Probability density functions of contact forces for cohesionless frictional granular materials

N.P. Kruyt a,*, L. Rothenburg b

a Department of Mechanical Engineering, University of Twente, P.O. Box 217, 7500 AE Enschede, Netherlands
b Department of Civil Engineering, University of Waterloo, Waterloo, Ont., Canada N2L 3G1

Received 19 March 2001; received in revised form 6 August 2001

Abstract

A theory is developed for the probability density functions of contact forces for cohesionless, frictional granular materials in quasi-static equilibrium. This theory is based on a maximum information entropy principle, with an expression for information entropy that is appropriate for granular materials. Entropy is maximized under the constraints of a prescribed stress and that the normal component of the contact force is compressive and that the tangential component of the contact force is limited by Coulomb friction. The theory results in a dependence of the probability density function for the tangential contact forces on the friction coefficient. The theoretical predictions are compared with results from discrete element simulations on isotropic, two-dimensional assemblies under hydrostatic stress. Good qualitative agreement is found for means and standard deviations of contact forces and the shape of the probability density functions, while the quantitative agreement is fairly good. Discrepancies between theory and simulations, such as the difference in shape of the probability density function for the normal force and the observed dependence on elastic properties of the exponential decay rate of tangential forces, are attributed to the fact that the method does not take into account any kinematics, which are essential in relation to elastic effects. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Granular materials; Force probability density function; Maximum entropy method

1. Introduction

The probability density functions of contact forces for cohesionless, frictional granular materials in quasi-static equilibrium have recently received considerable attention. Besides providing basic knowledge of the behaviour of granular materials, the probability density functions of contact forces are also important in more practical, probabilistic studies of failure and fracture.

Previous studies dealing with these probability density functions were either experimental (Mueth et al., 1998; Løvoll et al., 1999; Makse et al., 2000), theoretical (Coppersmith et al., 1996; Bagi, 1997; Socolar,
Many theoretical lattice-type models have been developed with various degrees of sophistication (for example Coppersmith et al. (1996), Socolar (1998) and Nguyen and Coppersmith (2000)). These lattice-type models are of hyperbolic type, corresponding to fully mobilized friction at the contacts. Generally, one expects models of elliptic type, due to the presence of elastic effects. A feature of these lattice models is that they lead to probability density functions with exponential tails that are also observed experimentally (Mueth et al., 1998; Løvoll et al., 1999; Makse et al., 2000). These models are based on various (and sometimes rather arbitrary) assumptions. They do not directly account for the geometrical characteristics of an assembly, which directly affect the macroscopic properties of an assembly (Kruyt and Rothenburg, 2001).

The theoretical model that will be developed here is based on information theory (Katz, 1967). Rothenburg (1980) gave a definition of information entropy that is appropriate for granular materials. A detailed motivation of this definition is given by Rothenburg and Kruyt (2001). A study similar to the current one was performed by Bagi (1997). It is formulated in the Cartesian components of the contact force. Since the Cartesian components do not possess natural cut off values (unlike the normal and tangential forces employed here that do possess such cut off values for cohesionless, frictional granular materials), this formulation is incorrect as will be explained in Section 4.1.

Theoretical probability density functions for the contact forces of cohesionless, frictional granular materials will be obtained by maximizing information entropy. This maximization is performed subject to constraints on the system that is provided by macroscopic knowledge, or information, on the system. The information given here consists of the stress tensor and some aspects of the contact constitutive relation, i.e. that normal forces are compressive for a cohesionless material and that tangential forces are limited by Coulomb friction for a frictional material. An important restriction of this macroscopic information is that it does not involve any kinematics. In particular, it does not take into account elastic deformation.

The theoretical predictions obtained from the entropy maximization are compared with results from discrete element simulations, as proposed by Cundall and Strack (1979), on two-dimensional, isotropic assemblies under hydrostatic loading. Quantities that are compared are means and (co)variances of normal and tangential components of the contact forces and their marginal probability density functions. For large forces, the theoretical and observed probability density functions show an exponential decay. These decay rates will also be compared.

The outline of this study is as follows. Section 2 contains the micromechanical expression for the average stress tensor, a description of the employed contact constitutive relation and the definition of the probability density function to be determined. In Section 3 the maximum entropy method is described. Section 4 gives the theoretical joint probability density functions for the normal and tangential contact forces, as obtained from the maximum entropy method, as well as its means, its (co)variances and the marginal probability density functions. The theoretical results are compared with those obtained from discrete element simulations in Section 5. Finally, findings from this study are discussed in Section 6.

