Semi-global stabilization and output regulation of constrained linear plants via measurement feedback

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Semi-global stabilization and output regulation of linear systems subject to state and/or input constraints have been studied in our earlier work by using state feedback. For the same problems, observer based measurement feedback control designs are the topics of this paper. High-gain observers are used in the feedback design in order to obtain accurate estimates of the state so that the constraint violation can be avoided. Due to the peaking phenomenon associated with a high-gain observer, a special saturation protection is built in the control laws to avoid possible constraint violation. The results in this paper show that the semi-global stabilization and semi-global output regulation problems for constrained linear systems are solvable via measurement feedback under solvability conditions similar to those in the state feedback.

1. Introduction

Systems with state and input constraints are prevalent in the practice of control engineering. Recently there have been renewed interest in the control of linear systems with state and input constraints. Historically, actuator constraints have been studied extensively in optimal control in the 60s. Important notions like maximum region of recoverability and maximum region of reachability were proposed at that time (LeMay 1964). Methods such as describing functions were already available at that time. But with the advent of state space methods in the 70s and 80s there was only a very limited research effort on the effects of actuator saturation. During the 90s we have witnessed intense research activities in the area of control of linear plants with saturating actuators. The two special issues (Bernstein and Michel 1995) and (Saberi and Stoorvogel 1999) document some significant contributions during the 90’s to the area of control of linear systems with saturating actuators. These special issues register the significance of the work while pointing out certain specific trends in the area.

We notice that most of the research during the past ten years focused on systems with input constraints. The research activities during this time can broadly be divided into two directions. The first direction follows the so called a priori design philosophy in which all the constraints are taken into account right at the onset of analysis and design. The second direction follows the so-called a posteriori design philosophy. In this philosophy, initially all the constraints are ignored and a satisfactory design method is used to to meet the design goals in the absence of constraints. Subsequent to this, certain compensations are designed and are provided to insure stability and to reduce the loss of performance in the presence of constraints. Anti-windup design methodology belongs to this second category.

In addition to input constraints, state constraints also widely exist in practical control systems. Although some research activity has appeared in the literature dealing with state constraints, little attention has been paid to the structural properties associated to the constraints. Recently a new approach to the state and/or input constraints has appeared, where the constraints on the state and/or input are modelled by a constrained output (Saberi et al. 2001, 2002). In Saberi et al. (2001, 2002), for the first time in the presence of state and input constraints, both stabilization and output regulation problems in a global framework as well as a semi-global framework are formulated. Solvability conditions for such problems are developed, and whenever the solvability conditions are satisfied, explicit design methodologies to arrive at appropriate controllers or regulators are presented. From the work in Saberi et al. (2001, 2002), a taxonomy of constraints emerged. The taxonomy of constraints presented in Saberi et al. (2001, 2002) delineates the constraints into distinct categories, such as right and non-right invertible constraints, minimum phase, at most weakly non-minimum phase, and strongly non-minimum phase constraints, etc. The work done in Saberi et al. (2001, 2002) focuses only on utilizing state feedback. For state feedback, and for the case of right invertible constraints, the results developed in Saberi et al. (2001, 2002) are complete and deal with
different facets of global and semi-global stabilization and output regulation. For the case of non-right invertible constraints, although certain partial results are provided and the complexities involved are pointed out, the complete development of necessary and sufficient conditions to achieve semi-global and global stabilization and output regulation turns out to be a very complex and challenging problem that is yet to be resolved.

This paper is a continuation of the work in Saberi et al. (2001, 2002). Our focus here is on utilizing measurement feedback controllers and regulators. In developing a measurement based design, some special elements are needed and these are discussed here. Since the global and semi-global stabilization and output regulation problems are not completely solved for non-right invertible constraints under state feedback, this paper considers only the case of right invertible constraints. Moreover, as will be explained in the text, the class of systems for which the global stabilization and output regulation problems can be studied under measurement feedback turns out to be very restrictive and uninteresting; as such, this paper considers only semi-global stabilization and output regulation problems.

The paper is organized as follows: After introducing certain preliminaries in §2, we present in §3 an observer based design methodology for measurement feedback control for the constrained semi-global stabilization problem. Following this, we present in §4 an observer based design methodology for measurement feedback control for the constrained semi-global output regulation problem. Section 5 provides some examples to illustrate the design procedures for stabilization and output regulation with measurement feedback. Conclusions are drawn in §6.

We use ‘im C’ to denote the image space of a matrix C, and ‘int X’ to denote the interior of a set X.

2. Preliminaries

In this section we describe the underlying system models and constraints and state the problem formulations for semi-global stabilization and semi-global output regulation. We also recall briefly the taxonomy of constraints from Saberi et al. (2001, 2002).

Consider a linear continuous-time system

\[ \Sigma: \begin{cases} \dot{x} = Ax + Bu \\ y = C_x x \\ z = C_z x + D_z u \end{cases} \] (1)

where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^r \) and \( z \in \mathbb{R}^p \) are respectively the state, input, measurement output and constrained output (see figure 1). The constrained output is subject to the constraint

\[ z(t) \in S, \quad \forall t \geq 0 \] (2)

where the set \( S \subset \mathbb{R}^p \) is a priori given and is referred to as a constraint set.

The following assumption on the constraint set \( S \) and the constrained output is used throughout the paper:

**Assumption 1:**

(i) The set \( S \) is compact, convex and contains 0 as an interior point.

(ii) \( C_z^T D_z = 0 \) and \( S = (S \cap \text{im } C_z) + (S \cap \text{im } D_z) \).

Given the constraint on the constrained output, the initial state of the system must obviously be restricted. Constraint violation cannot be avoided when the initial state of the system is arbitrary. For this reason, we need to define an admissible set of initial conditions. It is straightforward to show that if the initial state does not belong to this set, then one can never avoid the constraint violation by choosing any type of feedback control law.

**Definition 1:** Let the system (1) and a constraint set \( S \) be given. We define

\[ \mathcal{A}(S) := \{ x \in \mathbb{R}^n \mid \exists u \text{ such that } C_z x + D_z u \in S \} \]

as the admissible set of initial conditions.

**Remark:** In view of Assumption 1, we observe that the admissible set \( \mathcal{A}(S) \) can be equivalently written as

\[ \mathcal{A}(S) := \{ x \in \mathbb{R}^n \mid C_z x \in S \} \]

In this paper we are only concerned with constrained semi-global stabilization and output regulation via measurement feedback. The reason that the global stabilization and output regulation problems are not discussed here is very simple. The class of systems for which these problems can be solved using measurement feedback in a global framework turns out to be very restrictive and uninteresting. In the global framework, a necessary condition for solvability is the existence of a static feedback \( u = f(y) \) such that if \( x(0) \in \mathcal{A}(S) \), then \( x(t) \in \mathcal{A}(S) \) for all \( t > 0 \) where

\[ \dot{x}(t) = Ax(t) + Bf(C_x x(t)) \]
In other words, the system must be able to satisfy the state constraints with a static output feedback. This is clearly very restrictive and therefore results for the global case are not of much interest.

