ON THE CONTINUED ERLANG LOSS FUNCTION

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We prove two fundamental results in teletraffic theory. The first is the frequently conjectured convexity of the analytic continuation $B(x, a)$ of the classical Erlang loss function as a function of x, $x \geq 0$. The second is the uniqueness of the solution of the basic set of equations associated with the 'equivalent random method'.

convexity • complete monotonicity • Erlang loss function • equivalent random method • peakedness factor • teletraffic theory.

1. Introduction and results

This paper focuses on the functions

$$B(x, a) \equiv \left\{ a \int_0^\infty e^{-at}(1+t)^x \mathrm{d}t \right\}^{-1}, \tag{1}$$

$$m(x, a) \equiv aB(x, a) \tag{2}$$

and

$$z(x, a) \equiv 1 - m(x, a) + \frac{a}{x + 1 + m(x, a) - a}. \tag{3}$$

For a > 0 and x a non-negative integer $B(x, a)$ is easily seen to equal Erlang's loss function,

$$B(x, a) = \left\{ \sum_{i=0}^{x} \frac{a^i}{i!} \right\}^{-1} \frac{a^x}{x!}, \tag{4}$$

which may be interpreted as the probability that an arriving customer finds all servers busy in an x-server loss system in equilibrium, where service times are exponentially distributed with mean ν^{-1}, say, and where inter-arrival times are exponentially distributed with mean $(a\nu)^{-1}$. In the same context $m(x, a)$ and $z(x, a)$ are the mean and peakedness factor (variance-to-mean ratio), respectively, of the stream of blocked customers (see Cooper [3] or Wilkinson [17]); as is usual in teletraffic theory these quantities refer to the distribution of the number of busy servers in an infinite server system to which the stream of blocked customers is offered.

In several teletraffic studies the need arose to extend the definition of the Erlang loss function to non-integral values of x. We mention approximation techniques like the 'equivalent random method' (Wilkinson [17]), 'Haywards approximation' (Fredericks [6]) and the 'decomposition method' (Sanders et al. [13]), and network dimensioning algorithms as in Rapp [12], Akimura et al. [2] and Kortanek et al. [9], where often also derivatives of $B(x, a)$ with respect to x and a are needed. Depending on the application one has in mind there are various ways to perform the interpolation: a simple linear approach was adopted in [9], while Rapp [12] introduced a parabolic interpolation. However, the most commonly used extension is the analytic continuation (1) of the Erlang loss function; cf. Jagerman [7, 8].

In this paper we shall prove some properties of the (continued) Erlang loss function which are generally believed to be valid but for which no proofs seem to exist in the literature. Our most significant result, given in Section 2, is a proof of the following theorem, the validity of which has frequently been conjectured (Smith and Whitt [14]).

Theorem 1. *$B(x, a)$ is a convex function of x in the interval* $[0, \infty)$ *for every* $a > 0$.

We remark that Syski [15, p. 603] claims convexity if $a \leq 1$, but offers no proof nor any refer-

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ence substantiating his claim. If one restricts the domain of x to the non-negative integers, however, then several proofs for the convexity of $B(x, a)$ exist (see Messerli [10] and references therein).

A nice application of Theorem 1 is that it enables one to give a very short proof of the efficiency of resource sharing for $M/M/s$ loss systems; the argument is given in the appendix of [14].

Our second result, given in Section 3, is a proof of the following.

Theorem 2. *If $\mu > 0$ and $\zeta \geq 1$, then there is a unique solution $x \geq 0$, $a > 0$ to the system of equations $\mu = m(x, a)$, $\zeta = z(x, a)$.*

Solving the system of equations mentioned in this theorem is an essential step in Wilkinson's 'equivalent random method' [17]. It is obvious that in general there will be no solution with integral x, so that an extension of the Erlang loss function to non-integral values of x is called for. In Wilkinson's original paper [17] this extension is not made explicit. In studies like [8], where explicit use is made of the analytic continuation (1) of Erlang's loss function, the existence of a unique solution is tacitly assumed.

As a by-product of the proof of Theorem 2 we will show the validity of the next result.

Theorem 3. *If $x > 0$ and $a > 0$, then $z(x, a) > 1$.*

Again this result is well known for positive integral values of x; different proofs were given by Frankel [5], Pearce [11] and Van Doorn [16].

2. Proof of Theorem 1

We start with a simple auxiliary lemma.

