ELSEVIER
Theoretical Computer Science 290 (2003) 2125–2136
Theoretical Computer Science
www.elsevier.com/locate/tcs

Note

On tiling under tomographic constraints

Marek Chrobak[a,1], Peter Couperus[b], Christoph Dürr[c,*], Gerhard Woeginger[d,2]

[a] *Department of Computer Science, University of California, Riverside, CA 92521, USA*
[b] *Department of Mathematics, University of Washington, Seattle, WA 98195-4350, USA*
[c] *LRI, Université Paris-Sud, Bâtiment 490, F-91405 Orsay cédex, Paris, France*
[d] *Department of Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, Netherlands*

Received 4 December 2001; received in revised form 13 February 2002; accepted 9 May 2002
Communicated by P. Gritzmann

Abstract

Given a tiling of a 2D grid with several types of tiles, we can count for every row and column how many tiles of each type it intersects. These numbers are called the *projections*. We are interested in the problem of reconstructing a tiling which has given projections. Some simple variants of this problem, involving tiles that are 1×1 or 1×2 rectangles, have been studied in the past, and were proved to be either solvable in polynomial time or $\mathbb{NP}$-complete. In this note, we make progress toward a comprehensive classification of various tiling reconstruction problems, by proving $\mathbb{NP}$-completeness results for several sets of tiles.

Keywords: Discrete tomography; Tiling

1. Introduction

In *discrete tomography* we want to reconstruct a discrete object from its projections. This paper is concerned with the reconstruction of *tilings*. We are given a collection of *tiles*, where each tile can have a different shape. A tiling is a placement of non-overlapping copies of the tiles on a $n \times n$ grid, where each copy is obtained by

* Corresponding author.
E-mail addresses: marek@cs.ucr.edu (M. Chrobak), couperus@math.washington.edu (P. Couperus), christoph.durr@lri.fr (C. Dürr), g.j.woeginger@math.utwente.nl (G. Woeginger).
[1] Research supported by NSF grant CCR-9988360.
[2] This work was conducted while the author was affiliated with the Institut fuer Mathematik, TU Graz, Austria.

PII: S0304-3975(02)00542-X

translating one of the tiles. (In this note we do not allow tile rotations, although one could also consider the variant with rotations.) *Projections* of a tiling determine the number of tiles intersected by each row and column. Given such projections, we wish to reconstruct a tiling consistent with these projections, or to report that such a tiling does not exist.

Formally, a tile t is defined to be a finite subset of $\mathbb{Z}^2$. In this paper, we only consider tiles that are hole-less polyominoes. By $(i,j)+t=\{(i+i',j+j'):\ (i',j')\in t\}$ we denote the translation of t by vector $(i,j)\in\mathbb{Z}^2$. Fix a finite multiset of tiles $\mathscr{T}=\{t_1,t_2,\dots,t_h\}$. Without loss of generality, we assume that every tile t_k contains $(0,0)$, the so-called *center of the tile*, and in this paper it will always be the upper-left corner. We refer to the index k as the *type* of the tile. The tiles are identified by their type, and different tiles may have the same shape. One can think of tiles which are of the same shape but of different types as being of different colors.

A $\mathscr{T}$*-tiling* of a grid $G=\mathbb{Z}_n\times\mathbb{Z}_n$ is a finite set $T\subseteq\mathbb{Z}_n\times\mathbb{Z}_n\times[1,h]$, such that the sets $(i,j)+t_k$, for all $(i,j,k)\in T$, are disjoint and contained in G. If $\mathscr{T}=\{t\}$, we will sometimes write simply t*-tiling* instead of $\{t\}$-tiling. The *center projections* of T are vectors $r,c\in\mathbb{N}^{n\times h}$, where

$$r_{i,k}=|\{(i,j,k)\in T\}|$$

and

$$c_{j,k}=|\{(i,j,k)\in T\}|.$$

The numbers $r_{i,k}$ count the number of tiles of type k whose center is in row i, the numbers $c_{j,k}$ count the same for column j. In a similar manner, we define the *cell projections* r',c' of T, where we count for each row, each column, and each type the number of cells covered by that type of tile.

If tilings T and T' are disjoint, then projections of $T\cup T'$ are the sums of the projections of T and of T'. (This is true for both types of projections.) Therefore, the set of projections of all tilings T with a single tile ($|T|=1$) spans the set of all projections. The canonical bijection between *single-tile center projections* and *single-tile cell projections* implies a bijection between all center and cell projections. From now on we will use the term "projection" for center projections, unless stated explicitly otherwise.

