Analysing divergent logistic networks with local (R, S) inventory control*

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Abstract

This paper deals with divergent logistic networks where the inventory at each node is controlled using a periodic review strategy with order-up-to level. An approximate method is presented to analyse the network performance (service levels, mean physical stock). The method is tested on a range of 2-echelon and 3-echelon networks by comparison to results from Monte Carlo simulation. We conclude that the approximation accuracy is sufficient for global network design in many practical situations.

1. Introduction

The optimal control of logistic networks has received increasing interest in the past decade. It is recognized that companies can gain significant cost reductions by increasing their logistic performance. In the literature, the attention has been mainly focused on multi-echelon control policies. Optimal control policies have been developed for some multi-echelon systems (cf. Clark and Scarf [1], Langenhoff and Zijm [2] and references therein). A prerequisite for applying these policies is full information transparency in the logistic chain, i.e. at any stage in the network complete information about downstream stock and demand should be available.

Unfortunately full informational transparency can be hard to accomplish in practice because it may require major organizational changes. In that case a local inventory control policy should be used at each stockpoint in the network. It is then possible to coordinate control such that overall costs are reduced: increasing the inventory level at one stage in the network may enable other stages to reduce their inventory levels without affecting the customer service level at the downstream stages of the logistic network (i.e. the stages serving the external customers).

The analysis of such logistic networks with local control is scarcely discussed in the literature. Some results are available for divergent (distribution) networks where a continuous review (b, Q) policy is used at each stockpoint (cf. van Beek [3], Deuermeyer and Schwarz [4], and Svoronos and Zipkin [5]). Under a (b, Q) policy a stockpoint releases a replenishment order of fixed size Q each time the inventory position, defined as physical stock plus goods on order minus backorders, drops below the level b . A key part in the analysis is the calculation of the order delay due to stock-outs using Little's formula for the relation between the number of backorders and the delay.

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However, with periodic review policies, the delay cannot be obtained using Little's formula. In this paper we discuss an alternative approach for the approximate analysis of divergent logistic networks with a periodic review (R, S) inventory control policy at each stockpoint. Our analysis involves two key phases. Firstly, we derive expressions for the performance characteristics at time t by choosing a suitable initial system condition. Letting $t \rightarrow \infty$ yields (complex) expressions for the performance characteristics as a function of the order-up-to level S in a stationary situation. Secondly, we approximate these complex expressions as a function of S using a two-moment approximation (see also Ref. [6]).

A clear advantage of our approach is the fact that the number of restrictions on model parameters is limited. For example, Deuermeyer and Schwarz [4], and Svoronos and Zipkin [5] discuss a one-warehouse – multiple-retailer model. They assume purely Poisson customer demand at the retailers and furthermore identical size and delay of replenishment orders issued by the retailers. Zipkin [7] discusses a multi-echelon model with *compound* Poisson demand, in which each location follows a simple one-for-one replenishment policy. The contribution of the analysis in this paper is the possibility to combine several complex elements in one model. Our approach allows compound Poisson demand, different retailer control strategies and different delay of the retailers' replenishment orders.

Our main result is an approximate method to calculate mean physical stock and service levels in a distribution network with local (R, S) inventory control. Simulation results indicate that the approximation is quite accurate for the most important performance measures (the mean physical stock in the network and service levels to the customers). Errors on other performance measures (internal service levels in the network and mean physical stock at specific stockpoints) may be larger.

The remaining part of this paper is structured as follows. In the next Section we give a detailed model description and we introduce some basic notation. In Section 3 we give an outline of the approach used to approximate the network performance. We apply this approach to find expressions for performance characteristics of the downstream stockpoints and the other stockpoints (Sections 4 and 5, respectively). An algorithm description is given in Section 6. This algorithm is tested by comparison to Monte Carlo simulation. Experimental results for 2-echelon and 3-echelon networks are discussed in Sections 7 and 8, respectively. We end with some conclusions in Section 9.

2. Model description and basic notation

We consider a divergent logistic network through which a single product type is distributed. The network consists of three stockpoint types: downstream, upstream and intermediate stockpoints. A *downstream* stockpoint delivers products to customers outside the network. Its replenishment orders are delivered by a (upstream or intermediate) stockpoint in the network. An *upstream* stockpoint delivers products to one or more (intermediate or downstream) stockpoints in the network. Its replenishment orders are delivered by a source outside the network. An *intermediate* stockpoint is supplier to one or more (intermediate or downstream) stockpoints, while it receives replenishment orders from an (upstream or intermediate) stockpoint. An example of a divergent logistic network is shown in Fig. 1. We describe our model further on the basis of this example.

The upstream stockpoint in the network is a manufacturer's warehouse containing finished products. Replenishment orders are issued to the factory. These orders arrive at the warehouse after some (stochastic) manufacturing lead time. The manufacturer supplies products to two distribution centres. Each distribution centre supplies three wholesalers, while each wholesaler supplies a group of local customers (e.g. retailers). The network distribution process is triggered by the demand of these local customer groups. We assume a compound Poisson process for the

Fig. 1. Example of a divergent logistic network.

demand of each customer group. That is, external customers arrive at a downstream stockpoint (a wholesaler) according to a Poisson process. The demand per customer is a random variable. Both the arrival rate and the demand per customer may be different for each downstream stockpoint.

The normal time required to deliver an order from one stage to another (transport, paperwork) is a random variable. This normal order delivery time only concerns the period between order receipt at the supplier and order delivery at the customer. In other words, we assume that the time between order release by the customer and order receipt at the supplier is negligible. Furthermore, we assume that orders are delivered in the same order as they are released, so they do not overtake each other.

Orders arriving at a stockpoint on the basis of “First Come, First Served” (FCFS). If an order arrives and the available physical stock is insufficient to complete the order, it has to wait until additional stock has arrived. This waiting time is called *delay*. Thus the total order lead time consists of two parts: the normal order delivery time and the delay. A key part in the analysis is the determination of the delay and, hence, the total order lead time between each pair of stockpoints in the network.

Each stockpoint controls its inventory level using some periodic review (R, S) policy. This policy means that the inventory position, defined as physical stock plus goods on order minus backorders, is reviewed each R th time unit. At each review, a replenishment order is issued so that the inventory position immediately after placing the order equals the order-up-to level S . Each stockpoint uses its own control parameters R and S . We assume that the relation between the review periods of a receiving stockpoint j and its supplier i is described by a number $r_j \in \mathbb{N}$, defined as

$$r_j = R_i / R_j. \quad (2.1)$$

Such a relation exists for all customers j of stockpoint i . So the review period of stockpoint i is a multiple of the review period of each customer j . This assumption will rarely be a restriction in practise. Typically, going from upstream to downstream in the distribution network, the planning frequency decreases from monthly to daily. Furthermore, a value $r_j \notin \mathbb{N}$ results in a non-stationary demand process at the supplying stockpoint, i.e. the demand in consecutive review periods is not identically distributed. If average demand is higher in one review period than in the other, it is

intuitively inappropriate to apply the same reorder policy for both periods. Therefore, a choice of the review periods which ensures stationary demand processes at all stockpoints in the network is preferred. Assumption (2.1) implies that the cumulative demand in consecutive review periods is independent and identically distributed since the demand process at downstream stockpoints is compound Poisson. Consequently, consecutive replenishment order sizes are also independent and identically distributed.

An important issue in a network where all stockpoints use periodic review inventory control strategies is the *timing* of reviews. Different stockpoints may have the same review period (e.g. 1 week), but replenishment orders may be issued on different days. For example, one wholesaler reviews its inventory position each Monday, the other one each Friday. This timing should be incorporated in the model, because it may cause unbalanced service levels of a distribution centre to different wholesalers. We indicate the timing of the reviews at stockpoint j by a parameter $\Delta_j \in \langle 0, R_j \rangle$. The supplier of j (stockpoint i) receives r_j orders from stockpoint j during its review period R_i , which arrive at the times $\Delta_j + kR_j$ for $k = 0, 1, \dots, r_j - 1$.

Now we have completed our model description. To analyse the network performance we need approximations for the following variables for each stockpoint j in the network:

- (a) The delay distribution for each customer of stockpoint i . This distribution is required to obtain the distribution of the order lead time, which is the sum of normal order delivery time and the delay. It is important to approximate this distribution accurately because the lead time is a key variable in the analysis. We focus on the first two moments of the lead time distribution, since research has shown that for many stochastic models the first two moments of random variables involved are sufficient for an accurate approximation (see, e.g., Tijms [8]). Therefore, we need a method to approximate the first two moments of the delay for each intermediate stockpoint in the network.
- (b) The service level for each customer of stockpoint i , i.e. the fraction of demand that is delivered directly from stock on hand (without delay). We distinguish *internal* and *external* service levels. An internal service level is the service level offered by an intermediate stockpoint to another stockpoint in the network. An external service level concerns the service given by a downstream stockpoint to an external customer group. Of course the external service levels are most important for the network performance. Internal service levels are only instruments to achieve a target external customer service level.
- (c) The mean physical stock at stockpoint i . This is important as a cost factor.

Having a method to obtain these performance measures for each stockpoint, it is in principle possible to optimize the network performance. For example, choose the various order-up-to-levels S_i such that overall physical stock is minimized while satisfying some target external customer service level.

Before we present the analysis of the model, we introduce the following basic notation:

- Ω = Set of all stockpoints in the network.
- Ω^D = Set of all downstream stockpoints in Ω .
- Ω^I = Set of all intermediate stockpoints in Ω .
- Ω^U = Set of all upstream stockpoints in Ω .
- Ω_i = Set of all stockpoints which are directly supplied by stockpoint i , so
 $\Omega_i = \{j \in \Omega \mid \text{stockpoint } j \text{ issues replenishment orders to stockpoint } i\}.$
- ω_i = Set of all downstream stockpoints which are, directly or indirectly, supplied by stockpoint i . For example, we have in Fig. 1,
 $\omega_{\text{distribution centre 1}} = \{\text{Wholesaler A, B and C}\},$
 $\omega_{\text{Manufacturer}} = \{\text{Wholesaler A, B, C, D, E and F}\}.$

R_i	= Length of the period between two consecutive reviews at stockpoint i .
r_j	= Factor, indicating the relation between the review periods at stockpoint j and its supplier stockpoint i (according to (1.1)).
Δ_j	= Shift parameter, indicating the relation between the timing of reviews at stockpoint j and its supplier stockpoint i . Reviews at stockpoint j are planned at times $\Delta_j + kR_j$ after each review at stockpoint i ($k = 0, 1, \dots, r_j - 1$).
S_i	= Order-up-to level at stockpoint i .
P_i	= Normal delivery (or processing) time of a replenishment order released by stockpoint i , having mean μ_{P_i} and standard deviation σ_{P_i} .
L_i	= Lead time of a replenishment order released by stockpoint i . This lead time has probability distribution function $G_i(y)$, mean μ_{L_i} and standard deviation σ_{L_i} .
D_i	= Demand of an arbitrary customer arriving at stockpoint $i \in \Omega^D$. This demand has probability distribution function $F_i(x)$ with density $f_i(x)$, mean μ_{D_i} and standard deviation σ_{D_i} .
λ_i	= Customer arrival rate at stockpoint $i \in \Omega^D$. The customers arrive according to a Poisson process.
O_i	= Size of a replenishment order released by stockpoint i , having mean μ_{O_i} and standard deviation σ_{O_i} .
$N_i[t_1, t_2]$	= The number of replenishment orders released by stockpoint i in the time interval $[t_1, t_2]$.
$W_{i,t}$	= Delay (or waiting time) of a replenishment order issued by stockpoint i at time t .
W_i	= Delay of an arbitrary replenishment order issued by stockpoint i .
W_i^c	= Delay of an arbitrary customer arriving at stockpoint i .
β_i	= Service level to stockpoint i , i.e. the fraction of stockpoint i 's demand that is satisfied directly from stock on hand by its supplier.
β_i^c	= Service level of stockpoint i to its customers, i.e. the demand that is satisfied directly from stock on hand by stockpoint i .
$I_{i,t}$	= Physical stock of stockpoint i at time t .
I_i	= $\lim_{t \rightarrow \infty} I_{i,t}$ (the physical stock of stockpoint i at an arbitrary point in time).
$Q_i[t]$	= The number of replenishment orders delivered to stockpoint i in the time interval $[0, t]$.
$V_i[t_1, t_2]$	= The cumulative demand at stockpoint i in the time interval $[t_1, t_2]$.

