction coefficient has never been calculated; the analogous problem for a sphere of uniform permeability was solved only a few years ago by Felderhof and Deutch [2].

Consider a cartesian system of coordinates (x, y, z) with the z-axis along the axis of the cylinder. (Presently, we shall also use cylindrical coordinates (r, ϕ, z).) Let V and P denote the average local velocity and pressure of the fluid and U the local velocity of the cylinder. The fluid flow has to be solved from the Debye–Brinkman–Bueche equation:

$$-\nabla P + \eta\Delta V - (\eta/k)(V - U) = 0\,, \tag{1}$$

together with the incompressibility equation:

$$\mathrm{div}\, V = 0\,. \tag{2}$$

In this note we only consider the case of a uniform cylinder for which the permeability $k(r) = k_0$ if $r < a$, $k(r) = \infty$ if $r > a$; a denotes the radius of the cylinder. A microscopic derivation of eq. (1) has been given by Felderhof and Deutch [2]; and a macroscopic derivation was given by Wiegel and Mijnlieff [3]. Applications of the Debye–Brinkman–Bueche equation to the flow of a solvent through a polymer coil are also found in refs. [4–10].

The x and y components of U are:

$$U_x = -\omega_0 r \sin\phi; \quad U_y = +\omega_0 r \cos\phi. \tag{3}$$

For the pressure and the velocity we make the ansatz:

$$P = \text{constant}, \tag{4}$$

$$V_x = -V(r)\sin\phi; \quad V_y = +V(r)\cos\phi\,, \tag{5}$$

where $V(r)$ denotes an unknown function – the magnitude of the velocity – which has cylindrical symmetry. Upon substitution of eqs. (4) and (5) into eqs. (1) and (2) one finds that all equations are satisfied provided $V(r)$ is the solution of the ordinary differential equation:

$$V'' + \frac{1}{r}V' - \frac{1}{r^2}V = \begin{cases} V/k_0 - \omega_0 r/k_0, & 0 < r < a, \\ 0, & a < r. \end{cases} \tag{6a,b}$$

The boundary conditions are: (i) $V(0)$ should be finite; (ii) $V(\infty) = 0$; (iii) V and V' should be continuous at $r = a$.

The solution of this differential equation is straightforward and one finds:

$$V(r) = \begin{cases} \omega_0 r + BI_1(r/\sqrt{k_0}), & 0 < r < a, \quad (7a) \\ A/r, & a < r, \quad (7b) \end{cases}$$

with:

$$A = \omega_0 a^2 I_2(\sigma)/I_0(\sigma), \tag{8}$$

$$B = -2\omega_0 a/\sigma I_0(\sigma). \tag{9}$$

In these formulae the dimensionless parameter $\sigma \equiv a/\sqrt{k_0}$ is the ratio of the radius of the cylinder and the distance $\sqrt{k_0}$ over which the fluid flow effectively penetrates the cylinder. The $I_\nu(\sigma)$ denote the modified Bessel functions.

The torque of the forces which the cylinder exerts on the fluid, per unit of length, equals:

$$T = -2\pi B \frac{\eta}{k_0} \int_0^a r^2 I_1(r/\sqrt{k_0})\,\mathrm{d}r = 4\pi\eta\omega_0 a^2 \frac{I_2(\sigma)}{I_0(\sigma)}. \tag{10}$$

Hence the rotational friction coefficient per unit of length is given by:

$$f_\mathrm{R} = 4\pi\eta a^2 I_2(\sigma)/I_0(\sigma). \tag{11}$$

In the limit $\sigma \to \infty$ the cylinder becomes impermeable and one recovers the trivial result $f_\mathrm{R} = 4\pi\eta a^2$ which holds for a hard cylinder. If, on the other hand, $\sigma \ll 1$ the expression (11) simplifies to $f_\mathrm{R} = \frac{1}{2}\pi\eta a^2\sigma^2$, as it should. Using tables of the modified Bessel functions [11], we have tabulated in table 1 the correction factor due to the finite permeability of the cylinder.

For the sake of completeness, we also give the translational friction coefficient (force per unit relative velocity) of a permeable cylinder of height h constrained to move in a sheet of fluid with viscosity η. The sheet has thickness h and is embedded on both sides in another fluid of a much lower viscosity η'. The translational friction coefficient is given by:

$$f_\mathrm{T} = 4\pi\eta h \left\{ -\gamma + \ln(h\eta/a\eta') + \frac{2}{\sigma^2} + \frac{I_0(\sigma)}{\sigma I_1(\sigma)} \right\}^{-1}, \tag{12}$$

provided $(h\eta/a\eta') \gg 1$. In this formula $\gamma = 0.5772$ denotes Euler's constant. The derivation of eq. (12) will be given elsewhere [12]. The correction term $(2/\sigma^2) + I_0(\sigma)/\sigma I_1(\sigma)$ has also been tabulated in table 1.

Table 1
The second column gives the correction factor to the hard disk rotational friction coefficient due to finite permeability, eq. (11); the third column gives the correction term to the inverse of the hard disk translational friction coefficient, eq. (12).

σ	$I_2(\sigma)/I_0(\sigma)$	$(2/\sigma^2) + I_0(\sigma)/\sigma I_1(\sigma)$
0	0	∞
1	0.1072	4.2402
2	0.3022	1.2166
3	0.4600	0.6337
4	0.5682	0.4145
5	0.6426	0.3039
6	0.6958	0.2382
7	0.7355	0.1952
8	0.7662	0.1649
9	0.7905	0.1426
10	0.8103	0.1254

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