Note that we do not require the tilings to cover the whole grid. Tilings that cover the whole grid are called *complete*. Each tiling problem can be mapped into an equivalent complete tiling by adding one "clear" 1×1 square tile whose row cell projections are n minus the total sums of the other tiles' row cell projections, and the column projections are defined analogously.

Fig. 1 illustrates this definition for the tile set $\mathscr{T}=\{t_1,t_2\}$, where t_1 is the 3×3 square and t_2 is the 2×2 square. The numbers on the left are the projections $r_{i,k}$ and the numbers on the top are the $c_{j,k}$. Columns are numbered from left to right and rows from top to bottom with indices ranging from 0 to 7. For example, $c_{4,1}=2$ because column 4 contains two centers of tile t_1.

Given a tiling T, the computation of its projections is straightforward. Consider now the inverse problem: given the vectors r,c, find a $\mathscr{T}$-tiling T with projections r,c. This is called the $\mathscr{T}$*-reconstruction problem*. The related decision problem (is there such a tiling T?) is called the $\mathscr{T}$*-consistency problem*, or simply the $\mathscr{T}$*-tiling problem*.

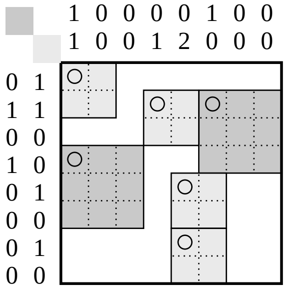
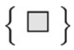
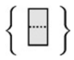
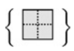
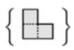
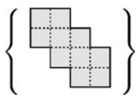
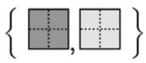
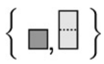
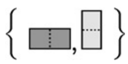
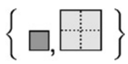


Fig. 1. A tiling of the 8×8 grid with its projections. The tile centers are marked by a circle.

Table 1
The complexity of different versions of the reconstruction/consistency problem

Type of tiles	Complexity	Reference
	$O(n^2)$	[7]
	$O(n^2)$	[2,5]
	$\geqslant$ "2-Atom problem"	Theorem 4
	$\mathbb{NP}$-complete	Theorem 6
	$\mathbb{NP}$-complete	Theorem 1
	Open	"2-Atom problem"
	$\mathbb{NP}$-complete	Theorem 3
	$\geqslant$"2-Atom problem"	(obvious)
	$\mathbb{NP}$-complete	Theorem 2
	$\mathbb{NP}$-complete	Theorem 5
	$\mathbb{NP}$-complete	[1]

Our results: For some types of tiles the reconstruction problem is easy to solve, while for other it may be hard. Table 1 summarizes the complexity of various tiling reconstruction problems, including both our results and previous work. In this table, by "ℕℙ-complete" we mean that the consistency problem is ℕℙ-complete.

The l-atom reconstruction problem: The simplest tile is a 1×1 square, which we call a *cell* or an *atom* (the original motivation for this problem came from the reconstruction of polyatomic structures). When $\mathscr{T}$ consists of l different cells, we will refer to the $\mathscr{T}$-tiling problem as the *l-atom problem*. Reconstructing 1-atom tilings is easy and can be solved in time linear in the size of the grid, as shown in 1957 by Ryser [6,7]. For 3 or more atoms (cells of different type), the reconstruction problem is ℕℙ-hard [1] (see also [3]). For 2 atoms, the complexity of the problem remains open.

One tile: For a single tile, it is known that if the tile is a horizontal *bar*, i.e. a rectangle of height one, the problem is as easy as reconstructing 1-atom tilings [2,5]. There exist other types of tiles, however, for which the problem is ℕℙ-complete. Two such tiles are given in Table 1. The problem remains open for rectangular tiles, even for the 2×2 square.

Two tiles: For pairs of tiles the situation is quite different. For horizontal and vertical dominoes—1×2 and 2×1 rectangles—the problem is ℕℙ-hard. However, the case when the domino tiling is required to be complete is open. The problem is also open for vertical dominoes and single cells. For squares the problem is ℕℙ-hard, both for two types of 2×2 squares and for a single 2×2 square and a cell.