3. Outline of the approach

To analyse the network, we take the following two steps:

Step A: Determine the demand process at each stockpoint. For downstream stockpoints this is a given compound Poisson process. For upstream and intermediate stockpoints the demand consists of replenishment orders from other stockpoints. Hence, we need the distribution of the replenishment order size for each stockpoint in the network. Because we have compound Poisson demand at the downstream stockpoints *and* because of assumption (2.1) it follows that consecutive replenishment orders of each stockpoint are independent and identically distributed. In fact, the replenishment order size of an (upstream or intermediate) stockpoint $i \in \Omega^U \cup \Omega^I$ is the superposition of the independent compound Poisson demand processes at the stockpoints $j \in \omega_i$ during a time period R_i . Application of Wald's equation yields for the first three central moments of the

replenishment order size O_i

$$E[O_i] = \sum_{j \in \omega_i} \lambda_j * R_i * E[D_j], \quad (3.1)$$

$$\text{VAR}[O_i] = \sum_{j \in \omega_i} \lambda_j * R_i * E[D_j^2], \quad (3.2)$$

$$m_3[O_i] = E[O_i - E[O_i]]^3 = \sum_{j \in \omega_i} \lambda_j * R_i * E[D_j^3]. \quad (3.3)$$

As will be seen later we only need the first three moments of the replenishment order size distribution for our approximations.

Step B: Determine the performance characteristics, based on the demand process as specified in step A. We start with the performance of the upstream stockpoint. We derive approximations for the mean physical stock at the upstream stockpoint, the first two moments of the delay incurred by each customer's orders and the service level to each customer. The first two moments of the delay are required to approximate the distribution of the lead time to each customer. Using this lead time distribution *and* the demand process as specified in step A we can determine the performance of each customer. In this way, working from upstream to downstream stockpoints, the network is analysed.

Now the crucial point is to determine the performance of some stockpoint $i \in \Omega$ given the following parameters:

- (a) The reorder policy parameters R_i , S_i and Δ_i .
- (b) The lead time distribution of replenishment orders issued by stockpoint i .
- (c) The demand process at stockpoint i , consisting of an arrival process and the demand size distribution. For downstream stockpoints, the demand process is a compound Poisson process. For upstream and intermediate stockpoints, the demand arrival process is determined by the timing of the reviews at the stockpoints $j \in \Omega_i$ (i.e. the parameters r_j and Δ_j). The demand size distribution is determined by the replenishment order size distribution of the stockpoints $j \in \Omega_i$ (given by (3.1)–(3.3)).

Because the demand process for upstream and intermediate stockpoints is clearly different from the demand process at downstream stockpoints, other approximate formulas have to be used. However, the basic approach is the same. In this Section, we give an outline of this basic approach. Application to downstream respectively upstream and intermediate stockpoints are discussed in the next two Sections.

To approximate the stockpoint performance, we apply the *PDF-method* as introduced by de Kok [6]. The essence of this method is the following:

- (a) Consider a performance characteristic in a stochastic process as a function of one key variable. In our model, we consider a performance characteristic of stockpoint i as a function of the order-up-to-level S_i . Call this function $A(S_i)$.
- (b) Find a function $H(S_i)$ which satisfies the following:
 - (i) $H(S_i)$ is a one-to-one transformation of $A(S_i)$.
 - (ii) $H(S_i)$ is a monotonically, non-decreasing function of S_i with

$$H(0) = 0 \quad \text{and} \quad \lim_{S_i \rightarrow \infty} H(S_i) = 1.$$

- (c) Approximate $H(S_i)$ by a simple probability distribution function (PDF) $H^*(S_i)$ using a two- or three-moment fit. A good candidate for such a PDE is a Coxian distribution. Expressions for a two- and three-moment fit can be found in Tijms [8] and van der Heijden [9].

(d) Now we can easily approximate the performance characteristic for a range of values S_i . Evaluation of $H^*(S_i)$ is simple and by applying the inverse of the transformation (see (b)) to $H^*(S_i)$ we can obtain the desired performance characteristic for a range of values S_i .

In fact there is in the case of a Coxian distribution no clear theoretical base for the PDF-method. However, de Kok [6] shows that it leads to reasonably accurate approximations for a range of single stockpoint inventory models. The power of the PDF-method is the possibility to convert complicated expressions to computationally attractive approximations. The method is only applicable if a simple transformation $H(S_i)$ can be found and the moments of $H(S_i)$ can be obtained easily.

What remains is the derivation of expressions for the relevant performance measures to which the PDF-method can be applied. We deal with this in the next two Sections.

4. Analysis of downstream stockpoints

For downstream stockpoints we use the approach as introduced by van der Heijden and de Kok [10] for a single-stockpoint (R, S) -model with compound Poisson demand. We proceed from an initial time $t = 0$ when we have the following situation: the inventory position at stockpoint i has just been reviewed and hence equals S_i . There are no replenishment orders yet to receive nor backorders to deliver, so that the physical stock equals S_i as well. Starting from this situation it is possible to derive expressions for the delay of a customer arriving at time $t > 0$ as well as the physical stock at time $t > 0$. Letting $t \rightarrow \infty$ we obtain expressions for the stationary delay distribution and the stationary mean physical stock. An expression for the customer service level is obtained using intermediate results for the mean physical stock. In this way van der Heijden and de Kok [10] obtain the following expressions for the performance of downstream stockpoints:

$$\begin{aligned}
 E[W_i^c] &= \frac{1}{R_i} * \int_0^{R_i} (R_i - t) * \Pr\{V_i^\infty[t] + D_i > S_i\} dt \\
 &\quad + \frac{1}{R_i} * \int_0^\infty [1 - G_i(w)] \int_w^{w+R_i} \Pr\{V_i^\infty[t] + D_i > S_i\} dt dw \\
 &\text{with } V_i^\infty[t] = \lim_{m \rightarrow \infty} V_i[mR_i, mR_i + t],
 \end{aligned} \tag{4.1}$$

$$\begin{aligned}
 E[(W_i^c)^2] &= \frac{1}{R_i} * \int_0^{R_i} (R_i - t)^2 * \Pr\{V_i^\infty[t] + D_i > S_i\} dt + 2 * \int_0^\infty [1 - G_i(w)] \int_0^{w+R_i} \Pr\{V_i^\infty[t] \\
 &\quad + D_i > S_i\} dt dw - \frac{2}{R_i} * \int_0^\infty [1 - G_i(w)] \int_w^{w+R_i} (t - w) * \Pr\{V_i^\infty[t] + D_i > S_i\} dt dw,
 \end{aligned} \tag{4.2}$$

$$\beta_i^c = \frac{1}{R_i E[D_i]} * \int_0^\infty [G_i(t) - G_i(t - R_i)] * \int_0^{S_i} y f_i(y) * \Pr\{V_i^\infty[t] \leq S_i - y\} dy dt, \quad (4.3)$$

$$\begin{aligned} E[I_i] &\approx \frac{1}{R_i} * \int_0^\infty [G_i(t) - G_i(t - R_i)] * \int_0^{S_i} \Pr\{V_i^\infty[t] \leq x\} dx dt \\ &\quad + \frac{E[D_i^2]}{2 * E[D_i]} * \int_0^\infty [G_i(t) - G_i(t - R_i)] * \Pr\{V_i^\infty[t] > S_i\} dt. \end{aligned} \quad (4.4)$$

The expressions (4.1)–(4.4) are suitable for the application of the PDF-method, but first we have to find a suitable transformation $H(S_i)$ for each performance measure. Such a transformation should satisfy conditions (i) and (ii) given in Section 3. This can be achieved in several ways. We focus on simple linear transformations, because then the first two moments of the random variable X associated with the distribution $H(S_i)$ can be obtained easily. Other transformations may lead to intractable expressions. Such a linear transformation is possible for the mean delay as a function of the order-up-to level S_i . Using elementary algebra we see that this function is monotonically non-increasing in S_i with

$$\lim_{S_i \rightarrow \infty} E[W_i^c(S_i)] = 0 \quad \text{and} \quad \lim_{S_i \downarrow 0} E[W_i^c(S_i)] = E[L_i] + \frac{1}{2} R_i.$$

Hence the required conditions for the function $H(S_i)$ are satisfied if we use the simple linear transformation

$$H_1(S_i) = 1 - \frac{E[W_i^c(S_i)]}{E[L_i] + \frac{1}{2} R_i}. \quad (4.5)$$

Similarly we obtain a simple linear transformation for the second moment of the delay as a function of the order-up-to level S_i :

$$H_2(S_i) = 1 - \frac{E[(W_i^c(S_i))^2]}{E[L_i^2] + R_i E[L_i] + \frac{1}{3} R_i^2}. \quad (4.6)$$

To apply the PDF-method to the service level, no transformation is required. We can simply use

$$H_3(S_i) = \beta_i^c(S_i). \quad (4.7)$$

Finally, the mean physical stock as a function of the order-up-to level S_i can be shown to be monotonically non-decreasing in S_i with

$$\lim_{S_i \rightarrow \infty} \frac{E[I_i(S_i)]}{S_i} = 1, \quad \lim_{S_i \rightarrow \infty} S_i - E[I_i(S_i)] = n_\infty = \lambda_i E[D_i] * (\frac{1}{2} R_i + E[L_i]),$$

$$\lim_{S_i \downarrow 0} E[I_i(S_i)] = n_0 = \frac{E[D_i^2]}{2E[D_i]}.$$

Because $E[I_i(S_i)]$ is proportional to S_i , we have to use a linear transformation of $S - E[I_i(S_i)]$ instead of $E[I_i(S_i)]$ itself. The following transformation satisfies all conditions of Section 3 and leads to tractable expressions for the first two moments of the random variable X_4 associated with $H_4(S_i)$:

$$H_4(S_i) = \frac{S_i - E[I_i(S_i)] + n_0}{n_\infty + n_0}. \quad (4.8)$$

The first two moments of the random variables X_1 – X_4 are given in Appendix 1.

5. Analysis of upstream and intermediate stockpoints

To derive expressions for the performance measures for upstream and intermediate stockpoints, we use a modification of the approach as described in Ref. [10]. Given the situation at the initial time $t = 0$ (see Section 4), they derive the following expression for the delay distribution of a replenishment order issued by stockpoint j (having supplier i) arriving at $t = T + mR_i$:

$$\Pr\{W_{j,T+mR_i} > w\} = \sum_{k=0}^m \Pr\{Q_i[T + mR_i + w] = k\} * \Pr\{V_i[kR_i, mR_i + T] > S_i\}, \quad w \geq 0. \quad (5.1)$$

Here we assume that replenishment orders arrive at the supplier (stockpoint i) one by one. Note that the delay may be different for the various customers of stockpoint i . The derivation of expression (5.1) for Poisson arrivals is given in Ref. [10]. This involves two steps. First, compound Poisson demand implies that arrivals occurring within a review period are continuously uniformly distributed over the time interval with length R_i . Hence T can be eliminated from (5.1) by weighting by this continuous uniform distribution. Second, letting $m \rightarrow \infty$ yields an expression for the stationary delay distribution. This expression is used to derive (4.1) and (4.2).