We first define the constrained semi-global stabilization of a system (1) via measurement feedback.

**Problem 1:** Consider a system of the form (1) with a constraint set \( \mathcal{S} \subseteq \mathbb{R}^p \) satisfying Assumption 1. Constrained semi-global stabilization via measurement feedback is concerned with finding (if possible) a family of measurement feedbacks of the form

\[
\begin{aligned}
\dot{v} &= g(v, y, w, t), \quad v \in \mathbb{R}^q \\
u &= h(v, y, w, t)
\end{aligned}
\]

such that for any compact set \( \mathcal{X} \subseteq \mathcal{A}(\mathcal{S}) \) and any compact set \( \mathcal{Y} \subseteq \mathbb{R}^q \) there exists a measurement feedback in this family such that the following conditions hold:

(i) The equilibrium point \( (x, v) = (0, 0) \) of the closed-loop system is asymptotically stable with \( \mathcal{X} \times \mathcal{Y} \) contained in its region of attraction.

(ii) For any \( (x(0), v(0)) \in \mathcal{X} \times \mathcal{Y} \), we have \( z(t) \in \mathcal{S} \) for all \( t \geq 0 \).

For semi-global output regulation with constraints we consider the system

\[
\begin{aligned}
\dot{x} &= Ax + Bu + Ew \\
w &= Sw \\
\Sigma_w : \\
\begin{aligned}
z &= C_zx + D_zu \\
y &= C_yx + D_yw \\
0 &= C_ex + D_ew 
\end{aligned}
\end{aligned}
\]

where the second equation is a model of the exosystem with state \( w \in \mathbb{R}^r \). The initial condition \( w(0) \) is assumed to be in some \( a \text{ priori} \) given compact set \( \mathcal{W} \). This exosystem plays a dual role. It generates an exogenous disturbance that affects the plant in terms of \( Ew \) which needs to be rejected but it also generates a reference signal \( D_zw \) which we need to track. The goal of disturbance rejection and tracking is achieved by requiring that \( e(t) \to 0 \) as \( t \to \infty \). We again impose Assumption 1 on the constraint for output regulation.

**Problem 2:** Consider a system of the form (4) with a constraint set \( \mathcal{S} \subseteq \mathbb{R}^p \) satisfying Assumption 1 and a compact set \( \mathcal{W} \subseteq \mathbb{R}^q \). Constrained semi-global output regulation via measurement feedback is concerned with finding (if possible) a family of measurement feedbacks of the form

\[
\begin{aligned}
\dot{v} &= g(v, y, w, t), \quad v \in \mathbb{R}^q \\
u &= h(v, y, w, t)
\end{aligned}
\]

such that for any compact set \( \mathcal{X} \subseteq \mathcal{A}(\mathcal{S}) \) and for any compact set \( \mathcal{Y} \subseteq \mathbb{R}^q \), there exists a measurement feedback in this family such that the following conditions are satisfied:

(i) In the absence of the exosystem, i.e. \( w(t) \equiv 0 \), the equilibrium point \( (x, v) = (0, 0) \) of the closed-loop system is asymptotically stable with \( \mathcal{X} \times \mathcal{Y} \) contained in its region of attraction.

(ii) For any \( (x(0), v(0)) \in \mathcal{X} \times \mathcal{Y} \) and \( w(0) \in \mathcal{W} \), we have \( z(t) \in \mathcal{S} \) for all \( t \geq 0 \) and \( \lim_{t \to \infty} e(t) = 0 \).

(iii) For any \( (x(0), v(0)) \in \mathcal{X} \times \mathcal{Y} \) and \( w(0) \in \mathcal{W} \), whenever we set \( w(t) = 0 \) for \( t \geq t_0 \), we have \( \lim_{t \to \infty} x(t) = 0 \), \( \lim_{t \to \infty} v(t) = 0 \), and \( z(t) \in \mathcal{S} \) for all \( t \geq t_0 \).

**Remark:** The condition (iii) of Problem 2 is included to guarantee the stability of the closed-loop system if the tracking reference signal is switched off at some time \( t_0 \geq 0 \).

### 2.1. Taxonomy of constraints

Before we proceed to the statement of our main results, we recall a taxonomy of constraints as emerged from the study of constrained global and semi-global stabilization and output regulation via state feedback (Saberi et al. 2001, 2002). We denote by \( \Sigma_{zu} := (A, B, C_z, D_z) \) the subsystem associated with the mapping from the input \( u \) to the constrained output \( z \) in system (1). For definitions of right invertibility and invariant zeros used in the following definition, the reader is referred to Trentelman et al. (2001).

**Definition 2:** The constraints are said to be

- **right invertible constraints** if the subsystem \( \Sigma_{zu} \) is right invertible.
- **non-right invertible constraints** if the subsystem \( \Sigma_{zu} \) is not right invertible.

**Definition 3:** The invariant zeros of the subsystem \( \Sigma_{zu} \) are called the constraint invariant zeros of system (1) associated with the constrained output \( z \).

In the next definition we denote by \( \mathbb{C}, \mathbb{C}^-, \mathbb{C}^0 \) and \( \mathbb{C}^+ \) respectively the set of complex numbers in the entire complex plane, open negative half complex plane, imaginary axis and open positive half complex plane.

**Definition 4:** The constraints are said to be

- **minimum phase constraints** if all the constraint invariant zeros of the plant are in \( \mathbb{C}^- \).
• weakly minimum phase constraints if all the constraint invariant zeros of the plant are in $\mathbb{C}^- \cup \mathbb{C}^0$ with at least one constraint invariant zero in $\mathbb{C}^0$ and those constraint invariant zeros in $\mathbb{C}^0$ are simple.
• weakly non-minimum phase constraints if all the constraint invariant zeros of the plant are in $\mathbb{C}^- \cup \mathbb{C}^0$ with at least one non-simple constraint invariant zero in $\mathbb{C}^0$.
• at most weakly non-minimum phase constraints if all the constraint invariant zeros of the plant are in $\mathbb{C}^- \cup \mathbb{C}^0$.
• strongly non-minimum phase constraints if at least one of the constraint invariant zeros of the plant is in $\mathbb{C}^+$.  

3. Constrained semi-global stabilization via measurement feedback

In this section we solve the constrained semi-global stabilization via measurement feedback, as formulated in Problem 1. Since the constrained semi-global stabilization for non-right invertible constraints remains an open problem even for state feedback, we focus on systems with right invertible constraints. The solvability conditions for constrained semi-global stabilization via measurement feedback are stated in the following theorem.