2. ℕℙ-hardness proofs

We now present our ℕℙ-hardness results. In our proofs we reduce the 3-atom problem to the given version of the $\mathscr{T}$-tiling problem. A similar strategy was used earlier by Dürr et al. [2] to show that reconstructing tilings of given sub-grids—grids with forbidden regions—with only vertical and horizontal dominoes is ℕℙ-hard (even if the tilings are required to be complete).

The general idea of the proofs can be summarized as follows. We think of the 3-atom problem as a 4-atom *complete tiling* problem, with an additional "clear" atom. For convenience, we name each possible tile in the 3-atom problem as *yellow*, *blue*, *red* or *clear*. Throughout this section, by $\langle r, c\rangle$ we will denote the given instance of the 3-atom problem. We will map $\langle r, c\rangle$ into an instance $\langle r', c'\rangle$ of the $\mathscr{T}$-tiling problem under consideration. In all proofs we assume, without loss of generality, that $\sum_i r_{i,k} = \sum_j c_{j,k}$ for all k. This assumption is valid, since we can extend any mapping to instances in which $\sum_i r_{i,k} \neq \sum_j c_{j,k}$, by mapping them into an arbitrary fixed negative instance $\langle r', c'\rangle$, say to one in which the totals of row sums are not equal to the totals of column sums. This does not affect the asymptotic running time nor the correctness of the transformation.

For simplicity, assume first that $\mathscr{T}$ contains just one tile. To construct $\langle r', c'\rangle$, we choose a small $d \times d$ grid $\mathscr{B}$, called a *block*, that can be tiled in only four possible ways (this restriction will be relaxed in some proofs). Each of these four so-called *admissible* block tilings will correspond to one atom. Instance $\langle r', c'\rangle$ will have grid dimensions $nd \times nd$. We view this grid as an $n \times n$ matrix consisting of $d \times d$ blocks. A segment of rows numbered $id, \ldots, (i+1)d-1$ will be referred to as a *block-row* i. The transformation maps $r_{i,1},\dots,r_{i,4}$, that is, the atom projections of row i, into a length-d vector which is a projection of block-row i. This vector is the linear combination of the projections of the admissible tilings of $\mathscr{B}$ with coefficients $r_{i,1},\dots,r_{i,4}$. The column projections are mapped in the same way.

If $\mathscr{T}$ has $h>1$ tiles, the transformation is the same, except that now the block projections are not length-d vectors but $d\times h$ matrices.

Obviously, for any tile set $\mathscr{T}$, the $\mathscr{T}$-consistency problem is in $\mathbb{NP}$. For any choice of $\mathscr{B}$ and its admissible tilings, the method outlined above can be implemented in polynomial time. It also has the property that if $\langle r,c\rangle$ has a solution then so does $\langle r',c'\rangle$. For if T is a solution of the 3-atom problem with projections $\langle r,c\rangle$, then the tiling T' obtained by replacing each atom by its corresponding admissible block is a $\mathscr{T}$-tiling with projections $\langle r',c'\rangle$. Thus, the above ingredients of $\mathbb{NP}$-completeness arguments will be omitted in the proofs below, and we will focus exclusively on proving the following implication: if $\langle r',c'\rangle$ has a solution then $\langle r,c\rangle$ has a solution.

The main difficulty is to construct $\mathscr{B}$ to make this latest implication work. In other words, we need the property that any tiling of the resulting instance $\langle r',c'\rangle$ can be transformed back into a solution of $\langle r,c\rangle$. To achieve this, we choose $\mathscr{B}$ and the admissible tilings so that the following two conditions hold:

(npc1) any tiling with projections $\langle r',c'\rangle$ consists only of admissible block tilings, and
(npc2) from the projections of the block-rows we can uniquely extract the projections of the atoms in the corresponding rows of the 3-atom problem.

To enforce condition (npc1), we use techniques inspired by classical structure theorems for realizations of 0–1 matrices with given projections [7]. Another useful method involves the total counts of different colors. By Y, B, R and C we will denote the total number of yellow, blue, red and clear atoms in $\langle r,c\rangle$. We have $B+Y+R+C=n^2$. The block projections impose additional restrictions on how many centers of the tiles in $\mathscr{T}$ can occur on certain positions in the blocks. These restrictions can be expressed in terms of numbers Y, B, R and C. By investigating these constraints, we prove that non-admissible tilings cannot occur.