Now consider the arrival process of replenishment orders from stockpoint j at the intermediate or upstream stockpoint i . It is clear that these orders arrive at fixed times which are determined by the time coordination parameters r_j and Δ_j (see Section 2). Hence arrivals within a review period are discretely uniformly distributed over the time interval with length R_i : orders arrive at times $\Delta_j + kR_j$ for $k = 0, \dots, r_j$. Weighting by this discrete uniform distribution and letting $m \rightarrow \infty$ yields

$$\begin{aligned} \Pr\{W_j > w\} &\approx \frac{1}{r_j} * \sum_{m=0}^{\infty} [G_i(\Delta_j + mR_j + w) \\ &\quad - G_i(\Delta_j + (m - c_j)R_j + w)] * \Pr\{V_i^\infty[\Delta_j + mR_j] > S_i\}, \quad w \geq 0. \end{aligned} \quad (5.2)$$

Integration over w gives the following expression for the mean delay:

$$E[W_j] \approx \frac{1}{r_j} * \sum_{m=0}^{\infty} \Pr\{Z_{j,m} > S_i\} * a_{j,m}, \quad (5.3)$$

where

$$Z_{j,m} = V_i^\infty[\Delta_j + mR_j] \quad \text{and} \quad a_{j,m} = \int_{\Delta_j + (m-r_j)R_j}^{\Delta_j + mR_j} [1 - G_i(x)] dx.$$

Similarly we can derive expressions for the second moment of the delay, the (internal) service level and the mean physical stock. After some algebra we obtain

$$E[W_j^2] \approx \frac{1}{r_j} * \sum_{m=0}^{\infty} \Pr\{Z_{j,m} > S_i\} * [b_{j,m} - 2 * (\Delta_j + (m - r_j)R_j) * a_{j,m} + 2r_jR_j * c_{j,m}], \quad (5.4)$$

where

$$b_{j,m} = \int_{\Delta_j + (m-r_j)R_j}^{\Delta_j + mR_j} 2x * [1 - G_i(x)] dx \quad \text{and} \quad c_{j,m} = \int_{\Delta_j + mR_j}^{\infty} [1 - G_i(x)] dx,$$

$$\beta_j \approx \frac{1}{r_j * E[O_j]} * \sum_{m=0}^{\infty} d_{j,m} * \int_0^{S_i} y f_j(y) * \Pr\{Y_{j,m} < S_i - y\} dy, \quad (5.5)$$

where

$$Y_{j,m} = V_i^\infty[\Delta_j + mR_j] \quad \text{and} \quad d_{j,m} = G_i(\Delta_j + mR_j) - G_i(\Delta_j + (m - r_j)R_j).$$

Note that the service level β_j of stockpoint j depends on its specific inventory control parameters. It is possible to specify different control parameters for different customers j of the same stockpoint i , as was mentioned in the introduction.

Finally, expression (4.4) for the mean physical stock remains valid, although application of the PDF-method leads to a different derivation. Both the transformation to a probability distribution function $H(S_i)$ and the derivation of the first two moments of the random variable associated with $H(S_i)$ are discussed in Appendix 2 for all performance measures.

6. Algorithm for network analysis

Now we have all the tools for an approximate analysis of a divergent logistic network. In this Section we summarize the steps to be taken.

- (A) Determine the first three moments of the replenishment order size for each stockpoint according to (3.1)–(3.3).
- (B) Initialize the current set of stockpoints to be analysed as $\Psi = \Omega^U$.
- (C) Analyse the performance of all stockpoints $i \in \Psi$ using the PDF-method. That is, approximate the probability distribution functions $H(S_i)$ by a Coxian distribution $H^*(S_i)$ using a two-moment fit for the following performance characteristics:

- (i) The mean delay for each customer of stockpoint i . If stockpoint i is a downstream stockpoint the customer is an external customer group, otherwise all stockpoints $j \in \Omega_i$ are customers. The first two moments of X are given by (A1.1) and (A1.2) if stockpoint i is a downstream stockpoint, and by (A2.2) and (A2.3) otherwise.
 - (ii) The second moment of the delay for each customer of stockpoint i . The first two moments of X are given by (A1.3) and (A1.4) in the case stockpoint i is a downstream stockpoint, and by (A2.9) and (A2.10) otherwise.
 - (iii) The service level for each customer of stockpoint i . The first two moments of X are given by (A1.5) and (A1.6) if stockpoint i is a downstream stockpoint, and by (A2.15) and (A2.16) otherwise.
 - (iv) The mean physical stock at stockpoint i . The first two moments of X are given by (A1.7) and (A1.8) in the case stockpoint i is a downstream stockpoint, and by (A2.21) and (A2.22) otherwise.
- (D) Determine the first two moments of the lead time from stockpoint i to stockpoint j for all $j \in \Omega_i$. Recall that the lead time is the sum of the normal order delivery time (an input parameter) and the delay, so

$$E[L_j] = E[P_j] + E[W_j], \quad E[L_j^2] = E[P_j^2] + 2E[P_j]E[L_j] + E[L_j^2],$$

Approximate the lead time distributions by Coxian distributions using a two-moment fit.

- (E) Determine the set of stockpoints to be analysed next as

$$\Psi_{\text{new}} := \bigcup_{i \in \Psi} \Omega_i.$$

If $\Psi = \emptyset$ then stop, else return to (C).

Note that the PDF-method is suitable for approximating the stockpoint performance for a range of order-up-to-levels S_i . We can use the PDF-method inversely too. For example we can fix the service level and find the required order-up-to-level S_i using a numerical routine. The approximating probability distribution functions $H^*(S_i)$ are evaluated quickly, so service level constraints can be handled easily.

Although we can evaluate the stockpoint performance for a range of order-up-to-levels S_i simply, we should be careful when analysing the entire network. A change in the order-up-to level at stockpoint i means a change in the delay and hence in the lead time distribution to all customers j of stockpoint i . Therefore, the approximating probability distribution functions $H^*(S_j)$ are no longer valid for all $j \in \Omega_i$ and these functions should be recalculated.

To test the approximate method as summarized in this Section, we make a comparison with results from Monte Carlo simulation. We consider two main model types, 2-echelon models and 3-echelon models. For each model a simulation of 100,000 time units is run. This means that at least 100,000 external customers per downstream stockpoint arrived during the simulation. Each simulation run is preceded by a run-in period of 1,000 time units to reach a stationary situation. No data are collected during this period.

We focus on the approximation accuracy for the service levels at various levels in the network and the mean physical stock in the entire network. The results of the comparison are discussed in the next two Sections (for 2-echelon and 3-echelon models respectively).

7. Numerical results for 2-echelon models

We consider the following type of 2-echelon models. There is one distribution centre, which supplies five wholesalers. Each wholesaler supplies one external customer group. We assume that these customer groups behave identically (i.e. the same customer arrival rate and the same demand size distribution). The distribution centre issues replenishment orders to a manufacturer which is not part of the network. This model is specified in detail by choosing the following parameters:

- (a) The length of the review period for the distribution centre (R_d) and for the wholesalers (R_w).
- (b) The timing of the reviews at the wholesalers relative to the reviews at the distribution centre (the values Δ_j).
- (c) The customer arrival rate at each wholesaler (λ). We assume a Poisson arrival process.
- (d) The distribution of the customer demand at each wholesaler. We assume a Coxian-2 distribution for the demand. Hence, the demand is characterized by the mean μ_D the square of the coefficient of variation $c^2[D]$. Note that the square of the coefficient of variation of a random variable X is defined as

$$c^2[X] = \frac{\text{VAR}[X]}{E^2[X]}.$$

- (e) The normal order delivery time distribution for the distribution centre and for the wholesalers. We assume this distribution is Coxian-2 or deterministic and so it is characterized by the mean (μ_{P_d} and μ_{P_w} respectively) and the square of the coefficient of variation ($c^2[P_d]$ and $c^2[P_w]$ respectively).
- (f) The service level offered to the wholesalers (β_w) by the distribution centre and the service level offered to the external customer groups (β_w^c) by the wholesalers. We assume that all wholesalers should offer the same service level to their customers. We use our approximate method to find the order-up-to levels S_i such, that the prespecified service levels are met.

Without loss of generality we may fix the mean customer demand $\mu_D = 10$ and the review period at the wholesalers $R_w = 5$. In fact, we choose a quantity dimension and a time dimension. To reduce the number of parameters involved, we fix the values $\Delta_j = 0.9, 1.9, 2.9, 3.9$ and 4.9 for wholesalers 1 to 5, respectively.

To examine the performance of our approximate method under various circumstances, we vary the remaining 9 model parameters as shown in Table 1. If we examine all possible combinations of parameter values, we have to carry out $2^9 = 512$ numerical experiments. This means a lot of work. To reduce the number of experiments we can choose between two options:

- (a) Reduce the number of parameters involved. Fixing a value for a few parameters implies that less experiments have to be carried out. However, the effects of those fixed parameters on the approximation accuracy cannot then be examined.
- (b) Carry out a selection of numerical experiments in which all parameter values are involved. By proper selection of experiments using the theory of experimental design (see e.g. Montgomery [11]) we can examine the most important parameter effects on the approximation accuracy using a limited number of experiments.

We choose for the second option and select a quarter fraction of a 2^9 factorial design. Hence, we carry out only $\frac{1}{4} * 2^9 = 128$ selected experiments. The main results of the numerical experiments are

TABLE 1

Parameter values for the numerical experiments with 2-echelon models

Parameter	Values	
R_d	5	20
μ_{P_d}/R_d	0.5	2
$c^2[P_d]$	0	0.25
β_w	0.6	0.9
λ	1	5
$c^2[D]$	0.5	1
μ_{P_w}	2.5	10
$c^2[P_w]$	0	0.25
β_w^c	0.95	0.99

shown in Figs. 2 and 3. A comparison between approximation and simulation in tabular form is shown in Appendix 3.

Figure 2 shows the relative deviation between the approximated and simulated mean physical stock in the network. This relative deviation is not very large (at most 5.1%). Detailed analysis shows that for some cases the mean physical stock at single stockpoints is approximated less accurately, but then errors compensate each other. We conclude that inventory holding costs can be approximated quite accurately unless the product value is clearly different at the various stages in the network (e.g. if final assembly takes place at the wholesalers).

The deviation between the approximated and simulated mean service level for the five external customer groups is shown in Fig. 3. The absolute value of the approximation error (simulation minus approximation) is mostly less than 0.02, but for some cases it may rise to 0.05. Detailed analysis of the experimental results show that this occurs for the combination of long, deterministic lead times to the upstream stockpoint (the distribution centre) and infrequent reviews at this stockpoint. An intuitive explanation for this is the following. In the case of frequent reviews at the distribution centre the interarrival times of replenishment orders are short. So the mean delay of the wholesalers' orders tends to be relatively small compared to the normal order delivery time. Hence, relatively large approximation errors in the delay lead to relatively small errors in the lead time of the wholesalers' orders. Conversely, *infrequent* reviews at the distribution centre lead to relatively *large* errors in the lead time of the wholesalers' orders. This lead time is a key variable in the approximation of the service level to the wholesalers' customers. Furthermore, it appears that the relative approximation accuracy of the delay is less for deterministic than for stochastic lead times. So the combination of infrequent reviews at the distribution centre and deterministic lead times of the distribution centres replenishment orders yields large approximation errors in the service level for external customers.

A similar comparison was made for the mean service level for the five wholesalers. It appears that the absolute value of the approximation error is somewhat larger than in the case of the service level for external customers, but this is not very troublesome. Firstly, the magnitude of internal service levels is clearly different from external service levels. An error of 0.05 when the service level equals 0.6 is not as bad as when the service level equals 0.95. Secondly, the internal service level is not important as a performance measure for the network. What counts is the service level for the final customers and the mean physical stock. The internal service level is only useful because it is easier to discuss the network design in terms of (internal) service levels rather than in terms of order-up-to levels.

Fig. 2. Relative deviation for the mean physical stock.

Fig. 3. Deviation in the mean service level for external customers.

Finally we remark that the waiting times of external customers are not always approximated well. Better approximations are obtained using the method described by van der Heijden and de Kok [10].

8. Numerical results for 3-echelon models

In the previous Section we have seen that approximation errors in the performance of the upstream stockpoint (delay of the wholesalers' replenishment orders) can be transmitted through the network (service levels for external customers). This raises the question whether approximation errors accumulate in the network and lead to unacceptable errors in the performance measures as the number of echelons in the network grows. Therefore, it is important to examine the approximation accuracy in networks with more than two echelons.

In this Section we examine 3-echelon networks with the following structure. We have a manufacturer's warehouse which supplies three distribution centres and issues replenishment orders to the factory. Each distribution centre supplies five wholesalers and each wholesaler supplies one

external customer group. Hence, the structure is similar to the one depicted in Fig. 1, only we have three distribution centres, and fifteen wholesalers and external customer groups in the network.

The number of model parameters in the 3-echelon network is considerably higher than in the 2-echelon network. Therefore we include only those parameters in our examination which are probably the most important. The analysis of 2-echelon networks showed that the approximation accuracy is especially influenced by the normal order delivery time distributions and the review frequencies, so we restrict ourselves to the parameters as shown in Table 2.

The notation is similar to the 2-echelon models. In addition to this notation, R_m denotes the review period at the manufacturer and β_d denotes the service level to each distribution centre. Further, μ_{P_m} and $c^2[P_m]$ denote, respectively, the mean and the square of the coefficient of variation of the manufacturers' replenishment normal order delivery time. Note that we have chosen ratios of review periods and mean normal order delivery times as parameters instead of the review periods and the mean normal order delivery times themselves. This is done to avoid the inclusion of unrealistic parameter combinations in the experiments (e.g. a higher review frequency at the manufacturer than at the distribution centres). In total we have $2^6 = 64$ combinations. This number is acceptable, so we carried out numerical experiments for all parameter combinations. The parameters which are not shown in Table 2 are fixed at the following values:

$$R_w = 5, \quad \mu_{P_w} = 5, \quad c^2[P_w] = 0.25, \quad \lambda = 1, \quad \mu_D = 10, \quad c^2[D] = 1, \\ \beta_w^s = 0.95, \quad \beta_w = 0.6, \quad \beta_d = 0.6,$$

$\Delta_j = 0.05, 2.05$ and 4.05 for the 3 distribution centres,

$\Delta_j = 0.9, 1.9, 2.9, 3.9$ and 4.9 for the 5 wholesalers at each distribution centre.