**Theorem 1:** Consider the plant $\Sigma$ as given by (1) and a constraint set $\mathcal{S}$ that satisfies Assumption 1. Assume that the constraints are right invertible. Then the constrained semi-global stabilization problem via measurement feedback as defined in Problem 1 is solvable if the following conditions hold:

(i) $(A, B)$ is stabilizable.

(ii) The constraints are at most weakly non-minimum phase.

(iii) The pair $(C_y, A)$ is observable.

Moreover, conditions (i) and (ii) are necessary.

**Remark:** It is shown in Saberi et al. (2002) that conditions (i) and (ii) are necessary and sufficient for semi-global stabilization (via state feedback) of the constrained systems with right invertible constraints. Hence, the necessity and sufficiency of conditions (i) and (ii) in Theorem 1 are natural requirements for measurement feedback as well. Detectability of the pair $(C_y, A)$ would be sufficient for measurement feedback control. But we shall use a high-gain observer in the proof of this theorem which we cannot do without the observability condition (iii). This theorem is also a generalization of the corresponding result in the case of input-only constraints. As a matter of fact, the input-only constraints can be modelled by an output equation with $C_z = 0$ and $D_z = I$. In this case, the set of invariant zeros of the mapping from input $u$ to the constrained output $z$ is equal to that of the open-loop poles of the system. Hence, the condition of at most weakly non-minimum phase constraint becomes that all the open-loop poles of the system have non-positive real parts. The measurement based stabilization for this special case was studied in Saberi et al. (1996).

The proof of this theorem follows by constructing an observer based measurement feedback controller for semi-global stabilization. So far, there is only one linear state feedback controller available in the literature (Saberi et al. 2002) by which the constrained semi-global stabilization problem is solved. For the sake of continuity of our presentation, we briefly recall from Saberi et al. (2002) the main procedure involved in the design of such a state feedback.

The design of state feedback law for a system with right invertible and non-minimum phase constraints is based on a transformation of the original system by choosing a special coordinate basis (scb) for the subsystem $\Sigma_{zu}$ of system (1). Because a system in scb displays explicitly its finite and infinite zero structure, this feature makes it a crucial step toward a complete design. The basic idea is to split the system into two parts: one consisting of the zero dynamics associated with the subsystem $\Sigma_{zu}$, and the other consisting of strongly controllable part of $\Sigma_{zu}$. Then, the design follows in two steps: we first design a suitable state feedback for the zero dynamics; then we construct a control input so that the output $z$ approaches the chosen stabilizing signal for the zero dynamics while not violating any constraints. It is shown in Saberi et al. (2002) that one can construct a linear and time-invariant control law.

More specifically, the design is accomplished in the following steps. For a system with right-invertible constraints, we can choose appropriate coordinates in the state space and the input space for the subsystem $\Sigma_{zu}$ so that the system (1) takes the form (Sannuti and Saberi 1987, Saberi and Sannuti 1990)

$$
\begin{align*}
\dot{x}_a &= A_ax_a + K_az \\
\dot{x}_c &= A_cx_c + B_cu_c + H_ax_a + K_cz \\
\dot{x}_d &= A_dx_d + B_d(u_d + G_ax_a + G_cx_c + G_dx_d) + K_dz \\
\bar{z} &= \begin{pmatrix} z_0 \\ z_d \end{pmatrix} = \begin{pmatrix} D_0u_0 \\ C_dx_d \end{pmatrix} \\
y &= C_{zu}x_a + C_{zc}x_c + C_{zd}x_d
\end{align*}
$$

Moreover, the eigenvalues of $A_a$ are the invariant zeros of the subsystem $\Sigma_{zu}$ and the matrix pair $(A_c, B_c)$ is...
forall
forall
F
z
represents the zero dynamics of subsystem such that
By Assumption 1 the set is controllable. System (5) can be split into two subsystems: the first one given by
\[
\Sigma_1: \dot{x}_a = A_u x_a + K_u z, \quad x_a \in \mathbb{R}^{n_a}
\]
and the second one by
\[
\Sigma_2: \begin{cases}
\dot{x}_d = A_d x_d + B_d (u_d + H_d x_a) + K_d z, & x_d \in \mathbb{R}^{n_d} \\
z = (z_0, z_d) = \begin{pmatrix} D_0 u_0 \\ C_d x_d \end{pmatrix}
\end{cases}
\]
(7)
By Assumption 1 the set \( S \) can be decomposed compatibly with the decomposition of \( z \) as
\[
S = S_0 \times S_d
\]
(8)
such that \( z \in S \) if and only if \( z_0 \in S_0 \) and \( z_d \in S_d \).
The condition that the constraints are at most weakly non-minimum phase implies that all the eigenvalues of \( A_2 \) are in the closed left-half plane because \( \Sigma_1 \) represents the zero dynamics of subsystem \( \Sigma_{w2} \). Hence if we view \( z \) as the input to this subsystem with constraint \( z(t) \in S \), the null controllability region of this subsystem is the entire \( x_a \) subspace, i.e. \( \mathbb{R}^{n_a} \). Let \( X \) be a compact set contained in the interior of \( A(S) \) and let
\[
X_1 = \{ x_a \in \mathbb{R}^{n_a} \mid \exists x_c \in \mathbb{R}^{n_c}, x_d \in \mathbb{R}^{n_d} \text{ such that } (x_a^T, x_c^T, x_d^T)^T \in X \}
\]
Then we can design a linear feedback \( z = F_d x_a \) to stabilize this subsystem \( \Sigma_1 \). Consider the closed loop system
\[
\dot{x}_a = (A_a + K_a F_a) x_a + K_a v
\]
For fixed \( M > 0 \), \( \delta > 0 \) and \( \rho \in (0, 1) \) we want \( F_a \) to be such that
\[
\lim_{t \to \infty} x_a(t) = 0
\]
and
\[
F_a x_a(t) \in \rho S
\]
for all \( t \geq 0 \), for all initial conditions \( x_a(0) \in X_1 \) and for all \( v \) satisfying
\[
\|v(t)\| \leq M e^{-\delta t}
\]
(10)
for all \( t \geq 0 \). Such a linear feedback law can be designed via a direct method or a Riccati equation based method (see, for example, Lin et al. 1996).
However, \( z \) is not a control variable. That means, we need to design \( u_0, u_e, u_d \) for the second subsystem (7) such that \( z \) approaches \( F_d x_a \) exponentially given any initial conditions in \( X \). As such, we let \( z = F_d x_a + v \), where \( v = z - F_d x_a \). It is shown in Saberi et al. (2002) that by an appropriate design, one can guarantee that \( v \) satisfies (10), the entire closed-loop system is asymptotically stable with \( X \) contained in the region of attraction, and the constraint on \( z \) is maintained. The details of this part of design can be found in Saberi et al. (2002).
With this linear state feedback at hand, our goal in this section is to implement it by an observer based measurement feedback controller; that is, the state in the state feedback needs to be replaced by its estimate obtained from an observer that utilizes the available measurements to arrive at an estimate of the state. It turns out that we need a high-gain observer for this goal. However, a high-gain observer usually has a peaking phenomenon associated with it, and such a peaking in the estimate of the state is dangerous as it can potentially cause constraint violation. Hence, it should be taken care of in our design. Therefore, the design task will be completed in three steps. First we show a lemma which states that any linear state feedback design that achieves the properties as we mentioned above is in fact robust in tolerating certain exponentially decaying disturbance of sufficiently small magnitude.
**Lemma 1:** Consider the system
\[
\Sigma_d: \begin{cases}
\dot{x} = A_d x + E_d \mu \\
z = C_d x + D_d \mu
\end{cases}
\]
with \( A_d \) Hurwitz stable. Let \( S \) be a given set satisfying Assumption 1. Let \( \mathcal{X} \subset \text{int } A(S) \) be a compact set in \( \mathbb{R}^n \). Assume there exists \( \rho \in (0, 1) \) such for all \( x(0) = x_0 \in \mathcal{X} \) and \( \mu(t) = 0 \), we have \( z(t) \in \rho S \) for all \( t \geq 0 \). Then there exist \( \varepsilon > 0 \), \( r > 0 \), and a compact set \( \Omega(\mathcal{X}) \subset \text{int } A(S) \) such that for all \( x_0 \in \mathcal{X} \) and for all \( \mu(t) \) satisfying
\[
\|\mu(t)\| \leq \varepsilon e^{-\rho t}, \quad \forall t \geq 0
\]
(12)
we have that \( z(t) \in S \) and \( x(t) \in \Omega(\mathcal{X}) \) for all \( t \geq 0 \), and \( x(t) \to 0 \) as \( t \to \infty \).
**Remark:** We state this lemma for any general linear state feedback design that meets the condition stated in the theorem. Although so far there is only one linear state feedback design that has been presented in Saberi et al. (2002), this lemma subsumes all other possible linear state feedback designs that solve the constrained semi-global stabilization problem.
**Proof:** We can decompose the output \( z \) in two components \( z = z_{x_0} + z_\mu \) where
\[
z_{x_0}(t) = C_d e^{A_d t} x_0
\]
\[
z_\mu(t) = \int_0^t C_d e^{A_d (t - \tau)} E_d \mu(\tau) d\tau + D_d \mu(t)
\]
By assumption, we have \( z_{x_0}(t) \in \rho S \) for all \( x_0 \in \mathcal{X} \). We can easily verify that for any \( r > 0 \) and \( \mu \) satisfying (12) we have:
\[
\|z_\mu\|_\infty \leq \varepsilon \|f * g\|_\infty
\]
where \( f * g \) is the convolution of \( f \) and \( g \) given by
\[ f(t) = \|C_d e^{A_d t} E_d\| + \|D_a\| \delta(t), \quad g(t) = e^{-rt} \]