We now discuss condition (npc2). Suppose that there is a $\mathscr{T}$-tiling T' with projections $\langle r',c'\rangle$. By (npc1), each block in T' is admissible. We transform T' into a solution T of the 3-atom problem by replacing each admissible block by its corresponding atom. To satisfy (npc2), we need to show that the projections of T are $\langle r,c\rangle$.

Number the admissible tilings from 1 to 4 and name their row projections $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4$. Let $\mathbf{b}_i$ be the projection of block-row i in T' and q_j the number of blocks in block-row i with the jth admissible tiling. Then the numbers q_j satisfy

$$q_1\mathbf{e}_1+q_2\mathbf{e}_2+q_3\mathbf{e}_3+q_4\mathbf{e}_4=\mathbf{b}_i. \tag{1}$$

By the construction, Eq. (1) has a solution $q_j=r_{i,j}$, for $j=1,2,3,4$. For (npc2) to hold, we need to ensure that (1) does not have any other solutions in non-negative integers. This can be easily accomplished by choosing the admissible tilings for which the projections $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4$ are linearly independent.

In fact (npc2) will hold even for a weaker condition. Note that the numbers $r_{i,j}$ satisfy $r_{i,1}+r_{i,2}+r_{i,3}+r_{i,4}=n$. So we extend each $\mathbf{e}_j$ by adding to it one coordinate

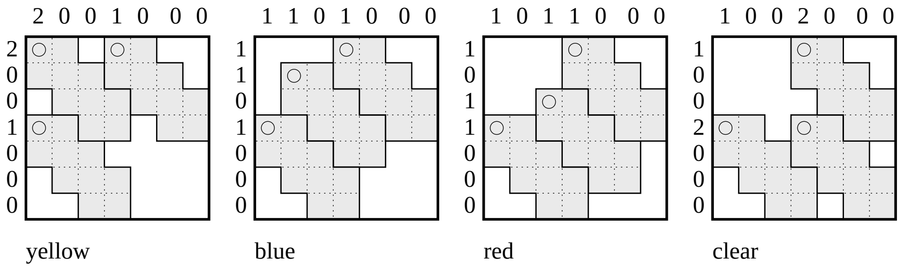
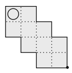


Fig. 2. The four admissible tilings of a block with 3 tiles t.

with value 1, and we similarly extend each vector $\mathbf{b}_i$ by adding to it one coordinate with value n. (Technically, the $\mathbf{e}_j$ and $\mathbf{b}_i$ can be $d \times h$ matrices, but for the purpose of the transformation we can as well treat them as vectors of length dh. Then the extended vectors will have length $dh+1$.) If these new vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ are linearly independent, we can use our admissible tilings for the transformation. Although we do not use it in the paper, it may be worth to mention that the linear independence of these extended vectors is equivalent to a condition called affine linear independence (see [4, p. 3]).

Theorem 1. *Let t be the tile*  *The consistency problem for t is $\mathbb{NP}$-complete.*

Proof. The proof follows the method outlined above. We reduce the 3-atom consistency problem to the t-tiling. We treat the 3-atom problems as the (equivalent) complete 4-atom problem, by adding an extra "clear" cell tile. We use a block of size 7×7. The admissible tilings are the tilings of the 7×7 block with three tiles t. There are exactly four admissible tilings. These tilings and their associations to different atoms are shown in Fig. 2. Using the projections of these admissible block tilings, we map any instance $r, c \in \mathbb{N}^{n \times 4}$ of the 3-atom consistency problem into an instance $r', c' \in \mathbb{N}^{7n}$ of the t-tiling problem.

The transformation works as follows. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \in \mathbb{N}^7$ denote the row projection vectors of the four tilings in Fig. 2. For every row i, its projections form a four-dimensional vector $(r_{i,1}, r_{i,2}, r_{i,3}, r_{i,4})$. We will map it into $\mathbb{N}^7$. The projections r' in the resulting instance are defined by

$$\begin{pmatrix} r'_{7i} \\ r'_{7i+1} \\ r'_{7i+2} \\ r'_{7i+3} \\ r'_{7i+4} \\ r'_{7i+5} \\ r'_{7i+6} \end{pmatrix} = r_{i,1}\mathbf{e}_1 + r_{i,2}\mathbf{e}_2 + r_{i,3}\mathbf{e}_3 + r_{i,4}\mathbf{e}_4.$$