The main results of the numerical experiments are shown in Figs. 4 and 5. Appendix 4 contains the results in tabular form.

Figure 4 shows the relative deviation in the approximation for the mean physical stock in the network. We see that the approximation is very accurate; the relative deviation never exceeds 2.5%. Again the mean physical stock is approximated less accurately for an individual stockpoint, but the errors compensate each other.

The approximation error in the mean service level for the 15 external customer groups is shown in Fig. 5. The absolute deviation is less than 0.015 in many cases, but may rise to 0.035 in specific

TABLE 2

Parameter values for the numerical experiments with 3-echelon models

Parameter	Values	
R_d/R_w	1	2
μ_{P_d}/μ_{P_w}	1	3
$c^2[P_d]$	0	0.25
R_m/R_d	1	2
μ_{P_m}/μ_{P_d}	1	3
$c^2[P_m]$	0	0.25

Fig. 4. Relative deviation for the mean physical stock.

Fig. 5. Deviation in the mean service level for external customers.

cases. Detailed analyses show that the approximation error is relatively large for relatively long, deterministic lead times at the manufacturer.

Finally we remark that the internal service levels for the distribution centres are approximated reasonably well, but the internal service levels to the wholesalers are approximated badly. The absolute error may increase to 0.18! Hence we conclude that the approximation of *internal* service levels is unreliable. However, the network performance in the sense of *external* service levels and mean physical stock is approximated well in most cases.

9. Conclusions

In this paper, we discussed an approximate method to analyse divergent logistic networks with local (R, S) inventory control. The approximations are based on the PDF-method by de Kok [6]. Comparison to results from Monte Carlo simulation shows that the network performance (service level for the final customers, mean physical stock) is approximated well in many cases. The method is less accurate in the case of relatively long, deterministic lead times at the upstream stockpoint.

Further the approximations for *internal* service levels are unreliable, but this is less important. We conclude that the method is accurate enough to be used for global design of distribution networks with local (R, S) inventory control. Possible subjects for further research can be:

- model extensions (orders are only partially backlogged, non-Poisson customer arrivals at the downstream stockpoint),
- finding design rules for stock allocation,
- the analysis of similar networks with other local inventory control strategies at the stockpoints.

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Appendix 1. The moments of X_n for downstream stockpoints

In this appendix we give the moments of the random variables X_n associated with the probability distributions $H_n(S_i)$ for $n = 1, \dots, 4$ as discussed in Section 4. These moments are required to apply the PDF-method for the approximation of the performance of downstream stockpoints.

To approximate the mean delay of customers arriving at stockpoint $i \in \Omega^D$ we need the first two moments of the random variable X_1 associated with probability distribution function $H_1(S_i)$ as defined by (4.5). Some elementary algebra yields

$$E[X_1] = E[D_i] + \frac{\lambda_i E[D_i] * \{E[L_i^2] + R_i * E[L_i] + \frac{1}{3} R_i^2\}}{2E[L_i] + R_i}, \quad (\text{A1.1})$$

$$E[X_1^2] = \frac{1}{6} a_2 R_i^2 + \frac{1}{3} a_1 R_i + a_0 + \frac{2a_2 E[L_i^3] + 3(a_2 R_i + a_1) E[L_i^2] + (a_2 R_i^2 + a_1 R_i) E[L_i]}{6E[L_i] + 3R_i},$$

$$\text{with } a_2 = (\lambda_i E[D_i])^2, \quad a_1 = \lambda_i (E[D_i^2] + 2E^2[D_i]) \quad \text{and} \quad a_0 = E[D_i^2]. \quad (\text{A1.2})$$

For the second moment of the delay we need the first two moments of a random variable X_2 associated with $H_2(S_i)$ as defined by (4.2) and (4.6). We obtain

$$E[X_2] = E[D_i] + \frac{\lambda_i E[D_i] * \{4E[L_i^3] + 6R_i E[L_i^2] + 4R_i^2 * E[L_i] + R_i^3\}}{12E[L_i^2] + 12R_i E[L_i] + 4R_i^2}, \quad (\text{A1.3})$$

$$E[X_2^2] = \frac{b_0 + b_1 E[L_i] + b_2 E[L_i^2] + b_3 E[L_i^3] + b_4 E[L_i^4]}{E[L_i^2] + R_i E[L_i] + \frac{1}{3} R_i^2}, \quad (\text{A1.4})$$

$$\text{with } b_0 = \frac{1}{30} a_2 R_i^4 + \frac{1}{12} a_1 R_i^3 + \frac{1}{3} a_0 R_i^2, \quad b_1 = \frac{1}{6} a_2 R_i^3 + \frac{1}{3} a_1 R_i^2 + a_0 R_i,$$

$$b_2 = \frac{1}{3} a_2 R_i^2 + \frac{1}{2} a_1 R_i + a_0, \quad b_3 = \frac{1}{3} a_2 R_i + \frac{1}{3} a_1 \quad \text{and} \quad b_4 = \frac{1}{6} a_2.$$

For the service level we have that the first two moments of a random variable X_3 associated with $H_3(S_i)$ as defined by (4.3) and (4.7) equal

$$E[X_3] = \frac{E[D_i^2]}{E[D_i]} + \lambda_i E[D_i] * \{E[L_i] + \frac{1}{2}R_i\}, \quad (A1.5)$$

$$E[X_3^2] = \frac{E[D_i^3]}{E[D_i]} + 3\lambda_i E[D_i^2] * \{E[L_i] + \frac{1}{2}R_i\} + \{\lambda_i E[D_i]\}^2 * \{R_i E[L_i] + \frac{1}{3}R_i^2 + E[L_i^2]\}. \quad (A1.6)$$

For the mean physical stock the first two moments of a random variable X_4 associated with $H_4(S_i)$ as defined by (4.4) and (4.8) are given by

$$E[X_4] = \frac{1}{2R_i^2 * (n_\infty + n_0)} * \left(R_i \text{VAR}[O_i] * (E[L_i] + \frac{1}{2}R_i) + E^2[O_i] * (R_i E[L_i] + \frac{1}{3}R_i^2 + E[L_i^2]) \right. \\ \left. + \frac{E[D_i^2]}{E[D_i]} * E[O_i] * (E[L_i] + \frac{1}{2}R_i) \right), \quad (A1.7)$$

$$E[X_4^2] = \frac{1}{R_i(n_\infty + n_0)} * \left(\frac{1}{3}m_3[O_i] * (E[L_i] + R_i) + \frac{E[O_i] * \text{VAR}[O_i]}{R_i} * (R_i E[L_i] \right. \\ \left. + \frac{1}{2}R_i^2 + E[L_i^2]) + \frac{E^3[O_i]}{3R_i^2} * (R_i^2 E[L_i] + \frac{1}{4}R_i^3 + \frac{3}{2}R_i E[L_i^2] + E[L_i^3]) \right. \\ \left. + \frac{E[D_i^2]}{2E[D_i]} * \left\{ \text{VAR}[O_i] * (E[L_i] + \frac{1}{2}R_i) + \frac{E^2[O_i]}{R_i} * (R_i E[L_i] + \frac{1}{3}R_i^2 + E[L_i^2]) \right\} \right). \quad (A1.8)$$

Appendix 2. The PDF-method for upstream and intermediate stockpoints

In Section 5 we present expressions for the performance measures of upstream and intermediate stockpoints. In this appendix we give the transformations of these expressions to probability distribution functions $H_n(S_i)$. Further we give the first two moments of the random variables associated with these distributions, which is sufficient to apply the PDF-method.

A2.1. The mean delay

We proceed from expression (5.3) for the mean delay of replenishment orders from stockpoint j at its supplier, stockpoint i . Using this expression it can be shown that the mean delay as a function of the order-up-to level S_i is monotonically non-increasing in S_i with

$$\lim_{S_i \rightarrow \infty} E[W_j(S_i)] = 0 \quad \text{and} \quad \lim_{S_i \downarrow 0} E[W_j(S_i)] = E[L_i] + \frac{1}{2}R_i + \frac{1}{2}R_j - \Delta_j,$$

Hence we can apply the PDF-method to the mean delay using the function

$$H_5(S_i) = 1 - \frac{E[W_j(S_i)]}{E[L_i] + \frac{1}{2}R_i + \frac{1}{2}R_j - \Delta_j}. \quad (\text{A2.1})$$

We can easily derive the first two moments of a random variable X_5 associated with distribution $H_5(S_i)$ from (5.3) and (A2.1):

$$E[X_5] = \frac{1}{r_j * (E[L_i] + \frac{1}{2}R_i + \frac{1}{2}R_j - \Delta_j)} * \sum_{m=0}^{\infty} E[Z_{j,m}] * a_{j,m}, \quad (\text{A2.2})$$

$$E[X_5^2] = \frac{1}{r_j * (E[L_i] + \frac{1}{2}R_i + \frac{1}{2}R_j - \Delta_j)} * \sum_{m=0}^{\infty} (\text{VAR}[Z_{j,m}] + E^2[Z_{j,m}]) * a_{j,m}. \quad (\text{A2.3})$$

Once we have expressions for $E[Z_{j,m}]$, $\text{VAR}[Z_{j,m}]$ and $a_{j,m}$ we can calculate the first two moments of X_5 by truncating the infinite sums. First note that

$$Z_{j,m} = \sum_{k \in \Omega_i} \sum_{l=1}^{N_k^z[\Delta_j + mR_j]} O_{k,l},$$

where $O_{k,l}$ is the l th replenishment order of stockpoint k and $N_k^\infty[x] = \lim_{n \rightarrow \infty} N_k[nR_i, nR_i + x]$.

Hence,

$$E[Z_{j,m}] = \sum_{k \in \Omega_i} N_k^\infty[\Delta_j + mR_j] * E[O_k], \quad (\text{A2.4})$$

$$\text{VAR}[Z_{j,m}] = \sum_{k \in \Omega_i} N_k^\infty[\Delta_j + mR_j] * \text{VAR}[O_k], \quad (\text{A2.5})$$

where the number of replenishment orders arriving from stockpoint k in the time interval with length $\Delta_j + mR_j$ is given by

$$N_k^\infty[\Delta_j + mR_j] = \begin{cases} m + 1, & k = j, \\ [(\Delta_j + mR_j - \Delta_k)/R_k + 1], & k \geq j, \end{cases}$$

with $[x]$ is the Entier function in x , i.e. the largest integer smaller than or equal to x .

In principle, the numbers $a_{j,m}$ can be obtained using numerical integration for which standard routines are widely available (e.g. Press, Flannery, Teukolsky and Vetterling [12]). However, this may require a lot of computer time because many factors $a_{j,m}$ may be involved in the calculation of the two moments of X_5 . Therefore we restrict ourselves to some specific cases, namely deterministic and Erlang distributed lead times.

For deterministic lead times L_i it is easy to see that

$$a_{j,m} = \begin{cases} r_j R_j, & L_i > \Delta_j + mR_j, \\ L_i - \Delta_j - (m - r_j)R_j, & \Delta_j + (m - r_j)R_j \leq L_i \leq \Delta_j + mR_j, \\ 0, & L_i < \Delta_j + (m - r_j)R_j. \end{cases} \quad (\text{A2.6})$$

The Erlang distribution with parameters $r \in \mathbb{N}$ and $\alpha > 0$ is defined by

$$1 - G_i(x) = \sum_{l=0}^{r-1} \frac{(\alpha x)^l * e^{-\alpha x}}{l!}.$$

Using mixtures of Erlang distributions we can cover a wide range of lead time distributions, with small as well as large variation (cf. Tijms [8] and van der Heijden [9]). However, for simplicity we present here only results for pure Erlang distributions. We can easily derive expressions for $a_{j,m}$ when the lead time has a pure Erlang distribution as defined above. For $m \geq r_j$ we have

$$a_{j,m} = \sum_{k=0}^{r-1} \frac{r-k}{\alpha} * \frac{[\alpha(\Delta_j + (m-r_j)R_j)]^k * e^{-\alpha(\Delta_j + (m-r_j)R_j)} - [\alpha(\Delta_j + mR_j)]^k * e^{-\alpha(\Delta_j + mR_j)}}{k!}. \quad (\text{A2.7a})$$

For $m \leq r_j - 1$ we have

$$a_{j,m} = (r_j - m)R_j - \Delta_j + \frac{1}{\alpha} * \left(r - \sum_{k=0}^{r-1} (r-k) * \frac{[\alpha(\Delta_j + mR_j)]^k * e^{-\alpha(\Delta_j + mR_j)}}{k!} \right). \quad (\text{A2.7b})$$

Combining (A2.2)–(A2.7) we are able to calculate the first two moments of X_5 using truncation of the infinite sum. We truncate this sum as soon as the relative change in the respective moments does not exceed some convergence criterion $\varepsilon > 0$.