with \( \delta \) the Dirac impulse. We can therefore for any fixed \( r \) choose \( \varepsilon \) such that

\[ z_{\mu}(t) \in \frac{1 - \rho}{2} S \]

for all \( t \geq 0 \). Combining this with \( z_{\nu_0}(t) \in \rho S \) shows that

\[ z(t) \in \frac{1 + \rho}{2} S \]

for all \( t \geq 0 \). Using the fact that the system is linear this implies that

\[ x(t) \in \frac{1 + \rho}{2} A(S) \]

There obviously exists a compact set \( \Omega(\mathcal{X}) \) such that

\[ \frac{1 + \rho}{2} A(S) \subset \Omega(\mathcal{X}) \subset \text{int} A(S) \]

and we find \( x(t) \in \Omega(\mathcal{X}) \) for all \( t \geq 0 \). \( \square \)

This lemma is the first preparatory step toward an observer design. However, if we adopt a fast observer to make an extremely accurate estimation of the state, there is an unavoidable annoying phenomenon called peaking associated with high-gain observer design. The peaking phenomenon must be taken care of seriously, for we are facing a control system with constraints. Fortunately, we have another lemma which provides us a mechanism to avoid the negative effect of peaking. As the second step, we recall the lemma here because it is instrumental to the high-gain observer design.

**Lemma 2:** Consider the system

\[ \dot{\eta} = (A - LC)\eta \]

where \( A \in \mathbb{R}^{n \times n} \) and \((C, A)\) is observable. Then, for any \( N > 0, \varepsilon > 0, r > 0, \) and \( \tau > 0 \) there exists a matrix \( L \) such that \( A - LC + \rho I \) is Hurwitz stable and

\[ \|\eta(t)\| \leq \varepsilon e^{-rt} \]

for all \( t \geq \tau \) and for all initial conditions \( \eta(0) \in \mathbb{R}^n \) satisfying \( \|\eta(0)\| \leq N \).

**Proof:** This lemma follows from the results in Izmailov (1987, 1988) and Theorem 8.2 in Sussmann and Kokotovic (1991). We simply sketch the proof for completeness. Since \((C, A)\) is observable, for any \( \rho > r \) there exists a matrix \( L \) such that \( A - LC + \rho I \) is Hurwitz stable. Let \( \nu \) be the largest observability index of the pair \((C, A)\). Then according to Theorem 8.2 in Sussmann and Kokotovic (1991) the state \( \eta \) has a peaking exponent \( \nu - 1 \). In other words, there exists a constant \( \alpha > 0 \) independent of \( \rho \) such that

\[ \|\eta(t)\| \leq \alpha \|\eta(0)\|(2\rho)^{\nu-1} e^{-2\rho t} \]

for all \( t \geq 0 \). For any given \( \varepsilon > 0, \tau > 0, \) and \( r > 0 \), we choose \( \rho > r \) large enough so that \( \alpha \|\eta(0)\|(2\rho)^{\nu-1} e^{-2\rho t} \leq \varepsilon \) for \( t \geq \tau \). Then this choice of \( L \) which guarantees Hurwitz stability of \( A - LC + 2\rho I \) leads to \( \eta \) satisfying (14) for all \( t \geq \tau \) and for any initial condition \( \eta(0) \in \mathbb{R}^n \) satisfying \( \|\eta(0)\| \leq N \). \( \square \)

So far we have prepared enough tools for the observer design. As the last step in the observer design, we construct the observer based controller in the proof of Theorem 1.

**Proof of Theorem 1:** The necessity of conditions (i) and (ii) is a consequence of the state feedback design (see Saberi et al. 2002). The sufficiency of the conditions are proven by an explicit design as presented below.