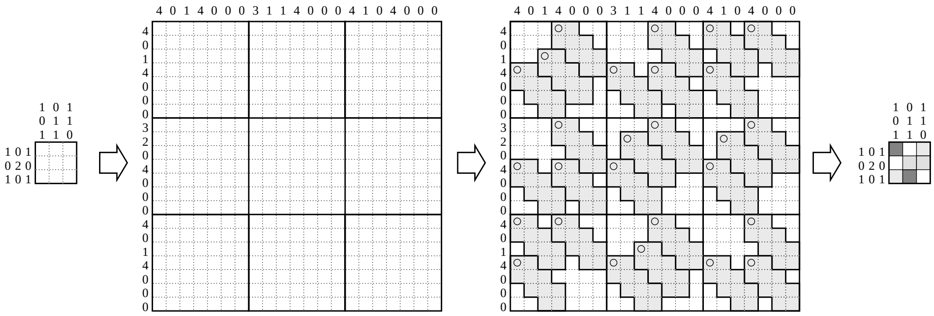


Fig. 3. Reducing the 3-atom reconstruction problem to the t-tiling reconstruction problem.

In other words, we set $r'_{7i} = r_{i,1} + n$, $r'_{7i+1} = r_{i,2}$, $r'_{7i+2} = r_{i,3}$, $r'_{7i+3} = r_{i,4} + n$ and $r'_{7i+l} = 0$ for $l = 4, 5, 6$. The column projections c' are determined in a similar manner.

We need to show that if $\langle r', c' \rangle$ has a solution then $\langle r, c \rangle$ also has a solution. To this end, let T be an arbitrary solution to $\langle r', c' \rangle$. We claim that in T every block is in one of the four configurations of Fig. 2. This is true because the rows and columns whose indices modulo 7 are greater than 3 have projection 0 and therefore all tiles t are strictly contained in a block. Further, $\langle r', c' \rangle$ requires $3n^2$ tiles t in total, and each of the n^2 blocks contains at most three tiles of type t.

The vectors $\mathbf{e}_k$ are linearly independent. So the projections $(r'_{7i+k})_{k \in \mathbb{Z}_7}$ of a block-row i can be uniquely written as $\sum_{k=1}^{4} r_{i,k} \mathbf{e}_k$. This ensures that T can be mapped into a 3-atom tiling with projections r, c (see Fig. 3). $\square$

For the next result, we need the following classical result on the structure of 0–1 matrices with given projections [7]. We state this fact in terms of tiling with cells.

Fact 1. *Let $r, c \in \mathbb{Z}^n$ be an $n \times n$ instance of the tiling problem with cells. Let $I \subseteq \mathbb{Z}_n$ be a row set and $J \subseteq \mathbb{Z}_n$ a column set. If*

$$\sum_{i \in I} r_i - \sum_{j \in \bar{J}} c_j = |I \times J|,$$

then in every solution the set $I \times J$ must be completely tiled and the set $\bar{I} \times \bar{J}$ must be completely empty.

Proof. Let T be a solution to $\langle r, c \rangle$. Let a be the number of cells in $I \times J$, b the number of cells in $I \times \bar{J}$ and c the number of cells in $\bar{I} \times \bar{J}$. Then

$$\sum_{i \in I} r_i - \sum_{j \in \bar{J}} c_j = (a + b) - (b + c) = a - c.$$

If $a - c = |I \times J|$ then $a = |I \times J|$ and $c = 0$, which concludes the proof. $\square$

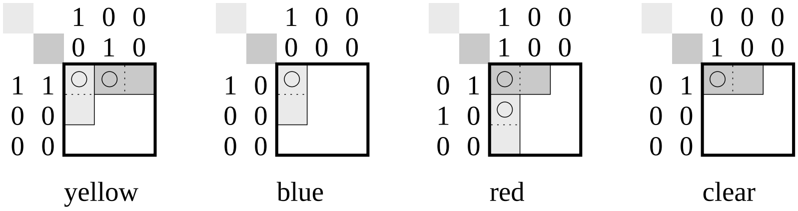


Fig. 4. Four admissible tilings of the 3×3 grid with dominoes.