2. Average stress tensor and contact constitutive relation

Consider a contact between two particles \( p \) and \( q \). The vector from the centre of particle \( p \) to the centre of particle \( q \) is the branch vector \( \mathbf{l}_{pq} \), while the normal and tangential vectors at the contact are \( \mathbf{n}_{pq} \) and \( \mathbf{t}_{pq} \) respectively (see also Fig. 1). The orientation of the contact normal is denoted by \( \theta_{pq} \), so \( \mathbf{n}_{pq} = \{ \cos \theta_{pq}, \sin \theta_{pq} \}^T \). The contact force \( f_{pq}^n \) is the force at the contact that is exerted by particle \( q \) on particle \( p \). The normal and tangential components of the contact force are \( f_{pq}^n \) and \( f_{pq}^t \).
For granular materials in quasi-static equilibrium the most important macroscopic quantity (together with the strain tensor) is the stress tensor $\sigma_{ij}$. The micromechanical expression for the average stress tensor in terms of contact quantities is (see for example, Drescher and De Josselin de Jong (1972), Strack and Cundall (1978) and Rothenburg and Selvadurai (1981))

$$\sigma_{ij} = \frac{1}{S} \sum_{c \in S} f_{i}^{c} f_{j}^{c}$$

(1)

where the summation is over the contacts $c$ in the region of interest with area $S$.

The contact constitutive relation that will be subsequently used in the discrete element simulations involves linear springs in normal and tangential direction with spring stiffnesses $k_n$ and $k_t$. The ratio $k_t/k_n$ is the stiffness ratio. Furthermore, Coulomb friction is present by which the tangential force is limited by $|f_t| \leq \mu f_n$, where $\mu$ is the friction coefficient. For cohesionless granular materials a contact is considered broken when the normal component of the contact force becomes tensile, i.e. only compressive normal forces are allowed. This defines the region of admissible contact forces, see also Fig. 2. This contact constitutive relation is described in detail by Cundall and Strack (1979).

For further theoretical manipulations it is advantageous to replace the discrete sum in Eq. (1) by an integral of a probability density function. For sufficiently random assemblies, this probability density function will be a continuous function. It will be assumed that all geometrical characteristics of the assemblies are known, so the geometrical probability density functions for contact orientation $\theta$ and branch
vector $l_i$ are known. For isotropic, two-dimensional assemblies, these were studied by Kruyt and Roths-
enburg (2001). Hence it is natural to express the joint probability density function of geometrical quantities and forces $P_{\text{det}}(\theta, l_i, f_n, f_t)$ as
\begin{equation}
P_{\text{det}}(\theta, l_i, f_n, f_t) = E(\theta) L(l_i|\theta) P_n(f_n, f_t|\theta, l_i)
\end{equation}
where the contact distribution function $E(\theta)$ (Horne, 1965) and the conditional probability density function $L(l_i|\theta)$ for the branch vector given the contact orientation $\theta$ are considered as known quantities. The conditional probability density function for the normal and tangential forces given the full contact ge-
ometry $P_n(f_n, f_t|\theta, l_i)$ is unknown. The determination this function is a major objective of this study.

The average over all forces $\overline{U}(\theta, l_i)$ of a quantity $U(\theta, l_i, f_n, f_t)$ is defined by
\begin{equation}
\overline{U}(\theta, l_i) = \int_{-\infty}^{\infty} df_n \int_{-\infty}^{\infty} P_n(f_n, f_t) U(\theta, l_i, f_n, f_t) df_t
\end{equation}
Note that the integration is over the region of admissible contact forces, see Fig. 2.

Using the decomposition of the probability density function (2), an overall average $\langle U \rangle$ of a quantity $U(\theta, l_i, f_n, f_t)$ is defined by
\begin{equation}
\langle U \rangle = \int_0^{2\pi} E(\theta) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L(l_i|\theta) \overline{U}(\theta, l_i) dl_1 dl_2 d\theta
\end{equation}

Using this notation, the expression for the stress tensor (1) in terms of probability density functions becomes
\begin{equation}
\sigma_{ij} = m_s \langle f_i l_j \rangle
\end{equation}
where $m_s$ is the contact density, i.e. the number of contacts per area.