We need a high-gain observer to estimate the state. The observer takes the standard form

\[ \dot{x} = A\hat{x} + Bu + L(y - C\hat{x}) \]

By the assumption that \((C_p, A)\) is observable, we can choose a gain matrix \( L \) such that the eigenvalues of the matrix \( A_{\text{obs}} := A - LC_p \) can be assigned anywhere in the left-half complex plane. The estimation error \( e := \hat{x} - x \) satisfies

\[ \dot{e} = A_{\text{obs}} e \] (15)

Our goal is to devise a measurement feedback such that the set \( \mathcal{X} \times \mathcal{V} \) is contained in the domain of attraction, meanwhile for all initial states in this set the constraints are satisfied (see the statement of Problem 1). Due to the possible peaking of the state estimate caused by the high-gain observer, the state estimate during the short period at the beginning of time is not useful. To insure that the constraints are not violated, we need to borrow an idea from Esfandiarri and Khalil (1992), and saturate the control so that the peaking signal does not enter the plant. The appropriate level of saturation is to be specified below. In this way the control law always generates bounded control signal, regardless of the peaking. Then, the design objective is to guarantee that the control law is functioning as closely as the state feedback law after the peaking is over. But during the short period of peaking, the state starting from \( \mathcal{X} \) may drift to a larger set, say \( \mathcal{X} \), because we are not controlling the plant, although the input is kept bounded. For this reason, we need two components in our design. One is to design from the beginning a state feedback \( u = Fx \) for a larger set of initial condition, say set \( \mathcal{X} \) which satisfies \( \mathcal{X} \subset \text{int} \mathcal{X} \subset \text{int} A(S) \), and make sure that \( \mathcal{X} \) is contained in the domain of attraction and for all initial conditions in the set \( \mathcal{X} \) the output \( z(t) \in \rho S \) for some \( \rho \in (0, 1) \). One can use the design technique provided in (Saberi et al. 2002) for this task. The other
component is a saturation element, the level of which is specified below.

Consider the system
\[ \dot{x} = (A + BF)x + BFe \]
where \( e \) is the estimation error. It follows from Lemma 1 that if \( e(t) \) satisfies
\[ \|e(t)\| \leq \epsilon e^{-rt} \] (16)
for certain \( \epsilon \in (0, 1) \), \( r > 0 \) and for all \( t > 0 \), then for all initial conditions in the set \( \mathcal{X} \) the constraints on \( z \) are satisfied, meanwhile the state trajectory remains in a compact set \( \Omega(\mathcal{X}) \subset \text{int } A(S) \) and the state converges to zero. Hence, there exists an \( M_1 > 0 \) such that \( \|Fx\|_\infty \leq M_1 \) for all \( x(0) \in \mathcal{X} \) and for all \( e \) satisfying (16), where
\[ M_1 = \sup_{x \in \Omega(\mathcal{X})} \|Fx\| \]
Define \( M_2 = \epsilon \|F\| \). Let \( \tau > 0 \) be such that
\[ \dot{x} = Ax + Bu \]
satisfies \( x(t) \in \mathcal{X} \) for all \( t \in [0, \tau] \) for all \( u \) satisfying \( \|u(t)\| \leq M_1 + M_2 \) and for all \( x(0) \in \mathcal{X} \subset \text{intr } \mathcal{X} \). Then, we can choose the observer gain matrix \( L \) so that the error bound (16) holds for \( t \geq \tau \), where \( \tau, \epsilon \) and \( r \) are as specified above. Consequently, the combination of the observer and the state feedback
\[ u = \text{sat}_{M_1 + M_2}(Fx) \]
where \( \text{sat}_M(\cdot) \) is a standard saturation function with saturation level \( M \), has the following properties: For any given initial state \( x(0) \in \mathcal{X} \), we have \( x(t) \in \mathcal{X} \) for all \( t \in [0, \tau] \); and for \( t \geq \tau \) we have
\[ u(t) = \text{sat}_{M_1 + M_2}(F\dot{x}(t)) = F\dot{x}(t) = Fx(t) + Fe(t) \]
Then stabilization follows from Lemma 2. \( \square \)

Note that an explicit construction of the state feedback relies on the results from Saberi et al. (2002) which explicitly used the property of right-invertible constraints. However, even in the case the constraints are not right-invertible, we can use the above methodology to solve the measurement feedback problem provided we have obtained somehow a linear state feedback which solves the constrained stabilization problem. The difficulty of course is that for the design of this state feedback there exist neither necessary and sufficient conditions for the existence of such a feedback nor a constructive method to obtain such a feedback. However, the above derivation yields the following theorem.

**Theorem 2:** Consider a system of the form (4) with a constraint set \( S \subset \mathbb{R}^p \) satisfying Assumption 1. Assume \( (C_y, A) \) is observable. The problem of constrained semi-global stabilization via measurement feedback is solvable if and only if the problem of constrained semi-global stabilization via linear state feedback is solvable.

4. **Constrained semi-global output regulation with measurement feedback**

In this section we solve the constrained semi-global output regulation via measurement feedback as defined in Problem 2. As in the previous section, we focus only on the right invertible constraints in this section.

In the study of classical output regulation problems without constraints, it is well known that the following assumptions are standard (see Saberi et al. 2000 for details).

**Assumption 2:** There exist matrices \( \Pi \) and \( \Gamma \) satisfying the regulator equations,
\[ \Pi S = A\Pi + B\Gamma + E \]
\[ 0 = C \Pi + D \]

**Assumption 3:** The matrix \( S \) has all its eigenvalues in the closed right-half plane.

**Assumption 4:** The pair \( (A, B) \) is stabilizable.

**Assumption 5:** The pair
\[ \begin{bmatrix} C_y & D_y \end{bmatrix}, \begin{bmatrix} A & E \\ 0 & S \end{bmatrix} \]
is observable.

Note that Assumptions 2 and 4 are necessary and Assumption 3 is natural. However, Assumption 5 is not necessary and can be relaxed. For a detailed discussion see Saberi et al. (2000).

Under these assumptions, the solvability conditions for the constrained semi-global output regulation via measurement feedback are stated in the following theorem.

**Theorem 3:** Consider system (4) with a constraint set \( S \subset \mathbb{R}^p \) satisfying Assumption 1 and a compact set \( \mathcal{W} \subset \mathbb{R}^q \). Assume that the constraint is right invertible and let Assumptions 2 to 5 be satisfied. Then the constrained semi-global output regulation via measurement feedback as defined in Problem 2 is solvable if the following conditions hold:

(i) The constraint is at most weakly non-minimum phase.

(ii) There exists a \( \rho \in (0, 1) \) such that for all \( w(0) \in \mathcal{W} \)
\[ (C\Pi + D\Gamma)w(t) \in (1 - \rho)S, \quad \text{for all } t \geq 0. \]
(iii) The signals $\Pi w$ and $\Gamma w$ are bounded. Moreover, condition (i) is necessary.