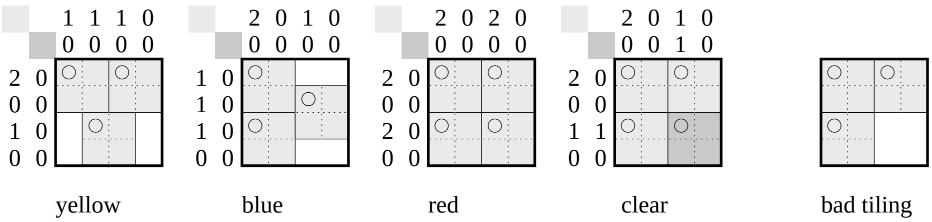


Fig. 5. Four tilings of the 4×4 grid with two types of squares.

Theorem 2. *The consistency problem for* 1×2 *and* 2×1 *rectangles* (*dominoes*) *is* $\mathbb{NP}$*-complete*.

Proof. We follow the idea outlined at the beginning of this section. We use a 3×3 block. The four admissible tilings of the block are shown in Fig. 4.

We need to show that if $\langle r', c' \rangle$ has a solution then $\langle r, c \rangle$ has a solution. Note that the row and column projection matrices of the four tilings in Fig. 4 are not linearly independent, but at least satisfy the weaker condition described on before, which is enough for the reduction. Therefore, to complete the proof we need to show that in any solution of the tiling instance every block is admissible.

Let $I = J = \{i \in \mathbb{Z}_{3n}\colon\ i \bmod 3 > 0\}$ be row and column sets. Denote the "clear" cell by t_3. Recall that Y, B, R and C denote the total numbers of yellow, blue, red and clear atoms in $\langle r, c \rangle$. Then we have $\sum_{i \in I} r'_{i,3} - \sum_{j \in \bar{J}} c'_{j,3} = (5Y + 5B + 4R + 6C) - (Y + B + 2C) = 4Y + 4B + 4R + 4C = 4n^2$. Fact 1 implies that in every block dominoes can only appear in the first row or first column of each block and the top-left cell is always covered by a domino. All the tilings that satisfy these condition are the admissible tilings in Fig. 4. □

Theorem 3. *The consistency problem for two types of* 2×2 *squares is* $\mathbb{NP}$*-complete*.

Proof. We refer to the two types of 2×2 squares as *light* and *dark* squares. We use the 4×4 block and four admissible block tilings shown in Fig. 5. The row and cell projection matrices of the admissible block tilings are linearly independent. Thus, to complete the proof, we have to show that any solution of $\langle r', c' \rangle$ uses only the four admissible block tilings.

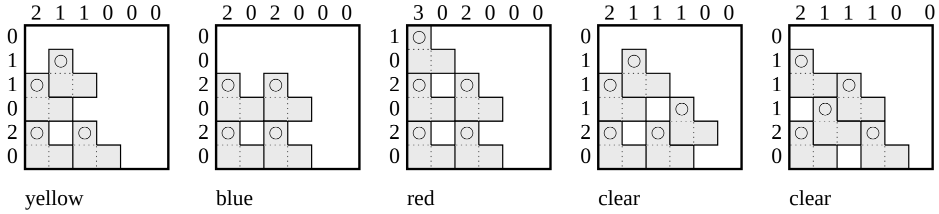


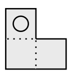
Fig. 6. Five admissible tilings of the 6×6 block with the L-shaped tile. The last two tilings have the same projections.

If we consider the cell projections rather than the center projections, we see that row 1 and column 1 of every block must be completely tiled with light squares (recall that rows and columns are numbered from 0). Therefore, a block can only be in one of the five tilings shown in Fig. 5. The fifth tiling—which is called *bad tiling*—has the same row projections as a "yellow" block and the same column projections as a "blue" block. By column projections for columns $j = 1 \pmod 4$, in any tiling there will be Y "yellow" blocks, and by row projections for rows $i = 1 \pmod 4$, there will be B "blue" blocks. There are $C + R$ remaining blocks, and $4(C + R)$ tiles must appear in these remaining blocks, so each of these remaining blocks must have four tiles. Thus, the bad tiling cannot occur. $\square$

The same technique can be used to reduce the 2-atom problem to the single-type square tiling problem. In this reduction, only the first 3 block tilings of Fig. 5 are used. It can also be used to prove $\mathbb{NP}$-completeness of the cell-and-square tiling problem. In this reduction, the dark square is replaced by a cell in Fig. 5, without modifying the projections.