A.2.2. The second moment of the delay

From expression (5.4) we see that the second moment of the delay is monotonically non-increasing in S_i with

$$\lim_{S_i \rightarrow \infty} E[W_j^2(S_i)] = 0,$$

$$\lim_{S_i \downarrow 0} E[W_j^2(S_i)] = E[W_j^2(0)] = \frac{1}{3}R_i^2 + E[L_i^2] + E[L_i] * (R_i + R_j - 2\Delta_j) + R_i * (\frac{1}{2}R_j - \Delta_j)$$

$$+ R_j * (\frac{1}{6}R_j - \Delta_j).$$

Hence we can apply the PDF-method to the mean delay using the function

$$H_6(S_i) = 1 - \frac{E[W_j^2(S_i)]}{E[W_j^2(0)]}. \quad (\text{A2.8})$$

We can easily derive the first two moments of a random variable X_6 associated with distribution $H_6(S_i)$ from (5.4) and (A2.8):

$$E[X_6] = \frac{1}{r_j * E[W_j^2(0)]} * \sum_{m=0}^{\infty} E[Z_{j,m}] * [b_{j,m} - 2 * (\Delta_j + (m-r_j)R_j) * a_{j,m} + 2 * r_j R_j * c_{j,m}], \quad (\text{A2.9})$$

$$E[X_6^2] = \frac{1}{r_j * [W_j^2(0)]} * \sum_{m=0}^{\infty} \{ \text{VAR}[Z_{j,m}] + E^2[Z_{j,m}] \} * [b_{j,m} - 2 * (\Delta_j + (m - r_j)R_j) * a_{j,m}] \\ + 2 * r_j R_j * c_{j,m}. \quad (\text{A2.10})$$

To evaluate these expressions, we need the numbers $b_{j,m}$ and $c_{j,m}$. Similarly to $a_{j,m}$, these numbers are easily obtained in the case of deterministic or Erlang distributed lead times. For deterministic lead times we have

$$b_{j,m} = \begin{cases} [\Delta_j + mR_j]^2 - [\Delta_j + (m - r_j)R_j]^2, & L_i > \Delta_j + mR_j, \\ L_i^2 - [\Delta_j + (m - r_j)R_j]^2, & \Delta_j + (m - r_j)R_j \leq L_i \leq \Delta_j + mR_j, \\ 0, & L_i < \Delta_j + (m - r_j)R_j, \end{cases} \quad (\text{A2.11})$$

$$c_{j,m} = \begin{cases} L_i - \Delta_j - mR_j, & L_i \geq \Delta_j + mR_j, \\ 0, & L_i < \Delta_j + mR_j. \end{cases} \quad (\text{A2.12})$$

In the case of Erlang distributed lead times we obtain for $m \geq r_j$

$$b_{j,m} = \sum_{k=0}^r \left(\frac{r * (r + 1) - k * (k - 1)}{\alpha^2} \right) \\ * \frac{[\alpha(\Delta_j + (m - r_j)R_j)]^k * e^{-\alpha(\Delta_j + (m - r_j)R_j)} - [\alpha(\Delta_j + mR_j)]^k * e^{-\alpha(\Delta_j + mR_j)}}{k!}. \quad (\text{A2.13a})$$

For $m < r_j$ we have

$$b_{j,m} = \frac{r(r + 1)}{\alpha^2} - [\Delta_j + (m - r_j)R_j]^2 \\ - \frac{1}{\alpha^2} * \sum_{k=0}^r \{ r * (r + 1) - k * (k - 1) \} * \frac{[\alpha(\Delta_j + mR_j)]^k * e^{-\alpha(\Delta_j + mR_j)}}{k!}. \quad (\text{A2.13b})$$

Finally, the numbers $c_{j,m}$ are obtained by

$$c_{j,m} = \sum_{k=0}^{r-1} \frac{(r - k)}{\alpha} * \frac{[\alpha(\Delta_j + mR_j)]^k * e^{-\alpha(\Delta_j + mR_j)}}{k!}. \quad (\text{A2.14})$$

A2.3. The (internal) service level

To use the PDF-method for the service level, we simply use the probability distribution function $H_7(S_i) = \beta_j(S_i)$. The first two moments of the random variable X_7 associated with $H_7(S_i)$ can be

obtained from (5.5):

$$E[X_7] = \frac{E[O_j^2]}{E[O_j]} + \frac{1}{r_j} * \sum_{m=0}^{\infty} d_{j,m} * E[Y_{j,m}], \quad (\text{A2.15})$$

$$E[X_7^2] = \frac{E[O_j^3]}{E[O_j]} + \frac{1}{r_j} * \sum_{m=0}^{\infty} d_{j,m} * E[Y_{j,m}^2] + \frac{2E[O_j^2]}{r_j E[O_j]} * \sum_{m=0}^{\infty} d_{j,m} * E[Y_{j,m}], \quad (\text{A2.16})$$

where

$$d_{j,m} = [G_i(\Delta_j + mR_j) - G_i(\Delta_j + (m - r_j)R_j)], \quad Y_{j,m} = V_i^\infty[\Delta_j + mR_j] = \sum_{k \in \Omega_i}^{N_k^\infty[\Delta_j + mR_j]} \sum_{l=1} O_{k,l},$$

with

$$N_k^\infty[\Delta_j + mR_j] = \begin{cases} m, & k = j, \\ \lceil (\Delta_j + mR_j - \Delta_k)/R_k + 1 \rceil, & k \leq j. \end{cases}$$

where $\lceil x \rceil$ is the largest integer $< x$.

To evaluate (A2.15) and (A2.16), we need $E[Y_{j,m}]$, $\text{VAR}[Y_{j,m}]$ and $d_{j,m}$. For the mean and variance of $Y_{j,m}$ we can derive

$$E[Y_{j,m}] = \sum_{k \in \Omega_i} N_k^\infty[\Delta_j + mR_j] * E[O_k], \quad (\text{A2.17})$$

$$\text{VAR}[Y_{j,m}] = \sum_{k \in \Omega_i} N_k^\infty[\Delta_j + mR_j] * \text{VAR}[O_k]. \quad (\text{A2.18})$$

The numbers $d_{j,m}$ for deterministic lead times are

$$d_{j,m} = \begin{cases} 1, & \Delta_j + (m - r_j)R_j < L_i \leq \Delta_j + mR_j, \\ 0, & \text{otherwise.} \end{cases} \quad (\text{A2.19})$$

For Erlang distributed lead times we have for $m \geq r_j$

$$d_{j,m} = \sum_{k=0}^{r-1} \frac{[\alpha(\Delta_j + (m - r_j)R_j)]^k * e^{-\alpha(\Delta_j + (m - r_j)R_j)} - [\alpha(\Delta_j + mR_j)]^k * e^{-\alpha(\Delta_j + mR_j)}}{k!}. \quad (\text{A2.19a})$$

For $m \leq r_j - 1$ we have

$$d_{j,m} = 1 - \sum_{k=0}^{r-1} \frac{[\alpha(\Delta_j + mR_j)]^k * e^{-\alpha(\Delta_j + mR_j)}}{k!}. \quad (\text{A2.19b})$$

A2.4. The mean physical stock

From (4.4) we can show that the mean physical stock is monotonically non-decreasing in S_i with

$$\lim_{S_i \rightarrow \infty} \frac{E[I_i(S_i)]}{S_i} = 1,$$

$$\lim_{S_i \rightarrow \infty} S_i - E[I_i(S_i)] = n_\infty = \sum_{k \in \Omega_i} \frac{E[O_k]}{R_k} * \left\{ \frac{1}{2} R_i + E[L_i] + \frac{1}{2} R_k - \Delta_k \right\},$$

$$\lim_{S_i \downarrow 0} E[I_i(S_i)] = n_0 = \frac{E[D_i^2]}{2E[D_i]}.$$

To use the PDF-method, we consider the random variable X_8 associated with the following probability distribution function $H_8(S_i)$:

$$H_8(S_i) = \frac{S_i - E[I_i(S_i)] + n_0}{n_\infty + n_0}, \quad (\text{A2.20})$$

with n_∞ and n_0 as defined above. Using (A2.20), we find for the first two moments of X_8

$$\begin{aligned} E[X_8] &= \frac{1}{2R_i * (n_\infty + n_0)} * \int_0^\infty [G_i(t) - G_i(t - R_i)] * \text{VAR}[V_i^\infty[t]] dt \\ &\quad + \frac{1}{2R_i * (n_\infty + n_0)} * \int_0^\infty [G_i(t) - G_i(t - R_i)] * E^2[V_i^\infty[t]] dt \\ &\quad + \frac{E[D_i^2]}{2R_i * (n_\infty + n_0) * E[D_i]} * \int_0^\infty [G_i(t) - G_i(t - R_i)] * E[V_i^\infty[t]] dt, \end{aligned} \quad (\text{A2.21})$$

$$\begin{aligned} E[X_8^2] &= \frac{1}{3R_i * (n_\infty + n_0)} * \int_0^\infty [G_i(t) - G_i(t - R_i)] * m_3[V_i^\infty[t]] dt \\ &\quad + \frac{1}{R_i * (n_\infty + n_0)} * \int_0^\infty [G_i(t) - G_i(t - R_i)] * E[V_i^\infty[t]] * \text{VAR}[V_i^\infty[t]] dt \\ &\quad + \frac{1}{3R_i * (n_\infty + n_0)} * \int_0^\infty [G_i(t) - G_i(t - R_i)] * E^3[V_i[t]] dt \\ &\quad + \frac{E[D_i^2]}{2R_i * (n_\infty + n_0) * E[D_i]} * \int_0^\infty [G_i(t) - G_i(t - R_i)] * \text{VAR}[V_i^\infty[t]] dt \\ &\quad + \frac{E[D_i^2]}{2R_i * (n_\infty + n_0) * E[D_i]} * \int_0^\infty [G_i(t) - G_i(t - R_i)] * E^2[V_i^\infty[t]] dt. \end{aligned} \quad (\text{A2.22})$$

To evaluate these two moments we need the following integrals:

$$\begin{aligned}
\int_0^{\infty} [G_i(t) - G_i(t - R_i)] * E[V_i^{\infty}] dt &= E[O_i] * E[L_i] + \int_0^{R_i} E[V_i^{\infty}[t]] dt, \\
\int_0^{\infty} [G_i(t) - G_i(t - R_i)] * \text{VAR}[V_i^{\infty}[t]] dt &= \text{VAR}[O_i] * E[L_i] + \int_0^{R_i} \text{VAR}[V_i^{\infty}[t]] dt, \\
\int_0^{\infty} [G_i(t) - G_i(t - R_i)] * E^2[V_i^{\infty}[t]] dt &= E^2[O_i] * E[L_i] + \int_0^{R_i} E^2[V_i^{\infty}[t]] dt \\
&\quad + 2E[O_i] * \int_0^{\infty} [1 - G_i(t)] * E[V_i^{\infty}[t]] dt, \\
\int_0^{\infty} [G_i(t) - G_i(t - R_i)] * m_3[V_i^{\infty}[t]] dt &= m_3[O_i] * E[L_i] + \int_0^{R_i} m_3[V_i^{\infty}[t]] dt, \\
\int_0^{\infty} [G_i(t) - G_i(t - R_i)] * E[V_i^{\infty}[t]] * \text{VAR}[V_i^{\infty}(t)] dt &= E[O_i] * \text{VAR}[O_i] * E[L_i] \\
&\quad + \int_0^{R_i} E[V_i^{\infty}[t]] * \text{VAR}[V_i^{\infty}[t]] dt + E[O_i] * \int_0^{\infty} [1 - G_i(t)] * \text{VAR}[V_i^{\infty}[t]] dt \\
&\quad + \text{VAR}[O_i] * \int_0^{\infty} [1 - G_i(t)] * E[V_i^{\infty}[t]] dt, \\
\int_0^{\infty} [G_i(t) - G_i(t - R_i)] * E^3[V_i^{\infty}[t]] dt &= E^3[O_i] * E[L_i] + \int_0^{R_i} E^3[V_i^{\infty}[t]] dt + 3E^2[O_i] \\
&\quad * \int_0^{\infty} [1 - G_i(t)] * E[V_i^{\infty}[t]] dt + 3E[O_i] \\
&\quad * \int_0^{\infty} [1 - G_i(t)] * E^2[V_i^{\infty}[t]] dt.
\end{aligned}$$