Since the function $P_n(f_n, f_t|\theta, l_i)$ is a probability density function, it must satisfy the normalizing con-
dition
\begin{equation}
\int P_n(f_n, f_t|\theta, l_i) df_n df_t = 1
\end{equation}

3. Maximum entropy principle

Using the notation for overall averages (4), the information entropy $I(P_n)$ of the conditional probability density function $P_n(f_n, f_t|\theta, l_i)$ for normal and tangential forces is defined by
\begin{equation}
I(P_n) = -\langle \ln P_n \rangle
\end{equation}

This definition of information entropy for granular materials in quasi-static equilibrium was first pro-
The central assumption made is that contact forces at contacts with similar orientations $\theta$ and branch vectors $l_i$ can be treated as independent realizations of a random variable with the corresponding conditional probability density function $P_n(f_n, f_t|\theta, l_i)$. By counting the multitude of all possible responses, it is found that the most probable outcome is found when this information entropy attains a maximum, like in many other applications of information theory (Katz, 1967; Kapur and Kesavan, 1992). Although the contact forces involving a single particle are not independent, since they satisfy the quasi-static force equilibrium conditions for the particle, contacts with similar orientations $\theta$ and branch vectors $l_i$ are practically independent (at least for assemblies with sufficient geometrical disorder), since such contacts are spatially separated. An analogy with a system consisting of a mixture of gases in kinetic theory is that contacts with similar orientations $\theta$ and branch vectors $l_i$ each constitute an independent subsystem.
The principle of maximum information entropy will be used to determine the conditional probability density function \( P_{nt}(f_n, f_t | \theta, l_i) \) for normal and tangential forces. The maximization of information entropy will be performed subject to constraints which constitute the macroscopic information on the system. These constraints are the given stress tensor (5) and the normalizing condition (6) for \( P_{nt}(f_n, f_t | \theta, l_i) \)

\[
\sigma_{ij} = m_S ((f_n n_i + f_t t_i) l_j) \quad \bar{1}(l_i, \theta) = 1
\]  

(8)

By maximizing the information entropy (7) subject to these constraints (using the method of Lagrangian multipliers), the usual exponential probability density function is obtained

\[
P_{nt}(f_n, f_t | \theta, l_i) = \frac{1}{Z(\theta, l_i)} e^{-|C_{ij}| f_n + C_{ij} f_t}
\]

(9)

where \( C_{ij} \) and \( Z(\theta, l_i) \) are the Lagrangian multipliers associated with the stress constraint and the normalizing constraint respectively. Here the summation convention has been adopted, which means that a summation over repeated subscripts is implied.

To keep the notation compact, some abbreviations are now introduced

\[
c_n = C_{ij} l_i n_i \quad c_t = C_{ij} l_i t_i
\]

(10)

Note that the parameters \( c_n \) and \( c_t \) depend on the geometrical quantities \( n_i, t_i \) and \( l_i \). Then the probability density function \( P_{nt}(f_n, f_t) \) can be written as

\[
P_{nt}(f_n, f_t) = \frac{1}{Z} e^{-(c_n f_n + c_t f_t)}
\]

(11)

4. Analysis of probability density functions and determination of Lagrangian multipliers

In this section the probability density function (11) will be analysed in detail and the Lagrangian multipliers will be determined for isotropic assemblies of disks under hydrostatic loading.

4.1. Analysis of probability density functions

The marginal probability density functions for the normal and tangential forces \( P_n \) and \( P_t \) are given by

\[
P_n(f_n) = \int_{f_n}^{\infty} \frac{d f_t}{P_{nt}(f_n, f_t)} = \frac{1}{Z} \left[ \frac{e^{-(c_n - \mu c_t)} f_n - e^{-(c_n + \mu c_t)} f_n}{c_t} \right]
\]

(12)

\[
P_t(f_t) = \int_{|f_t|}^{\infty} \frac{d f_n}{P_{nt}(f_n, f_t)} = \frac{1}{Z} \left[ \frac{e^{-c_t f_t} e^{-c_n f_t}}{c_n} \right]
\]

(13)

For these marginal probability density functions to remain convergent, it is necessary that the term \( c_n^2 - \mu^2 c_t^2 \) is positive.