Remark: It is shown in Saberi et al. (2001) that conditions (i), (ii), and (iii) are sufficient for constrained semi-global output regulation of linear plants with right invertible constraints via state feedback. Condition (ii) is the so-called constraint compatibility condition of the reference/disturbance signal. The necessity of condition (i) is due to the requirement of internal stabilization (see Theorem 1). Similar to the remark after Theorem 1, this theorem is also a generalization of the corresponding problem with input-only constraints.

We need some preparation before the proof of this theorem. For the purpose of design, it would be helpful to build a connection between the output regulation problem and the stabilization problem by utilizing the regulator equations (17) in Assumption 2. Let $(\Pi, \Gamma)$ be a solution to the regulator equations (17). Define the new variables

$$\tilde{x} = x - \Pi w, \quad \tilde{u} = u - \Gamma w, \quad \tilde{z} = z - \Lambda w$$

where $A = C_2 \Pi + D_2 \Gamma$. Then it is readily verified that system (4) in terms of the new variables becomes

$$\begin{cases}
\dot{x} = Ax + Bu \\
\dot{z} = C_2 \tilde{x} + D_2 \tilde{u} \\
e = C_0 \tilde{x}
\end{cases}$$

Note that in (19) the dynamics of exosystem has been absorbed in the dynamics of the new variables. System (19) has a remarkable property, it tells us that the output regulation can be achieved by designing a controller that stabilizes this system. That is, if we design $\tilde{u}$ such that the closed-loop system is internally stable, then we have $e(t) \to 0$ as $t \to \infty$, which is one of the goals for output regulation. However, this does not solve the whole problem, for we have constraints on the constrained output. In other words, in addition to achieving regulation, the control law constructed must also guarantee that $z = z + Aw$ satisfies the constraints, i.e. $z(t) + \Lambda w(t) \in S$ for all $t \geq 0$.

Based on the idea said above, a state feedback design for constrained output regulation has been proposed in Saberi et al. (2001). In that design some special care has been taken to guarantee that the constraint is not violated. In order to implement such a state feedback regulation law with an observer, we need a robustness lemma similar to Lemma 1 for output regulation.

Lemma 3: Consider the system

$$\begin{cases}
\dot{x} = A_d \tilde{x} + E_d \mu \\
\dot{z} = C_d \tilde{x} + D_d \mu
\end{cases}$$

where $\tilde{x}$ and $\tilde{z}$ are as defined in (18), and $A_d$ is a Hurwitz stable matrix. Let $S$ be a given set satisfying Assumption 1. Let $\mathcal{X} \subset \text{int} \mathcal{A}(S)$ be a compact set in $\mathbb{R}^n$ and $\mathcal{W}$ a compact set in $\mathbb{R}^m$. Let the condition (ii) in Theorem 3 hold. Assume there exists $\rho_1 \in (\rho, 1)$ such for all $x(0) \in \mathcal{X}$, $w(0) \in \mathcal{W}$ and $\mu(t) = 0$, we have $z(t) \in \rho_1 S$ for all $t \geq 0$. Then there exist $\varepsilon > 0$, $r > 0$, and a compact set $\Omega(\mathcal{X}) \subset \text{int} \mathcal{A}(S)$ such that for all $x_0 \in \mathcal{X}$ and for all $\mu(t)$ satisfying

$$\|\mu(t)\| \leq \varepsilon e^{-rt}, \quad \forall t \geq 0$$

we have that $z(t) \in S$ and $x(t) \in \Omega(\mathcal{X})$ for all $t \geq 0$ and $x(t) \to 0$ as $t \to \infty$.

Proof: The proof is a slight modification of the proof of Lemma 1. By the definition of $\tilde{z}$ we have

$$z(t) = \tilde{z}(t) + \Lambda w(t) = C_d \tilde{x}(t) + \Lambda w(t) + D_d \mu(t)$$

We can decompose the output $z$ in two components $z = z_0 + z_\mu$, where

$$z_0(t) = C_d e^{A_d t} \tilde{x}(0) + \Lambda w(t)$$

$$z_\mu(t) = \int_0^t C_d e^{A_d (t-\tau)} E_d \mu(\tau) d\tau + D_d \mu(t)$$

As in the proof of Lemma 1, for any fixed $r$ we can choose $\varepsilon \in (0, 1)$ such $z_\mu(t) \in (1 - \rho_1) S$. By assumption, we have $z_0(t) \in \rho_1 S$ for all $x(0) \in \mathcal{X}$ and $w(0) \in \mathcal{W}$. Putting together, we get that $z(t) \in S$ for all $t \geq 0$. Similarly to the proof of Lemma 1, we can show that there exists an $\Omega(\mathcal{X}) \subset \text{int} \mathcal{A}(S)$ such that $x(t) \in \Omega(\mathcal{X})$ for all $t \geq 0$.

Now we are ready to present a proof for Theorem 3.

Proof of Theorem 3: The necessity of condition (i) of Theorem 3 follows from the requirement of stability (see Saberi et al. 2002). We prove the sufficiency of the conditions in this theorem by an explicit design. In this proof we will only show that the condition (iii) of Problem 2 is satisfied for a sufficiently large time $t_0 > 0$. The requirement for arbitrary $t_0 > 0$ can be achieved by a slight modification of the proof. A state feedback law under the same solvability conditions (without the boundedness assumption on $\Gamma w$) has been designed in Saberi (2001). In this proof we replace the state used in the state feedback by its estimate.

As in the stabilization case, we introduce a high-gain based observer to obtain accurate estimate of the state for feedback. By Assumption 5, there exists an observer
gain matrix \( L = (L_x \ T \ L_w \ T) \) such that the eigenvalues of the matrix
\[
A_{\text{obs}} := \begin{pmatrix} A & E \\ 0 & S \end{pmatrix} - \begin{pmatrix} L_x \\ L_w \end{pmatrix} \begin{pmatrix} C_y & D_y \end{pmatrix}
\]
(22)
can be assigned anywhere in the left-half complex plane. The observer for the state variables \( x \) and \( w \) takes the standard form
\[
\begin{align*}
\dot{\hat{x}} &= \begin{pmatrix} A & E \\ 0 & S \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{w} \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u \\
&\quad + \begin{pmatrix} L_x \\ L_w \end{pmatrix} [y - C_y \hat{x} - D_y \hat{w}]
\end{align*}
\]
Let \( e_x = \hat{x} - x \) and \( e_w = \hat{w} - w \) be the estimation errors, and denote \( e = (e_x^T, e_w^T)^T \). It is clear that the error vector \( e \) is governed by the exponentially stable system:
\[
\dot{e} = A_{\text{obs}} e
\]
with initial conditions \( e(0) \) in a bounded set determined by \( \mathcal{X}, \mathcal{W}, \) and \( \mathcal{V} \) (see the formulation of Problem 2).