These expressions can be evaluated as follows:

$$\int_0^{R_i} E[V_i^\infty[t]] dt = \frac{1}{2} R_i * E[O_i] + \sum_{k \in \Omega_i} r_k E[O_k] * (\frac{1}{2} R_k - \Delta_k),$$

$$\int_0^{R_i} \text{VAR}[V_i^\infty[t]] dt = \frac{1}{2} R_i * \text{VAR}[O_i] + \sum_{k \in \Omega_i} r_k * \text{VAR}[O_k] * (\frac{1}{2} R_k - \Delta_k),$$

$$\int_0^\infty [1 - G_i(t)] * E[V_i^\infty[t]] dt = \sum_{k \in \Omega_i} E[O_k] \sum_{m=0}^\infty c_{k,m},$$

$$\begin{aligned} \int_0^{R_i} E^2[V_i^\infty[t]] dt &= \sum_{k \in \Omega_i} r_k E^2[O_k] * (\frac{1}{3} R_i r_k + \frac{1}{2} R_i + \frac{1}{6} R_k - r_k \Delta_k) \\ &\quad + 2 \sum_{k \in \Omega_i} \sum_{\substack{j \in \Omega_i \\ j > k}} [\alpha_{k,j}(r_k, r_j) - r_k r_j * \min\{\Delta_k, \Delta_j\}] * E[O_k] * E[O_j], \end{aligned}$$

$$\begin{aligned} \text{with } \alpha_{k,j}(l, m) &= \frac{1}{E[O_k] * E[O_j]} * \int_0^{\min\{\Delta_k + l R_k, \Delta_j + m R_j\}} E[V_{i,k}^\infty[t]] * E[V_{i,j}^\infty[t]] dt \\ &= \int_0^{\min\{\Delta_k + l R_k, \Delta_j + m R_j\}} N_k[t] N_j[t] dt, \end{aligned}$$

$$\int_0^{R_i} m_3[V_i^\infty[t]] dt = \frac{1}{2} R_i * m_3[O_i] + \sum_{k \in \Omega_i} r_k m_3[O_k] * (\frac{1}{2} R_k - \Delta_k),$$

$$\int_0^\infty [1 - G_i(t)] * \text{VAR}[V_i^\infty[t]] dt = \sum_{k \in \Omega_i} \text{VAR}[O_k] \sum_{m=0}^\infty c_{k,m},$$

$$\begin{aligned} \int_0^{R_i} E[V_i^\infty[t]] * \text{VAR}[V_i^\infty[t]] dt &= \sum_{k \in \Omega_i} r_k E[O_k] * \text{VAR}[O_k] * (\frac{1}{3} R_i r_k + \frac{1}{2} R_i + \frac{1}{6} R_k - r_k \Delta_k) \\ &\quad + \sum_{k \in \Omega_i} \sum_{\substack{j \in \Omega_i \\ j > k}} [\alpha_{k,j}(r_k, r_j) - r_k r_j * \min\{\Delta_k, \Delta_j\}] \\ &\quad * \{E[O_k] \text{VAR}[O_j] + \text{VAR}[O_k] E[O_j]\}, \end{aligned}$$

$$\begin{aligned}
\int_0^{R_i} E^3[V_i^\infty[t]] dt &= \sum_{k \in \Omega_i} r_k^2 E^3[O_k] * (\tfrac{1}{4}R_i r_k + \tfrac{1}{2}R_i + \tfrac{1}{4}R_k - r_k \Delta_k) \\
&\quad + 3 \sum_{k \in \Omega_i} \sum_{\substack{j \in \Omega_i \\ j \leq k}} [\beta_{k,j}(r_k, r_j) - r_k^2 r_j * \min\{\Delta_k, \Delta_j\}] * E^2[O_k] * E[O_j] \\
&\quad + 6 \sum_{i \in \Omega_i} \sum_{\substack{j \in \Omega_i \\ j > k}} \sum_{\substack{h \in \Omega_i \\ h > j}} [\delta_{k,j,h}(r_k, r_j, r_h) - r_k r_j r_h * \min\{\Delta_k, \Delta_j, \Delta_h\}] * E[O_k] E[O_j] E[O_h],
\end{aligned}$$

with

$$\begin{aligned}
\beta_{k,j}(l, m) &= \frac{1}{E^2[O_k] * E[O_j]} * \int_0^{\min\{\Delta_k + lR_k, \Delta_j + mR_j\}} E^2[V_{i,k}^\infty[t]] * E[V_{i,j}^\infty[t]] dt \\
&= \int_0^{\min\{\Delta_k + lR_k, \Delta_j + mR_j\}} N_k^2[t] N_j[t] dt
\end{aligned}$$

and

$$\begin{aligned}
\delta_{k,j}(l, m, n) &= \int_0^{\min\{\Delta_k + lR_k, \Delta_j + mR_j, \Delta_h + nR_h\}} \frac{E[V_{i,k}^\infty[t]] E[V_{i,j}^\infty[t]] E[V_{i,h}^\infty[t]]}{E[O_k] E[O_j] E[O_h]} dt \\
&= \int_0^{\min\{\Delta_k + lR_k, \Delta_j + mR_j, \Delta_h + nR_h\}} N_k[t] N_j[t] N_h[t] dt,
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty [1 - G_i(t)] E^2[V_i^\infty[t]] dt &= \sum_{k \in \Omega_i} E^2[O_k] \sum_{m=0}^\infty (2m+1) c_{k,m} \\
&\quad + 2 \sum_{k \in \Omega_i} \sum_{\substack{j \in \Omega_i \\ j > k}} \eta_{k,j}(\infty, \infty) E[O_k] E[O_j],
\end{aligned}$$

$$\begin{aligned}
\eta_{k,j}(l, m) &= \int_0^{\min\{\Delta_k + lR_k, \Delta_j + mR_j\}} [1 - G_i(t)] * \frac{E[V_{i,k}^\infty[t]] E[V_{i,j}^\infty[t]]}{E[O_k] * E[O_j]} dt \\
&= \int_0^{\min\{\Delta_k + lR_k, \Delta_j + mR_j\}} [1 - G_i(t)] * N_k[t] * N_j[t] dt.
\end{aligned}$$

TABLE A1

Target $\beta_w^c = 0.95$ and target $\beta_w = 0.6$

λ	$c^2[D]$	R_d	μ_{P_d}	$c^2[P_d]$	μ_{P_w}	$c^2[P_w]$	Mean stock		β_w^c		β_w	
							sim.	appr.	sim.	appr.	sim.	appr.
1	0.5	5	2.5	0.25	2.5	0.00	488	493	0.939	0.950	0.577	0.600
1	0.5	5	10.0	0.00	2.5	0.00	527	537	0.950	0.950	0.588	0.600
1	0.5	5	10.0	0.00	2.5	0.25	548	558	0.950	0.950	0.588	0.600
1	0.5	20	10.0	0.25	2.5	0.00	840	847	0.937	0.951	0.557	0.599
1	0.5	20	10.0	0.25	2.5	0.25	853	861	0.938	0.951	0.557	0.599
1	0.5	20	40.0	0.25	10.0	0.00	2023	2025	0.942	0.953	0.608	0.609
1	0.5	20	40.0	0.25	10.0	0.25	2120	2130	0.948	0.954	0.608	0.609
1	1.0	5	2.5	0.00	2.5	0.25	574	580	0.938	0.950	0.595	0.600
1	1.0	5	2.5	0.00	10.0	0.00	687	698	0.942	0.950	0.595	0.600
1	1.0	5	2.5	0.25	10.0	0.25	899	905	0.946	0.949	0.581	0.600
1	1.0	5	10.0	0.00	2.5	0.25	643	643	0.947	0.950	0.589	0.600
1	1.0	5	10.0	0.00	10.0	0.00	742	756	0.949	0.950	0.589	0.600
1	1.0	20	10.0	0.25	2.5	0.00	915	920	0.937	0.951	0.560	0.599
1	1.0	20	10.0	0.25	10.0	0.00	994	1002	0.939	0.951	0.560	0.599
1	1.0	20	40.0	0.25	10.0	0.00	2094	2092	0.942	0.953	0.609	0.608
5	0.5	5	2.5	0.00	2.5	0.00	1277	1330	0.936	0.950	0.599	0.600
5	0.5	5	2.5	0.00	2.5	0.25	1459	1508	0.942	0.950	0.599	0.600
5	0.5	5	2.5	0.25	10.0	0.25	3048	3079	0.948	0.949	0.571	0.600
5	0.5	5	10.0	0.00	2.5	0.00	1599	1645	0.969	0.950	0.595	0.600
5	0.5	5	10.0	0.00	2.5	0.25	1764	1809	0.967	0.950	0.595	0.600
5	0.5	5	10.0	0.00	10.0	0.00	1756	1845	0.964	0.950	0.595	0.600
5	0.5	5	10.0	0.25	10.0	0.25	3867	3866	0.953	0.950	0.603	0.599
5	0.5	20	10.0	0.25	2.5	0.25	3472	3481	0.940	0.951	0.553	0.599
5	0.5	20	10.0	0.25	10.0	0.00	3378	3395	0.933	0.951	0.553	0.599
5	0.5	20	40.0	0.00	2.5	0.00	4352	4471	0.999	0.950	0.563	0.600
5	0.5	20	40.0	0.25	10.0	0.25	9913	9895	0.950	0.954	0.613	0.609
5	1.0	5	2.5	0.25	10.0	0.00	1717	1777	0.938	0.950	0.572	0.600
5	1.0	5	10.0	0.00	2.5	0.00	1699	1744	0.966	0.950	0.595	0.600
5	1.0	5	10.0	0.00	10.0	0.00	1902	1986	0.961	0.950	0.595	0.600
5	1.0	5	10.0	0.00	10.0	0.25	3312	3362	0.953	0.949	0.595	0.600
5	1.0	20	10.0	0.00	10.0	0.00	2891	2952	0.944	0.950	0.551	0.600
5	1.0	20	10.0	0.25	2.5	0.00	3467	3473	0.937	0.951	0.553	0.599
5	1.0	20	40.0	0.25	2.5	0.25	9424	9441	0.941	0.953	0.612	0.609

The values $\alpha_{k,j}(l, m)$, $\beta_{k,j}(l, m)$, $\delta_{kjh}(l, m, n)$ and $\eta_{kj}(l, m)$ can be obtained recursively as follows:

A. The values $\alpha_{k,j}(l, m)$

basis: $\alpha_{i,j}(0, 0) = 0$.

iteration: Suppose we know $\alpha_{k,j}(l, m)$. Then there are two possibilities:

(i) $\Delta_k + lR_k < \Delta_j + mR_j$. Then we derive $\alpha_{k,j}(l+1, m)$ from

$$\alpha_{k,j}(l+1, m) = \alpha_{k,j}(l, m) + (l+1) * m * \min\{R_k, \Delta_j + mR_j - \Delta_k - lR_k\},$$

(ii) $\Delta_k + lR_k > \Delta_j + mR_j$. Then we derive $\alpha_{k,j}(l, m+1)$ from

$$\alpha_{k,j}(l, m+1) = \alpha_{k,j}(l, m) + l * (m+1) * \min\{\Delta_k + lR_k - \Delta_j - mR_j, R_j\}.$$

After $r_k + r_j$ iterations we have $\alpha_{k,j}(r_k, r_j)$.