After some algebra it follows from Eq. (12) or Eq. (13) that

\[
\bar{1} = \frac{2\mu}{Z (c_n - \mu c_t)(c_n + \mu c_t)}
\]

(14)

From this equation the Lagrangian multiplier \( Z \) associated with the normalizing condition for the probability density function \( \bar{1} = 1 \) can be determined
The average normal and tangential force are given by
\[
\bar{f}_n = \frac{2c_n}{(c_n - \mu c_t)(c_n + \mu c_t)} \quad (16)
\]
and
\[
\bar{f}_t = \frac{-2\mu c_t}{(c_n - \mu c_t)(c_n + \mu c_t)} \quad (17)
\]

The (co)variances can be determined after some algebra
\[
\Sigma_{nn} = \bar{f}_n^2 - \bar{f}_n^2 = \frac{2(c_n^2 + \mu^2 c_t^2)}{(c_n - \mu c_t)^2(c_n + \mu c_t)^2} \quad (18)
\]
\[
\Sigma_{tt} = \bar{f}_t^2 - \bar{f}_t^2 = \frac{2\mu^2(c_n^2 + \mu^2 c_t^2)}{(c_n - \mu c_t)^2(c_n + \mu c_t)^2} \quad (19)
\]
\[
\Sigma_{nt} = \bar{f}_n\bar{f}_t - \bar{f}_n\bar{f}_t = \frac{-2\mu^2 c_n c_t}{(c_n - \mu c_t)^2(c_n + \mu c_t)^2} \quad (20)
\]

The correlation coefficient \(\rho_{nt}\) between normal and tangential forces becomes
\[
\rho_{nt} = \frac{\Sigma_{nt}}{\sqrt{\Sigma_{nn}\Sigma_{tt}}} = \frac{-2\mu^2 c_n c_t}{(c_n^2 + \mu^2 c_t^2)} \quad (21)
\]

The maximum information entropy \(I\) is
\[
I = \ln \left( \frac{2\mu}{c_n^2 - \mu^2 c_t^2} \right) + 2 \quad (22)
\]

The expressions (16) and (17) for the averages \(\bar{f}_n\) and \(\bar{f}_t\) in terms of the coefficients \(c_n\) and \(c_t\) can be inverted to give expressions for \(c_n\) and \(c_t\) in terms of the averages \(\bar{f}_n\) and \(\bar{f}_t\)
\[
c_n = \frac{2\mu^2 \bar{f}_n}{\mu^2 \bar{f}_n^2 - \bar{f}_t^2} \quad c_t = \frac{-2\bar{f}_t}{\mu^2 \bar{f}_n^2 - \bar{f}_t^2} \quad (23)
\]

Therefore it follows that the term \(c_n^2 - \mu^2 c_t^2\), which has been assumed to be positive, equals \(4\mu^2 / (\mu^2 \bar{f}_n^2 - \bar{f}_t^2)\), which is indeed always positive.

Using the expressions (18)–(20), the (co)variances can be expressed in terms of average normal and tangential forces
\[
\Sigma_{nn} = \frac{1}{2} \left( \bar{f}_n^2 + \frac{\bar{f}_t^2}{\mu^2} \right) \quad (24)
\]
\[
\Sigma_{tt} = \frac{1}{2} (\mu^2 \bar{f}_n^2 + \bar{f}_t^2) \quad (25)
\]
\[
\Sigma_{nt} = \bar{f}_n \bar{f}_t \quad (26)
\]

Eq. (26) shows that normal and tangential forces are only uncorrelated when the average tangential force equals zero. Superficially it appears from Eq. (11) that normal and tangential forces are always uncorrelated, since it seems that the joint probability density function can be written as the product of two terms,
each involving only one variable. This is not correct, since the term with the tangential force depends on the normal forces through the region of admissible forces: the tangential force is limited by Coulomb friction.

In terms of average normal and tangential forces the maximum information entropy $I$ becomes

$$I = \ln \left( \frac{\mu^2 \bar{f}_n^2 - \bar{f}_r^2}{2\mu} \right) + 2$$

(27)

Since the constraints (8) are linear in the contact forces, it should be pointed out that it is essential to take into account the region of admissible forces in order to obtain a maximum entropy probability density function. Without this extra information, the maximum entropy method does not provide a probability density function for this case (Kapur and Kesavan, 1992, p. 67). Therefore the method of Bagi (1997) that is formulated in the Cartesian components of the contact force, and does not use the information on the region of admissible forces, is incorrect.