By Lemma 2, one can choose the observer gain matrix \( L \) so that the estimation error satisfies \( \|e(t)\| \leq \varepsilon e^{-r(t-t)} \), for all \( t \geq t \) and all initial states. The choice of such an \( L \) is dictated by \( \varepsilon, r, \) and \( \tau \) to be specified below.

The philosophy is the same as that in stabilization. We design a linear stabilizing feedback regulation law \( F \) for all initial conditions in the set \( \mathcal{X} \), which satisfies \( \mathcal{X} \subset \mathcal{Y} \subset \mathcal{A}(\mathcal{S}) \), so that the output \( z \) satisfies \( z(t) \in \mathcal{P}_t \mathcal{S} \) and we achieve asymptotic tracking whenever \( e \) satisfies (16) and \( \rho_1 \in (\rho, 1) \). This can be done by using the state feedback law developed in (Saberi et al. 2001) in combination with Lemma 3.

Because of the connection between the output regulation and stabilization, the remaining argument is very similar to that in the proof of Theorem 1. By Lemma 3 and condition (iii), for all \( x(0) \in \mathcal{X} \) and \( w(0) \in \mathcal{W} \) there exists an \( M_1 > 0 \) such that \( \|u\|_\infty \leq M_1 \), where
\[
u = Fx + Gw \text{ with } G = F - F\Pi,
\]
and \( \mathcal{W} \) is the set where all trajectories \( w(t) \) lie. Define \( M_2 = \varepsilon (\|F\| + \|G\|) \). For the given \( M_1 \) and \( M_2 \), we choose a \( \tau > 0 \) such that \( x(t) \in \mathcal{X} \) for all \( t \in [0, \tau] \) regardless of \( x(0) \in \mathcal{X}, \) \( w(0) \in \mathcal{W} \), and any \( u \) satisfies \( \|u\|_\infty \leq M_1 + M_2 \). With the specified \( \varepsilon, r, \) and \( \tau, \) we choose an appropriate observer gain matrix \( L \) so that the inequality (3.10) in Lemma 2 holds. Then it is easily verified that the observer, combined with the state feedback \( \hat{u} = \text{sat}_{M_1 + M_2} (Fx + G\hat{w}) \), solves the output regulation problem with constraints. This completes the proof.

We can obtain a similar result as in Theorem 2 for the problem of constrained semi-global regulation. Again an explicit construction of the state feedback relies on the results from Saberi et al. (2001) which used the property of right-invertible constraints. However, even in case the constraints are not right-invertible then we can use the above methodology to solve the measurement feedback problem provided we have obtained somehow a linear state feedback which solves the constrained stabilization problem. The difficulty of course is that for the design of this state feedback there exist neither necessary and sufficient conditions for the existence of such a feedback nor a constructive method to obtain such a feedback. Hence, the above derivation yields the following theorem.

**Theorem 4:** Consider a system of the form (4) with a constraint set \( \mathcal{S} \subset \mathbb{R}^p \) satisfying Assumption 1 and a compact set \( \mathcal{W} \subset \mathbb{R}^q \). Assume \( (C_y, A) \) is observable. The problem of constrained semi-global output regulation via measurement feedback is solvable if and only if the problem of constrained semi-global output regulation via linear state feedback is solvable.

5. Illustrative examples

In this section we give two examples to illustrate the design procedures presented in the proofs of Theorems 1 and 3. The first example is used to illustrate the constrained semi-global stabilization via measurement feedback, while the second one is for the constrained semi-global output regulation via measurement feedback. In connection with simulation, we also make some comments about the freedom in choosing a design.

**Example 1:** Consider the system
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= u \\
y &= x_1 \\
z &= x_3 \in [-1, 1] =: \mathcal{S}
\end{align*}
\]
This system is already in the format of scb and has a second order zero dynamics given by
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= z
\end{align*}
\]
and the remaining part of the system having relative degree one is given by
Clearly the constraint is weakly non-minimum phase and right invertible. Moreover, the constraint is of type one. We first design a measurement feedback law for semi-global stabilization starting from a linear state feedback law for the zero dynamics. Then, for comparison purpose, we design another measurement feedback law based on a non-linear state feedback law for the zero dynamics, by which we indicate that a non-linear feedback law has a potential to improve the performance.

Since we are concerned with semi-global stabilization and $z = x_3$ is constrained to $[-1, 1]$, for illustration purpose we shall design a control law so that any $(x_1, x_2, x_3) \in [-12, 12] \times [-8, 8] \times [-0.8, 0.8]$ is contained in the domain of attraction. In simulation, we always choose zero initial conditions for the observer.

The linear state feedback law for the zero dynamics is chosen to be $z = -k_1 x_1 - k_2 x_2$, where $k_1, k_2 > 0$ are gain parameters. To make sure that $z$ does not violate the constraint, we need to choose small enough gains if the initial conditions for $x_1$ and $x_2$ are possibly large. Then following the procedure in Saberi et al. (2002), we obtain

$$u = -\delta (x_3 + k_1 x_1 + k_2 x_2) - (k_1 x_2 + k_2 x_3)$$

where $\delta > 0$ is a design parameter.

The standard observer takes the form:

$$\dot{x}_1 = x_2 + L_1 (y - x_1)$$
$$\dot{x}_2 = x_3 + L_2 (y - x_1)$$
$$\dot{x}_3 = u + L_3 (y - x_1)$$

In simulation we choose $(L_1, L_2, L_3)$ so that the three repeated observer poles are placed at $s = \alpha$. Replacing the state by their estimates and adding a saturation with level one, we obtain the control input as

$$u = \text{sat}_1 (-\delta (x_3 + k_1 x_1 + k_2 x_2) - (k_1 x_2 + k_2 x_3))$$

Figure 2 shows the simulation result of the design by choosing $\delta = 5$, a moderate value, all the three observer poles at $s = -12$, and gain parameters $(k_1, k_2) = (-0.01, 0.1)$. The initial condition for the plant is fixed at $(12, 8, -0.8)$ and the initial condition for the observer is $(0, 0, 0)$.

In this paper we have restricted the design only to linear measurement feedback law. However, for the sake of improving performance, one might want to use non-linear measurement feedback law. It is not difficult to show that any non-linear semi-global stabilizing feedback law under which the closed loop system satisfies Lemma 1 can also be implemented using our special observer. We use the same example to illustrate this.