Theorem 4. *If the consistency problem for* 2×2 *squares can be solved in polynomial time*, *then the* 2-*atom problem can be solved in polynomial time*.

Theorem 5. *The consistency problem for* 1×1 *cells and* 2×2 *squares is* $\mathbb{NP}$-*complete*.

Theorem 6. *Let* t *be the* L-shaped *tile* . *The consistency problem for* t *is* $\mathbb{NP}$-*complete*.

Proof. We reduce the 3-atom problem to the t-tiling problem using the admissible block tilings of Fig. 6. The first three tilings correspond to the colored atoms, and the two last tilings (with identical projections) correspond to the clear atom. The row and column projections of the admissible tilings are linearly independent.

It is sufficient to show that in any solution T to $\langle r', c' \rangle$ every block must be admissible. We define the matrix $M \in \mathbb{N}^{6\times 6}$ where $m_{i,j}$ is the number of tiles in T whose center equals $(i, j) \bmod 6$. Row and column sums of T imply that M must have the following form (the numbers on the left and on top are the row and column sums of M):

$$
\begin{array}{r|cccccc}
 & 2n^2+R & Y+C & n^2+B+R & C & 0 & 0 \\
\hline
R & m_{0,0} & m_{0,1} & m_{0,2} & m_{0,3} & 0 & 0 \\
Y+C & m_{1,0} & m_{1,1} & m_{1,2} & m_{1,3} & 0 & 0 \\
n^2+B+R & m_{2,0} & m_{2,1} & m_{2,2} & m_{2,3} & 0 & 0 \\
C & m_{3,0} & m_{3,1} & m_{3,2} & m_{3,3} & 0 & 0 \\
2n^2 & m_{4,0} & m_{4,1} & m_{4,2} & m_{4,3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

The tiles centered at (i,j) and $(i-1,j-1)$ overlap, and they both overlap with the tile centered at $(i-1,j)$ or $(i,j-1)$. This introduces two additional constraints on the matrix: for every i,j

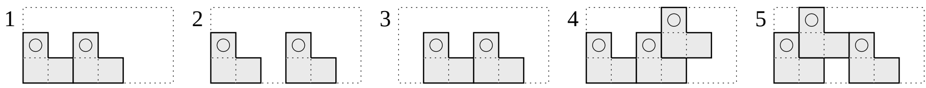
$$m_{i-1,j-1} + m_{i-1,j} + m_{i,j} \leqslant n^2 \tag{2}$$

and

$$m_{i-1,j-1} + m_{i,j-1} + m_{i,j} \leqslant n^2, \tag{3}$$

where we set $m_{i,j} = 0$ for $i = -1$ or $j = -1$.

Every block must have 2 centers in row 4 (recall that rows and columns are numbered from 0). So row 3 can have at most 1 center. We now consider sub-blocks that consist of rows 3–5. By the above argument, the tiles that are fully contained in these sub-blocks must have one of the following configurations:



This immediately gives $m_{3,0} = m_{3,2} = 0$, $m_{4,0} = n$. Let a_s be the number of blocks whose last three rows are in configuration of type s above, for $s = 1,\ldots,5$. Then $a_4 = m_{3,3}$ and $a_2 + a_3 + a_5 = m_{4,3}$. Since the projections of row 3 and column 3 of M are equal C, we get $a_4 + a_5 = C$ and $a_2 + a_3 + a_4 + a_5 + m_{0,3} + m_{1,3} + m_{2,3} = C$. Thus $a_2 = a_3 = m_{0,3} = m_{1,3} = m_{2,3} = 0$. This means that sub-blocks of types 2 and 3 cannot occur and that M has the form

$$
M = \begin{pmatrix}
m_{0,0} & m_{0,1} & m_{0,2} & 0 & 0 & 0 \\
m_{1,0} & m_{1,1} & m_{1,2} & 0 & 0 & 0 \\
m_{2,0} & m_{2,1} & m_{2,2} & 0 & 0 & 0 \\
0 & a & 0 & C-a & 0 & 0 \\
n^2 & 0 & n^2-a & a & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

where we write $a = a_5$, for simplicity.