4.2. Lagrangian multipliers

In this subsection the Lagrangian multiplier $C_{ij}$ associated with the stress constraint will be determined for the simple case where the particles are disks, $l_i = l_n n_i$, and the assembly is isotropic, $E(\theta) = 1/(2\pi)$, and it is loaded by a hydrostatic stress $\sigma_{ij} = \sigma \delta_{ij}$, where $\delta_{ij}$ is the Kronecker symbol. Then it follows from the stress constraint (5) that

$$C_{ij} = C \delta_{ij}, \quad C = \frac{mS}{\sigma}$$

(28)

The corresponding average normal and tangential force are

$$\bar{f}_n = \frac{2\sigma}{mS} \bar{m}_n, \quad \bar{f}_r = 0$$

(29)

Now the marginal probability density functions can be simplified. For hydrostatic stresses it was shown that $\bar{f}_r = 0$ or $c_t = 0$, see Eq. (23). The marginal probability density functions are found from Eqs. (12) and (13) by considering the limit case of $c_t \rightarrow 0$. It is found that the non-dimensional normal and tangential forces $\xi$ and $\eta$ have marginal probability density functions $P_\xi$ and $P_\eta$ that are given by

$$\xi = \frac{\bar{f}_n}{\bar{f}_n}, \quad P_\xi(\xi) = 4\xi e^{-2\xi}, \quad \xi \geq 0$$

(30)

$$\eta = \frac{\bar{f}_r}{\mu \bar{f}_n}, \quad P_\eta(\eta) = e^{-2|\eta|}, -\infty < \eta < \infty$$

(31)

Surprisingly, the same probability density function for the non-dimensional normal force was obtained from a lattice-type model for specific choices for some of its parameters (Coppersmith et al., 1996). This lattice-type model does not provide a probability density function for the non-dimensional tangential force.

5. Comparison with discrete element simulations

Discrete element simulations, as proposed by Cundall and Strack (1979), were performed with isotropic, two-dimensional assemblies of 50,000 disks from a wide lognormal particle-size distribution with average particle radius $R_{avg}$. Starting from a complete loose state without contacts, the assemblies were compressed until a preset hydrostatic stress $\sigma$ was obtained and static equilibrium had been obtained. The hydrostatic
stress $\sigma$ was such that $\sigma/k_n = 6 \times 10^{-3}$. Hence, it did not result in significant “overlap” of the particles. Periodic boundaries were employed to minimize boundary effects.

Various simulations were performed, all starting from the same loose state, for different values of friction coefficient $\mu$ and stiffness ratio $k_l/k_n$. The resulting assemblies all were isotropic, as characterized by a very small anisotropy of the contact distribution function $E(\theta)$. Due to the different stiffness ratios and friction coefficients, the assemblies had different packing densities $v$ and coordination numbers $\Gamma$, i.e. the average number of contacts per particle. A simple relation between $\Gamma$ and $v$ was given by Kruyt and Rothenburg (1996). The cases considered and the resulting geometrical characteristics of the assemblies are summarized in Table 1. As noted and analysed by Kruyt and Rothenburg (2001), the average branch vector $l_n$ is not necessarily equal to the average particle diameter: $l_n \neq 2R_{avg}$ in general.

The difference between the theoretical average normal force (29) and that observed from the discrete element simulations was 2%. This difference was independent of friction coefficient and stiffness ratio. Theoretical and observed average tangential forces were both zero.

A comparison of the predicted and observed standard deviations of normal and tangential forces is given in Fig. 3. The observed standard deviations of the normal forces are within 15% of that predicted theoretically, see Eq. (24). These standard deviations show a mild dependence on friction coefficient and are practically independent of stiffness ratio. According to the theory, see Eqs. (18) and (19), the ratio of the tangential over the normal standard deviation equals the friction coefficient (even for anisotropic assemblies!). The simulations show that this ratio of standard deviations increases with friction coefficient, but the rate of increase depends significantly on stiffness ratio $k_l/k_n$. The observed correlation coefficients between normal and tangential forces were zero, as predicted theoretically, see Eq. (29).