In order to generate a non-linear semi-global stabilizing feedback we follow the procedure suggested in Saberi (2002); however, with a non-linear state feedback

![Figure 2. Simulation result with a linear feedback law for the zero dynamics.](image-url)
for the zero dynamics by choosing $z = f(x_1, x_2) = \tanh(-k_1 x_1 - k_2 x_2)$, where $k_1, k_2 > 0$ are gain parameters. It gives rise to the semi-global stabilizing feedback

$$u = -\delta [x_3 + \tanh(k_1 x_1 + k_2 x_2)] - [1 - \tanh^2(k_1 x_1 + k_2 x_2)](k_1 x_2 + k_2 x_3)$$

In fact, for this example this non-linear state feedback achieves semi-global stabilization for any $k_1, k_2 > 0$ and $\delta > 0$. Moreover, the resulting closed-loop system satisfies the robustness properties given by Lemma 1.

Next, by replacing the states by their estimates and adding a saturation with level one, we obtain the observer based measurement feedback control input law as

$$u = \text{sat}_1(-\delta [\hat{x}_3 + \tanh(\hat{k}_1 \hat{x}_1 + \hat{k}_2 \hat{x}_2)]) - [1 - \tanh^2(\hat{k}_1 \hat{x}_1 + \hat{k}_2 \hat{x}_2)](\hat{k}_1 \hat{x}_2 + \hat{k}_2 \hat{x}_3)$$

We choose the same values for $\delta$, $\sigma$ and the initial conditions as in the previous design, except that the gain parameters are now chosen to be $(k_1, k_2) = (0.2, 2)$. The simulation result is shown in figure 3. Note that the constraint on $z = x_3 \in [-1, 1]$ is never violated. Also one may observe the faster settling of the state $x_3$ compared to the first design.

The next example is used to illustrate the design of a measurement feedback for constrained output regulation. We added some complexity to the system to make the simulation more interesting. We also introduce an exosystem so that the plant tracks a sinusoid.

**Example 2:** Consider the following plant

$$\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= u \\
y &= x_1
\end{align*}$$

with constraint

$$z = x_3 \in [-1, 1] =: \mathcal{S}$$

The tracking signal is generated by the exosystem

$$\begin{align*}
\dot{w}_1 &= -w_2 \\
\dot{w}_2 &= w_1
\end{align*}$$

with initial condition $w_1(0) = 0.5$, $w_2(0) = 0$. We are to design an observer based regulator so that $x_1$ tracks

![Figure 3. Simulation result with a non-linear feedback law for the zero dynamics.](image)
where \( w_1(t) = 0.5 \cos(t) \). For this goal, we introduce the tracking error equation

\[
e = x_1 - w_1
\]

In our standard terminology, this system has a second order zero dynamics with input \( z \)

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= z
\end{align*}
\]

and the second part with relative degree two as

\[
\begin{align*}
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= u
\end{align*}
\]

Clearly, the constraint on the given plant is right invertible and weakly non-minimum phase.

Solving the regulator equations yields,

\[
\Pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma = (1 \ 0)
\]

This leads to

\[
(C_2 \Pi + D_2 \Gamma)w(t) = -w_1(t) = -0.5 \cos(t) \in (1 - \rho)S
\]

for any \( \rho \in (0, 0.5] \). Thus, all the conditions of Theorem 3 are satisfied.

The design is accomplished by introducing the following variable changes: \( \dot{x} = x - \Pi w \) and \( \bar{u} = u - \Gamma w \), so that the output regulation problem can be treated as a stabilization problem. It is easy to verify that \( \bar{u} = -k_1 \bar{x}_1 - k_2 \bar{x}_2 \)

\[
\bar{x}_1 = x_1 - w_1, \quad \bar{x}_2 = x_1 + w_2, \quad \bar{x}_3 = x_3 + w_1, \quad \bar{x}_4 = x_4 - w_2 \quad \text{and} \quad \bar{u} = u - w_1.
\]

We outline our design as follows. First we choose a low gain feedback law for the zero dynamics as

\[
g_1 = -k_1 \bar{x}_1 - k_2 \bar{x}_2
\]

where \( k_1, k_2 > 0 \) are gains to be chosen small enough so that the domain of attraction can be sufficiently large. Here, a guideline for choosing \( k_1 \) and \( k_2 \) is to guarantee that \( |g_1| < \rho/2 \) for all possible initial states of \( \bar{x}_1 \) and \( \bar{x}_2 \) in a compact set containing zero. Then, we carry out the sequential design:

\[
g_2 = -\delta_1 (\bar{x}_3 - g_1) - \frac{dg_1}{dt}
\]

\[
g_3 = -\delta_2 (\bar{x}_4 - g_2) - \frac{dg_2}{dt}
\]

and the control input is chosen to be

\[
u = g_3 + w_1
\]

The next task is to design an observer. We assume that the tracking signal is available. Thus we only need to design an observer to estimate the plant state \( x = (x_1, x_2, x_3, x_4)^T \). We use the standard observer

\[
\dot{x} = A \dot{x} + Bu + L(y - \dot{x}_1)
\]

where the observer gain \( L \) is chosen so that the observer has repeated poles at \( s = a \). As we have discussed previously, the high gain observer may cause peaking estimation error. Hence, when we plug in the state estimates in the control law designed above, the peaking of the state estimate may propagate to state \( x_3 \) and cause constraint violation. For this reason, we need to cut off the peaking by adding a saturation element to \( u \). Thus, the control law becomes

\[
u = \text{sat}_M(g_3 + w_1)
\]

where \( M \) is a saturation level to be chosen.

In the simulation we have chosen the following parameters: \((k_1, k_2) = (0.01, 0.03)\) and \((\delta_1, \delta_2) = (7, 2)\). The repeated observer pole is assigned at \( a = -25 \). The saturation level is chosen to be \( M = 20 \). It can be verified that any initial condition of state \((x_1, x_2, x_3, x_4) \in [-3, 3] \times [-3, 3] \times [-0.5, 0.5] \times [-0.5, 0.5] \) is in the domain of attraction. Figure 4 shows the simulation result of the closed-loop system with the above designed measurement feedback law. The initial condition for plant is \((3, -3, -0.5, -0.5)\) and the initial condition for observer is zero. Figure 5 shows the transient details which is not clear from figure 4.

6. Conclusion

In this paper we developed the design of observer based measurement feedback controllers for the constrained semi-global stabilization and constrained semi-global output regulation for linear systems with right invertible constraints. The observers use high-gain feedback. To avoid the possible constraint violation caused by peaking phenomenon associated with an high-gain observer, appropriate saturation mechanisms are employed in the control laws. It turns out that the solvability conditions for the two problems under the measurement feedback are similar to those developed in Saberi et al. (2001, 2002) where only state feedback laws were considered. Some examples are given to illustrate the design procedures.

A major open problem is the case of non-right invertible constraints which remains a challenging and difficult problem.

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Figure 4. Simulation of tracking: $e$ is the tracking error, $x_3$ is the constrained state and $u$ is the saturated control.

Figure 5. The transient part of tracking.
References