TABLE A2

Target $\beta_w^c = 0.95$ and target $\beta_w = 0.9$

λ	$c^2[D]$	R_d	μ_{P_d}	$c^2[P_d]$	μ_{P_w}	$c^2[P_w]$	Mean stock		β_w^c		β_w	
							sim.	appr.	sim.	appr.	sim.	appr.
1	0.5	5	2.5	0.00	10.0	0.00	663	675	0.949	0.950	0.896	0.900
1	0.5	5	2.5	0.25	2.5	0.25	595	600	0.948	0.950	0.890	0.900
1	0.5	5	2.5	0.25	10.0	0.25	911	919	0.949	0.950	0.890	0.900
1	0.5	5	10.0	0.25	10.0	0.00	955	966	0.951	0.950	0.898	0.899
1	0.5	5	10.0	0.25	10.0	0.25	1170	1178	0.951	0.950	0.898	0.899
1	0.5	20	10.0	0.25	10.0	0.00	1143	1157	0.957	0.950	0.902	0.901
1	0.5	20	10.0	0.25	10.0	0.25	1352	1364	0.954	0.950	0.902	0.901
1	0.5	20	40.0	0.00	2.5	0.25	1199	1227	0.993	0.950	0.886	0.900
1	0.5	20	40.0	0.25	2.5	0.00	2393	2405	0.967	0.954	0.906	0.905
1	1.0	5	2.5	0.00	2.5	0.00	639	643	0.947	0.950	0.897	0.900
1	1.0	5	10.0	0.25	2.5	0.00	939	943	0.951	0.950	0.899	0.899
1	1.0	5	10.0	0.25	10.0	0.25	1260	1267	0.950	0.950	0.899	0.899
1	1.0	20	10.0	0.00	10.0	0.25	1331	1341	0.955	0.950	0.892	0.900
1	1.0	20	10.0	0.25	2.5	0.25	1152	1158	0.957	0.950	0.901	0.901
1	1.0	20	40.0	0.00	2.5	0.00	1276	1298	0.990	0.950	0.890	0.900
1	1.0	20	40.0	0.00	2.5	0.25	1289	1312	0.989	0.950	0.890	0.900
1	1.0	20	40.0	0.00	10.0	0.25	1526	1548	0.974	0.950	0.890	0.900
5	0.5	5	2.5	0.00	10.0	0.00	1784	1863	0.951	0.950	0.897	0.900
5	0.5	20	10.0	0.00	2.5	0.00	3526	3567	0.988	0.950	0.887	0.900
5	0.5	20	10.0	0.25	10.0	0.25	5702	5744	0.957	0.951	0.907	0.900
5	0.5	20	40.0	0.00	2.5	0.25	5056	5156	1.000	0.950	0.883	0.900
5	0.5	20	40.0	0.25	10.0	0.00	11143	11110	0.964	0.955	0.906	0.905
5	1.0	5	2.5	0.00	2.5	0.00	1703	1736	0.952	0.950	0.898	0.900
5	1.0	5	2.5	0.00	2.5	0.25	1881	1912	0.953	0.950	0.898	0.900
5	1.0	5	2.5	0.25	10.0	0.25	3552	3592	0.949	0.949	0.887	0.900
5	1.0	5	10.0	0.25	2.5	0.25	3475	3488	0.956	0.949	0.898	0.899
5	1.0	20	10.0	0.00	2.5	0.25	3771	3809	0.976	0.950	0.888	0.900
5	1.0	20	10.0	0.25	10.0	0.25	5796	5839	0.956	0.951	0.905	0.901
5	1.0	20	40.0	0.00	2.5	0.00	5056	5152	1.000	0.950	0.886	0.900
5	1.0	20	40.0	0.25	10.0	0.00	11224	11198	0.965	0.955	0.906	0.905
5	1.0	20	40.0	0.25	10.0	0.25	12027	12038	0.966	0.954	0.906	0.905

B. The values $\beta_{k,j}(l, m)$ basis: $\beta_{k,j}(0, 0) = 0$.iteration: Suppose we know $\beta_{k,j}(l, m)$. Then there are two possibilities:(i) $\Delta_k + lR_k < \Delta_j + mR_j$. Then we derive $\beta_{k,j}(l+1, m)$ from

$$\beta_{k,j}(l+1, m) = \beta_{k,j}(l, m) + (l+1)^2 * m * \min\{R_k, \Delta_j + mR_j - \Delta_k - lR_k\},$$

(ii) $\Delta_k + lR_k > \Delta_j + mR_j$. Then we derive $\beta_{k,j}(l, m+1)$ from

$$\beta_{k,j}(l, m+1) = \beta_{k,j}(l, m) + l^2 * (m+1) * \min\{\Delta_k + lR_k - \Delta_j - mR_j, R_j\},$$

After $r_k + r_j$ iterations we have $\beta_{k,j}(r_k, r_j)$.

TABLE A3

Target $\beta_w^c = 0.99$ and target $\beta_w = 0.6$

λ	$c^2[D]$	R_d	μ_{P_d}	$c^2[P_d]$	μ_{P_w}	$c^2[P_w]$	Mean stock		β_w^c		β_w	
							sim.	appr.	sim.	appr.	sim.	appr.
1	0.5	5	2.5	0.25	2.5	0.00	682	687	0.987	0.990	0.577	0.600
1	0.5	5	10.0	0.25	2.5	0.25	1012	1017	0.984	0.990	0.599	0.599
1	0.5	5	10.0	0.25	10.0	0.00	1069	1078	0.984	0.990	0.599	0.599
1	0.5	5	10.0	0.25	10.0	0.25	1393	1402	0.990	0.990	0.599	0.599
1	0.5	20	10.0	0.00	10.0	0.25	1432	1445	0.992	0.990	0.555	0.600
1	0.5	20	10.0	0.25	2.5	0.00	1190	1199	0.985	0.990	0.557	0.599
1	0.5	20	10.0	0.25	10.0	0.00	1240	1253	0.984	0.990	0.557	0.599
1	0.5	20	40.0	0.00	2.5	0.00	1494	1517	0.999	0.990	0.567	0.600
1	0.5	20	40.0	0.00	10.0	0.25	1767	1790	0.998	0.990	0.567	0.600
1	0.5	20	40.0	0.25	2.5	0.25	3255	3279	0.989	0.991	0.608	0.609
1	1.0	5	2.5	0.00	2.5	0.00	776	784	0.987	0.990	0.595	0.600
1	1.0	5	10.0	0.25	2.5	0.25	1114	1120	0.985	0.990	0.597	0.599
1	1.0	5	10.0	0.25	10.0	0.00	1209	1217	0.986	0.990	0.597	0.599
1	1.0	20	10.0	0.25	2.5	0.25	1320	1329	0.986	0.990	0.560	0.599
1	1.0	20	10.0	0.25	10.0	0.25	1664	1674	0.991	0.991	0.560	0.599
1	1.0	20	40.0	0.00	2.5	0.25	1613	1632	0.998	0.990	0.574	0.600
5	0.5	5	2.5	0.00	10.0	0.25	4431	4484	0.988	0.990	0.598	0.600
5	0.5	5	10.0	0.00	2.5	0.25	2446	2482	0.996	0.990	0.595	0.600
5	0.5	5	10.0	0.25	2.5	0.00	3738	3733	0.980	0.990	0.603	0.599
5	0.5	5	10.0	0.25	10.0	0.25	5600	5611	0.990	0.990	0.603	0.599
5	0.5	20	10.0	0.00	2.5	0.25	3977	4009	0.996	0.990	0.550	0.600
5	0.5	20	40.0	0.00	2.5	0.00	6421	6481	1.000	0.990	0.563	0.600
5	0.5	20	40.0	0.25	10.0	0.00	13709	13695	0.982	0.991	0.613	0.609
5	1.0	5	2.5	0.00	2.5	0.25	2204	2250	0.990	0.990	0.598	0.600
5	1.0	5	2.5	0.25	10.0	0.00	2321	2367	0.986	0.990	0.572	0.600
5	1.0	5	10.0	0.00	2.5	0.00	2312	2348	0.996	0.990	0.595	0.600
5	1.0	5	10.0	0.00	10.0	0.00	2540	2600	0.994	0.990	0.592	0.600
5	1.0	20	10.0	0.00	2.5	0.00	3908	3939	0.995	0.990	0.550	0.600
5	1.0	20	10.0	0.00	2.5	0.25	4090	4121	0.995	0.990	0.551	0.600
5	1.0	20	40.0	0.00	10.0	0.00	6064	6129	1.000	0.990	0.564	0.600
5	1.0	20	40.0	0.00	10.0	0.25	7638	7699	0.999	0.990	0.564	0.600

C. The values $\delta_{k,j,h}(l, m, n)$ basis: $\delta_{k,j,h}(0, 0, 0) = 0$.iteration: Suppose we know $\delta_{k,j,h}(l, m, n)$. Then there are three possibilities:

- (i) $\Delta_k + lR_k < \Delta_j + mR_j$ and $\Delta_k + lR_k < \Delta_h + nR_h$. Then we derive $\delta_{k,j,h}(l+1, m, n)$ from

$$\delta_{k,j,h}(l+1, m, n) = \delta_{k,j,h}(l, m, n) + (l+1) * m * n * \min\{R_k, \Delta_j + mR_j - \Delta_k - lR_k, \Delta_h + nR_h - \Delta_k - lR_k\},$$

- (ii) $\Delta_j + mR_j < \Delta_k + lR_k$ and $\Delta_j + mR_j < \Delta_h + nR_h$. Then we derive $\delta_{k,j,h}(l, m+1, n)$ from

$$\delta_{k,j,h}(l, m+1, n) = \delta_{k,j,h}(l, m, n) + l * (m+1) * n * \min\{R_j, \Delta_k + lR_k - \Delta_j - mR_j, \Delta_h + nR_h - \Delta_j - mR_j\},$$

TABLE A4

Target $\beta_w^c = 0.99$ and target $\beta_w = 0.9$

λ	$c^2[D]$	R_d	μ_{P_d}	$c^2[P_d]$	μ_{P_w}	$c^2[P_w]$	Mean stock		β_w^c		β_w	
							sim.	appr.	sim.	appr.	sim.	appr.
1	0.5	5	2.5	0.00	10.0	0.25	1259	1266	0.989	0.990	0.896	0.900
1	0.5	5	2.5	0.25	2.5	0.25	792	798	0.991	0.990	0.890	0.900
1	0.5	5	2.5	0.25	10.0	0.00	915	925	0.991	0.990	0.890	0.900
1	0.5	5	10.0	0.00	2.5	0.00	810	817	0.993	0.990	0.894	0.900
1	0.5	20	10.0	0.25	2.5	0.25	1310	1318	0.991	0.990	0.902	0.901
1	0.5	20	40.0	0.25	10.0	0.00	2880	2887	0.982	0.991	0.906	0.905
1	1.0	5	2.5	0.00	2.5	0.25	893	898	0.990	0.990	0.897	0.900
1	1.0	5	2.5	0.00	10.0	0.25	1392	1398	0.990	0.990	0.897	0.900
1	1.0	5	2.5	0.25	10.0	0.00	1073	1081	0.991	0.990	0.892	0.900
1	1.0	5	10.0	0.00	2.5	0.00	944	950	0.992	0.990	0.895	0.900
1	1.0	5	10.0	0.00	10.0	0.25	1473	1482	0.991	0.990	0.896	0.900
1	1.0	20	10.0	0.25	2.5	0.00	1397	1403	0.991	0.990	0.902	0.900
1	1.0	20	10.0	0.25	10.0	0.00	1551	1560	0.991	0.990	0.901	0.901
1	1.0	20	40.0	0.25	2.5	0.00	3009	3017	0.984	0.991	0.905	0.905
1	1.0	20	40.0	0.25	10.0	0.00	3001	3004	0.983	0.991	0.905	0.905
1	1.0	20	40.0	0.25	10.0	0.25	3251	3258	0.987	0.991	0.905	0.905
5	0.5	5	2.5	0.00	2.5	0.00	2064	2093	0.995	0.990	0.897	0.900
5	0.5	5	2.5	0.00	2.5	0.25	2376	2405	0.994	0.990	0.897	0.900
5	0.5	5	2.5	0.25	10.0	0.00	2427	2488	0.991	0.990	0.888	0.900
5	0.5	5	10.0	0.00	10.0	0.00	2611	2670	0.996	0.990	0.898	0.900
5	0.5	20	10.0	0.00	2.5	0.00	4198	4224	0.999	0.990	0.887	0.900
5	0.5	20	10.0	0.00	10.0	0.00	4292	4343	0.998	0.990	0.887	0.900
5	0.5	20	10.0	0.25	10.0	0.25	7279	7310	0.991	0.990	0.907	0.901
5	0.5	20	40.0	0.00	2.5	0.25	6674	6727	1.000	0.990	0.883	0.900
5	0.5	20	40.0	0.00	10.0	0.25	7957	8013	0.998	0.990	0.883	0.900
5	1.0	5	2.5	0.25	2.5	0.00	2359	2390	0.992	0.990	0.887	0.900
5	1.0	5	2.5	0.25	10.0	0.25	5042	5077	0.989	0.990	0.887	0.900
5	1.0	5	10.0	0.00	10.0	0.25	5184	5218	0.990	0.990	0.897	0.900
5	1.0	5	10.0	0.25	2.5	0.25	4270	4282	0.986	0.990	0.898	0.899
5	1.0	20	10.0	0.00	10.0	0.00	4497	4545	0.997	0.990	0.888	0.900
5	1.0	20	10.0	0.00	10.0	0.25	6741	6771	0.992	0.991	0.888	0.900
5	1.0	20	40.0	0.00	2.5	0.00	6618	6672	1.000	0.990	0.886	0.900
5	1.0	20	40.0	0.00	2.5	0.25	6777	6831	1.000	0.990	0.886	0.900

(iii) $\Delta_h + nR_h < \Delta_k + lR_k$ and $\Delta_h + nR_h < \Delta_j + mR_j$. Then we derive $\delta_{k,j,h}(l, m, n + 1)$ from

$$\begin{aligned} \delta_{k,j,h}(l, m, n + 1) &= \delta_{k,j,h}(l, m, n) \\ &+ l * m * (n + 1) * \min\{R_h, \Delta_j + mR_j - \Delta_h - nR_h, \Delta_k + lR_k - \Delta_h - nR_h\}. \end{aligned}$$

After $r_k + r_j + r_h$ iterations we have $\delta_{k,j,h}(r_k, r_j, r_h)$.