<table>
<thead>
<tr>
<th>$k_l/k_n$</th>
<th>$\mu$</th>
<th>$v$</th>
<th>$l_n/R_{avg}$</th>
<th>$\sqrt{l_n^2 - l_n^2/R_{avg}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.125</td>
<td>0.844</td>
<td>2.065</td>
<td>0.364</td>
</tr>
<tr>
<td>0.25</td>
<td>0.25</td>
<td>0.842</td>
<td>2.067</td>
<td>0.364</td>
</tr>
<tr>
<td>0.5</td>
<td>0.50</td>
<td>0.841</td>
<td>2.067</td>
<td>0.364</td>
</tr>
<tr>
<td>1.0</td>
<td>0.125</td>
<td>0.844</td>
<td>2.065</td>
<td>0.364</td>
</tr>
<tr>
<td>0.25</td>
<td>0.25</td>
<td>0.842</td>
<td>2.067</td>
<td>0.364</td>
</tr>
<tr>
<td>0.5</td>
<td>0.50</td>
<td>0.841</td>
<td>2.067</td>
<td>0.364</td>
</tr>
</tbody>
</table>

Fig. 3. Comparison of theoretical and observed standard deviations of normal and tangential forces for various friction coefficients and stiffness ratios.
A comparison of the theoretical non-dimensional marginal probability density functions (30) and (31) with those obtained from discrete element simulations is given in Fig. 4.

The observed marginal probability density function for the non-dimensional normal force is independent of stiffness ratio \( k_t / k_n \) and only weakly dependent on friction coefficient. The qualitative agreement with the theoretical probability density function (30) is reasonable. The simulations show that the probability density function remains nonzero for small forces, while the theory predicts that the probability density function becomes zero.

The qualitative agreement between the theoretical probability density function (31) for the non-dimensional tangential force and that obtained from the simulations is very good. The results from the
simulations show that the non-dimensional probability density function is only weakly dependent on friction coefficient (mainly for low $\mu$), but that it is significantly dependent on stiffness ratio $k_t/k_n$. The quantitative agreement between the probability density functions is very good for some of the cases considered.

For large values of the non-dimensional forces, the marginal probability density functions behave exponentially

$$P_\xi \approx e^{-r_\xi \xi} \quad P_\eta \approx e^{-r_\eta |\eta|} \quad \text{for } |\xi| \gg 1 \quad |\eta| \gg 1$$

where $r_\xi$ and $r_\eta$ are the exponential decay rates of the non-dimensional normal and tangential forces. According to the developed theory, we have $r_\xi = 2$ and $r_\eta = 2$, see Eqs. (30) and (31). Note that a decay rate of the non-dimensional tangential force $r_\eta = 2$ means that the decay rate of the dimensional tangential force depends on the friction coefficient as $2/(\mu f^*_n)$.

Results for the exponential decay rates from the discrete element simulations are shown in Fig. 5. The decay rate for the non-dimensional normal force $r_\xi \approx 2$ as predicted theoretically, independent of stiffness ratio and with only a mild increase with increasing friction. The theoretical decay rate for the non-dimensional tangential force is of the correct order of magnitude, but the results of the simulations show that the decay rate increases with friction coefficient, and more importantly, it strongly depends on the stiffness ratio.

Results from simulations on the exponential decay rates have been reported by Radjaï et al. (1996), Thornton (1997), Makse et al. (2000) and Antony (2000). Based on three-dimensional discrete element simulations, Thornton observed a decay rate of the non-dimensional normal force $r_\xi \approx 2.2$ which depended slightly on the level of confining stress, Antony obtains $r_\xi \approx 1.4$ independently of elastic properties, while Makse et al. find $r_\xi \approx 1.0$ for low confining stresses. Makse et al. (2000) found that for very high confining stresses the probability density function changes character from an exponential type to a Gaussian type. From their two-dimensional simulations Radjaï et al. (1996) obtained $r_\xi \approx 1.4$ and $r_\eta \approx 2.0$. It should be pointed out that their simulations use a different method, see for example (Moreau, 1994), that does not include elastic properties.

Experiments on three-dimensional systems were performed by Mueth et al. (1998), Løvoll et al. (1999) and Makse et al. (2000). Mueth et al. (1998) found a decay rate $r_\xi \approx 1.5$, Løvoll et al. (1999) observed
r_\xi \approx 1.8\), while Makse et al. (2000) found \(r_\xi \approx 1.0\) for low levels of confining stress. The decay rate of the tangential force \(r_g\) could not be measured.