Projections of row 0 and column 0 imply $m_{0,0} \leqslant R$ and $m_{0,0} + m_{1,0} + m_{2,0} = n^2 + R$, and from (2) we have $m_{1,0} + m_{2,0} \leqslant n^2$. Therefore, $m_{0,0} = R$ and $m_{1,0} + m_{2,0} = n^2$. The first equation forces $m_{0,1} = m_{0,2} = 0$, while the second forces $m_{2,1} = 0$, by (3). Hence

$$M = \begin{pmatrix} R & 0 & 0 & 0 & 0 & 0 \\ b & m_{1,1} & m_{1,2} & 0 & 0 & 0 \\ n^2 - b & 0 & m_{2,2} & 0 & 0 & 0 \\ 0 & a & 0 & C - a & 0 & 0 \\ n^2 & 0 & n^2 - a & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for some integer $b \geqslant 0$. Projections of column 1 and rows 1, 2 imply that

$$M = \begin{pmatrix} R & 0 & 0 & 0 & 0 & 0 \\ b & Y + C - a & a - b & 0 & 0 & 0 \\ n^2 - b & 0 & B + R + b & 0 & 0 & 0 \\ 0 & a & 0 & C - a & 0 & 0 \\ n^2 & 0 & n^2 - a & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

By inequality (2) for $(i,j) = (3,1)$ we have $n^2 - b + a \leqslant n^2$, so $a = b$, because all entries are non-negative. Thus $m_{1,2} = 0$.

Write $(i,j) \bowtie (i',j')$ if each block has a center at exactly one of the positions (i,j) or (i',j'). Clearly, if $m_{i,j} + m_{i',j'} = n^2$ and the tiles centered at (i,j), (i',j') overlap, then $(i,j) \bowtie (i',j')$. Since $a = b$, by the form of M above, we get the following relations:

$$(1,0) \bowtie (2,0) \bowtie (3,1) \bowtie (4,2) \bowtie (4,3) \quad \text{and} \quad (1,1) \bowtie (2,2).$$

Write $(i,j) \equiv (i',j')$ if each block has a tile centered at (i,j) iff it has a tile centered at (i',j'). By the above, we get

$$(1,0) \equiv (3,1) \equiv (4,3) \quad \text{and} \quad (2,0) \equiv (4,2)$$

By extending the three allowed configurations (number 1,4,5) of the rows 3–5, and using the above constraints, we obtain that the only block tilings that meet these conditions are the admissible tilings in Fig. 6. □

3. Conclusion

We proved $\mathbb{NP}$-completeness for several variants of the tiling problem, but what can be said about the infinitely many variants for which the complexity remains open?

A tile t of width w and height h is said to be *interlocking* if there is a box of width $<2w$ and height $<2h$ which contains two disjoint copies of t. For example, the tiles from Theorems 1 and 6 are interlocking, while rectangles are not. There are non-rectangular tiles that are not interlocking, for example the U-shaped tiles. For instances consisting of one interlocking tile we believe the tiling problem to be $\mathbb{NP}$-complete.

We are also confident that the problem is $\mathbb{NP}$-complete for all variants involving at least two different tiles, one of width $\geqslant 2$ and one (possibly the same one) of height $\geqslant 2$. This condition ensures that the problem is not invariant under column or row permutations.

We believe that the techniques developed in this paper will be useful in developing generic transformations that can show $\mathbb{NP}$-completeness of wide classes of tiling problems.

References

[1] M. Chrobak, C. Dürr, Reconstructing polyatomic structures from discrete X-rays: NP-completeness proof for three atoms, Theoret. Comput. Sci. 259 (2001) 81–98.
[2] C. Dürr, E. Goles, I. Rapaport, E. Rémila, Tiling with bars under tomographic constraints, Theoret. Comput. Sci. 290 (this Vol.) (2003) 1317–1329.
[3] R. Gardner, P. Gritzmann, D. Prangenberg, On the computational complexity of determining polyatomic structures by X-rays, Theoret. Comput. Sci. 233 (2000) 91–106.
[4] M. Grötschel, L. Lovász, A. Schrijver, Geometric Algorithms and Combinatorial Optimization, Springer, Berlin, 1988.
[5] C. Picouleau, Reconstruction of domino tiling from its two orthogonal projections, Theoret. Comput. Sci. 255 (2001) 437–447.
[6] H.J. Ryser, Combinatorial properties of matrices of zeroes and ones, Canad. J. Math. 9 (1957) 371–377.
[7] H.J. Ryser, Combinatorial Mathematics, Mathematical Association of America and Quinn & Boden, Rahway, NJ, 1963.