D. The values $\eta_{k,j}(l, m)$
basis: $\eta_{k,j}(0, 0) = 0$.

TABLE A5

Target $R_d = 5$

R_m	R_d	μ_{P_m}	μ_{P_d}	$c^2[P_m]$	$c^2[P_d]$	Mean stock		β_w^c		β_w		β_d	
						sim.	appr.	sim.	appr.	sim.	appr.	sim.	appr.
5	5	5	5	0.00	0.00	2312	2335	0.943	0.949	0.526	0.600	0.600	0.600
5	5	5	5	0.00	0.25	2442	2460	0.942	0.949	0.584	0.598	0.600	0.600
5	5	5	5	0.25	0.00	2465	2482	0.940	0.949	0.564	0.600	0.596	0.600
5	5	5	5	0.25	0.25	2595	2607	0.942	0.949	0.580	0.595	0.596	0.600
5	5	15	5	0.00	0.00	2473	2503	0.952	0.950	0.600	0.600	0.605	0.600
5	5	15	5	0.00	0.25	2604	2629	0.952	0.949	0.632	0.597	0.605	0.600
5	5	15	5	0.25	0.00	3261	3249	0.943	0.950	0.654	0.601	0.610	0.601
5	5	15	5	0.25	0.25	3384	3377	0.946	0.949	0.627	0.599	0.610	0.601
5	5	15	15	0.00	0.00	2667	2703	0.959	0.950	0.606	0.600	0.605	0.600
5	5	15	15	0.00	0.25	3451	3464	0.948	0.947	0.626	0.599	0.605	0.600
5	5	15	15	0.25	0.00	3316	3310	0.942	0.950	0.657	0.601	0.610	0.601
5	5	15	15	0.25	0.25	4156	4152	0.951	0.951	0.631	0.626	0.610	0.601
5	5	45	15	0.00	0.00	3312	3358	0.982	0.950	0.780	0.600	0.612	0.600
5	5	45	15	0.00	0.25	4058	4095	0.969	0.948	0.693	0.599	0.612	0.600
5	5	45	15	0.25	0.00	5954	5890	0.903	0.908	0.648	0.543	0.527	0.519
5	5	45	15	0.25	0.25	6646	6635	0.926	0.923	0.606	0.584	0.527	0.519
10	5	5	5	0.00	0.00	2436	2454	0.939	0.949	0.521	0.600	0.568	0.600
10	5	5	5	0.00	0.25	2551	2565	0.940	0.947	0.556	0.584	0.568	0.600
10	5	5	5	0.25	0.00	2559	2574	0.939	0.949	0.565	0.600	0.567	0.598
10	5	5	5	0.25	0.25	2669	2700	0.941	0.949	0.573	0.594	0.567	0.598
10	5	15	5	0.00	0.00	2593	2619	0.950	0.949	0.580	0.600	0.580	0.600
10	5	15	5	0.00	0.25	2711	2736	0.950	0.948	0.600	0.588	0.580	0.600
10	5	15	5	0.25	0.00	3306	3300	0.943	0.951	0.650	0.602	0.601	0.598
10	5	15	5	0.25	0.25	3427	3426	0.946	0.951	0.622	0.601	0.601	0.598
10	5	15	15	0.00	0.00	2771	2804	0.957	0.949	0.589	0.600	0.580	0.600
10	5	15	15	0.00	0.25	3555	3567	0.948	0.947	0.615	0.599	0.580	0.600
10	5	15	15	0.25	0.00	3354	3352	0.941	0.951	0.653	0.603	0.601	0.593
10	5	15	15	0.25	0.25	4203	4304	0.953	0.953	0.631	0.630	0.601	0.598
10	5	45	15	0.00	0.00	3420	3466	0.982	0.950	0.760	0.600	0.598	0.600
10	5	45	15	0.00	0.25	4187	4222	0.970	0.949	0.693	0.605	0.598	0.600
10	5	45	15	0.25	0.00	6633	6552	0.930	0.942	0.727	0.592	0.615	0.599
10	5	45	15	0.25	0.25	7269	7232	0.948	0.945	0.668	0.621	0.607	0.593

iteration: Suppose we know $\eta_{k,j}(l, m)$. Then there are two possibilities:

(i) $\Delta_k + lR_l < \Delta_j + mR_j$. Then we derive $\eta_{k,j}(l+1, m)$ from

$$\begin{aligned} \eta_{k,j}(l+1, m) &= \eta_{k,j}(l, m) \\ &+ (l+1) * m * K(\Delta_k + lR_k, \min\{R_k, \Delta_j + mR_j - \Delta_k - lR_j\}), \end{aligned}$$

(ii) $\Delta_k + lR_k > \Delta_j + mR_j$. Then we derive $\eta_{k,j}(l, m+1)$ from

$$\begin{aligned} \eta_{k,j}(l, m+1) &= \eta_{k,j}(l, m) \\ &+ l * (m+1) * K(\Delta_j + mR_j, \min\{\Delta_k + lR_k - \Delta_j - mR_j, R_j\}). \end{aligned}$$

TABLE A6

Target $R_d = 10$

R_m	R_d	μ_{P_m}	μ_{P_d}	$c^2[P_m]$	$c^2[P_d]$	Mean stock		β_w^c		β_w		β_d	
						sim.	appr.	sim.	appr.	sim.	appr.	sim.	appr.
10	10	5	5	0.00	0.00	2995	3035	0.941	0.949	0.528	0.600	0.603	0.600
10	10	5	5	0.00	0.25	3097	3132	0.939	0.949	0.562	0.599	0.603	0.600
10	10	5	5	0.25	0.00	2828	2850	0.934	0.949	0.535	0.600	0.544	0.598
10	10	5	5	0.25	0.25	2933	2954	0.934	0.949	0.553	0.598	0.544	0.598
10	10	15	5	0.00	0.00	3106	3147	0.948	0.949	0.565	0.600	0.602	0.600
10	10	15	5	0.00	0.25	3216	3254	0.948	0.949	0.600	0.599	0.602	0.600
10	10	15	5	0.25	0.00	3502	3495	0.939	0.951	0.598	0.598	0.599	0.598
10	10	15	5	0.25	0.25	3614	3613	0.942	0.951	0.597	0.599	0.599	0.598
10	10	15	15	0.00	0.00	3308	3356	0.958	0.949	0.582	0.600	0.602	0.600
10	10	15	15	0.00	0.25	4040	4068	0.948	0.950	0.614	0.601	0.602	0.600
10	10	15	15	0.25	0.00	3593	3593	0.942	0.951	0.615	0.603	0.599	0.598
10	10	15	15	0.25	0.25	4290	4291	0.945	0.950	0.594	0.601	0.599	0.598
10	10	45	15	0.00	0.00	3872	3929	0.981	0.950	0.731	0.600	0.609	0.600
10	10	45	15	0.00	0.25	4584	4632	0.970	0.950	0.684	0.601	0.609	0.600
10	10	45	15	0.25	0.00	6745	6641	0.932	0.939	0.719	0.589	0.611	0.593
10	10	45	15	0.25	0.25	7315	7255	0.943	0.943	0.648	0.596	0.611	0.593
20	10	5	5	0.00	0.00	3209	3238	0.941	0.949	0.532	0.600	0.564	0.600
20	10	5	5	0.00	0.25	3319	3344	0.942	0.949	0.568	0.598	0.564	0.600
20	10	5	5	0.25	0.00	3123	3135	0.936	0.949	0.556	0.600	0.550	0.599
20	10	5	5	0.25	0.25	3232	3243	0.938	0.949	0.565	0.595	0.550	0.599
20	10	15	5	0.00	0.00	3358	3392	0.951	0.949	0.569	0.600	0.571	0.600
20	10	15	5	0.00	0.25	3473	3506	0.952	0.949	0.598	0.598	0.571	0.600
20	10	15	5	0.25	0.00	3728	3726	0.941	0.951	0.610	0.602	0.591	0.600
20	10	15	5	0.25	0.25	3828	3829	0.943	0.951	0.606	0.599	0.591	0.600
20	10	15	15	0.00	0.00	3535	3574	0.958	0.949	0.588	0.600	0.571	0.600
20	10	15	15	0.00	0.25	4249	4272	0.950	0.951	0.609	0.598	0.571	0.600
20	10	15	15	0.25	0.00	3780	3782	0.941	0.951	0.628	0.603	0.591	0.600
20	10	15	15	0.25	0.25	4476	4482	0.946	0.951	0.597	0.603	0.591	0.600
20	10	45	15	0.00	0.00	4125	4179	0.981	0.950	0.719	0.600	0.582	0.600
20	10	45	15	0.00	0.25	4812	4870	0.970	0.951	0.671	0.598	0.579	0.600
20	10	45	15	0.25	0.00	6863	6774	0.936	0.950	0.719	0.599	0.609	0.599
20	10	45	15	0.25	0.25	7418	7369	0.944	0.950	0.650	0.604	0.609	0.599

Here the value $K(x, a)$ is defined by

$$K(x, a) = \int_x^{x+a} [1 - G_i(y)] dy.$$

For deterministic lead time L_i we have

$$K(x, a) = \begin{cases} a, & L_i > x + a, \\ L_i - x, & x \leq L_i \leq x + a, \\ 0, & L_i < x. \end{cases}$$

For Erlang distributed lead time we have

$$K(x, a) = \sum_{k=0}^{r-1} \frac{(r-k)}{\alpha} * \frac{(\alpha x)^k * e^{-\alpha x} - [\alpha(x+a)]^k * e^{-\alpha(x+a)}}{k!}.$$

Appendix 3: Tables for 2-echelon models

This appendix contains a comparison between simulated and approximated performance characteristics for the 2-echelon models as described in Section 7. The results are divided over 4 tables (Tables A1–A4), one table for each combination of target service level for external customers β_w^c and target service level for the wholesalers β_w .

As the tables show, the approximate service levels may be slightly different from the target values. The reason for this is the following.

Implementation of the assumption that orders do not overtake each other in a simulation program is difficult. If an order is about to overtake a preceding order, it is rescheduled just after its predecessor. Therefore, the resulting normal order delivery time distribution is different from the distribution specified in the input. It is possible to correct the input distribution to allow for this phenomenon, but a minor difference will remain. Hence, simulation parameters are different from the parameters used for the approximation and, therefore, the results are not fully comparable.

To overcome this, we choose to recalculate the approximations using the first two moments of the resulting normal order delivery time distributions from the simulation. As a consequence, simulation and approximation are better comparable, but the approximate service levels may differ from the targets (as can be seen from the tables).

Appendix 4. Tables for 3-echelon models

A comparison between simulated and approximated performance characteristics for the 3-echelon models as described in Section 8 is given in Tables A5 and A6.

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