The experimentally determined behaviour of the probability density function of the normal forces for small forces is contradictory: Løvoll et al. (1999) found \(P_n \rightarrow 0\) (as predicted by the developed theory), while Mueth et al. (1998) observe \(P_n \rightarrow K > 0\) (as found from the simulations).

The developed theory predicts that under hydrostatic stress, the joint probability density function is independent of tangential force, as follows from Eqs. (11), (23) and (29). This in turn implies that the conditional probability density function \(P_{g|n}(g|n)\) for the non-dimensional tangential force \(g\) given non-dimensional normal force \(n\) is uniform. A comparison with results from discrete element simulations is shown in Fig. 6. The results show a nearly uniform distribution, but with (sharply) increased probabilities near the range of tangential forces where friction is fully mobilized. A similar distribution was described by Radjaï et al. (1996).

### 6. Discussion

A straightforward theory was developed for predicting the probability density functions of contact forces for cohesionless, frictional granular materials in quasi-static equilibrium. It is based on a maximum entropy principle that uses a definition of entropy that is appropriate for granular materials. The only physical constraints involved in the entropy maximization are the prescribed stress tensor and a region of admissible forces that is defined by the requirements that normal forces are compressive and that tangential forces are limited by Coulomb friction. The theory is oversimplified in the aspect that it does not involve kinematics, which are important for elastic effects.

The results of the theory are compared with results from discrete element simulations with large isotropic, two-dimensional assemblies under hydrostatic stress. The quantitative agreement for the first moments of the probability density functions is reasonable (standard deviation of tangential force) to good (means of normal and tangential force, standard deviation of normal force and covariance of normal and tangential forces).

Qualitatively, the agreement between theoretical and observed marginal probability density functions is good. For some combinations of friction coefficient and stiffness ratio, there even is very good quantitative agreement for the tangential forces. The theory further predicts a uniform conditional probability density
function for the tangential force given the normal force. This is also observed from the simulations, although these show a (sharp) increase near the range of tangential forces where friction is fully mobilized.

As noted by many others, the probability density functions exhibit an exponential decay for large forces, which is also predicted here theoretically for normal and tangential forces. The decay rate for the non-dimensional normal force is reasonably well predicted by the theory, while the observed decay rate of the non-dimensional tangential force depends on the elastic properties, contrary to what is predicted by the theory. The increase of the decay rate with friction coefficient is predicted qualitatively.

A limitation of the developed theory is that it does not involve any kinematics, which are essential in relation to elastic effects. Rothenburg and Kruyt (2001) use the maximum entropy principle to predict the elastic properties of assemblies of non-rotating particles with bonded contacts. Hence, in this case tensile forces are admissible and friction is not included. In this case there are two macroscopic constraints that must be satisfied, the given stress tensor and the work constraint, which states that the work at the internal springs must equal the work done at the boundary. This latter constraint, which does involve kinematics, is a quadratic constraint. Hence for this case Gaussian probability density functions are obtained theoretically. These are also found from the corresponding discrete element simulations.

These elastic effects, and the corresponding Gaussian probability density functions, are mostly important for normal forces in general and for high levels of confining stress. High levels of confining stress correspond to increased coordination numbers. For high coordination numbers kinematics becomes dominant, while for low coordination numbers statics is more important in determining the macroscopic material response (Kruyt and Rothenburg, 1996). This argument may explain the change in character of the probability density functions, from exponential to Gaussian, when the confining stress is increased. This phenomenon was found both experimentally and from simulations by Makse et al. (2000).

It is therefore expected that more complete theories, that match the results from experiments and simulations (even) closer, must incorporate not only the statics of the system, but also its kinematics.

Contrary to the lattice-type models with their many parameters, the developed theory, with its minimal set of assumptions, indicates which assumptions are really required to obtain theoretically the desired behaviour, such as exponential decay of the probability density functions for large forces. These assumptions are that information entropy is maximized and that the region of admissible forces is restricted: the normal force is compressive for a cohesionless material and the tangential force is limited by Coulomb friction.

Future work will deal with the extension of the proposed information entropy method to anisotropic assemblies and the inclusion of elastic effects.

References