Lemma 1. *Let $y \geq 0$, $p(u) \geq 0$ and both $p(u)$ and $q(u)$ be increasing functions of u for $u \geq 0$. Then $\int_0^y p(u)q(u)\mathrm{d}u \geq 0$ as soon as $\int_0^y q(u)\mathrm{d}u \geq 0$.*

Proof. Let $\int_0^y q(u)\mathrm{d}u \geq 0$. Choose $y_0 \in [0, y]$ such that $q(u) \leq 0$ for $u \in [0, y_0)$ and $q(u) \geq 0$ for $u \in (y_0, y]$. Then $\int_0^y p(u)q(u)\mathrm{d}u \geq p(y_0)\int_0^y q(u)\mathrm{d}u \geq 0$. □

Let h denote any increasing, concave and continuously differentiable function on $[0, \infty)$ satisfying $h(0) = 0$ and $h(t) = O(t)$ as $t \to \infty$. For $a > 0$, $x \geq 0$ we define

$$f(x, a) \equiv \int_0^\infty \exp(-at + xh(t))\mathrm{d}t \tag{5}$$

and

$$\phi(x, a) \equiv 2(f'(x, a))^2 - f(x, a)f''(x, a), \tag{6}$$

where a prime indicates differentiation with respect to x. The conditions imposed on h imply that these derivatives exist and that

$$f^{(n)}(x, a) = \int_0^\infty h(t)^n \exp(-at + xh(t))\mathrm{d}t \tag{7}$$

for $n = 0, 1, 2, \ldots$. From now on we shall tacitly assume that $x \geq 0$ and $a > 0$.

Proposition 1. *$\phi(x, a)$ is a completely monotone function of a.*

Proof. By (6) and (7) and the convolution theorem for Laplace transforms $\phi(x, a) = \int_0^\infty e^{-at}g(x, t)\mathrm{d}t$, where

$$g(x, t) \equiv \int_0^t \left(2h(u)h(t-u) - h(u)^2\right) \times \exp(x(h(u) + h(t-u)))\mathrm{d}u.$$

By symmetry arguments,

$$g(x, t) = \int_0^{t/2} \left(2h(u)h(t-u) - (h(u) - h(t-u))^2\right) \times \exp(x(h(u) + h(t-u)))\mathrm{d}u.$$

To prove the proposition we must show that $g(x, t) \geq 0$ for all $x, t \geq 0$ (see Feller [4, Sect. XIII.4]). Now, since h is concave, $h(u) + h(t-u)$ is increasing in u for $0 \leq u \leq t/2$, and hence, so is $p(u) \equiv \exp(x(h(u) + h(t-u)))$. The same is valid for $-(h(u) - h(t-u))^2$ and $2h(u)h(t-u)$, as is easily verified. Therefore Lemma 1 applies and it suffices to show that $g(0, t) \geq 0$ for $t \geq 0$. By (6), (7) and integration by parts we obtain

$$\int_0^\infty e^{-at}g(0, t)\mathrm{d}t = \phi(0, a)$$

$$= 2\left\{\int_0^\infty e^{-at}h(t)\mathrm{d}t\right\}^2 - a^{-1}\int_0^\infty e^{-at}h(t)^2\mathrm{d}t$$

$$= 2a^{-2}\left\{\int_0^\infty e^{-at}h'(t)\mathrm{d}t\right\}^2 - 2a^{-2}\int_0^\infty e^{-at}h(t)h'(t)\mathrm{d}t,$$

where we have used the initial and final conditions on h. Finally using the convolution theorem again we get

$$\tfrac{1}{2}a^2 g(0, t) = \int_0^t h'(u)h'(t-u)\mathrm{d}u - h(t)h'(t)$$

$$= \int_0^t h'(u)(h'(t-u) - h'(t))\mathrm{d}t \geq 0,$$

since $h'(t-u) \geq h'(t)$ by the concavity of h. □

The next proposition, a counterpart to Proposition 1, is a direct consequence of $h(u)^2 + h(t-u)^2 - 2h(u)h(t-u) = (h(u) - h(t-u))^2 \geq 0$.

Proposition 2. *If* $\psi(x, a) \equiv f(x, a)f''(x, a) - (f'(x, a))^2$, *then* $\psi(x, a)$ *is a completely monotone function of* a.

Thus not only $(-\partial/\partial a)^n \phi(x, a) \geq 0$ but also $(-\partial/\partial a)^n \psi(x, a) \geq 0$ for $n = 0, 1, \ldots$. In particular for $n = 0$,

$$(f'(x, a))^2 \leq f(x, a)f''(x, a) \leq 2(f'(x, a))^2. \tag{8}$$

In passing we note that the easy part of (8) – the inequality on the left, which corresponds to $\psi(x, a) \geq 0$ – is just a Cauchy–Schwarz inequality. Our next result follows at once from $(\partial/\partial x)^2 f(x, a)^{-1} = \phi(x, a)f(x, a)^{-3} \geq 0$ and $(\partial/\partial x)^2 \log f(x, a) = \psi(x, a)f(x, a)^{-2} \geq 0$.

Corollary. (*i*) $f(x, a)^{-1}$ *is a convex function of* x.
(*ii*) $f(x, a)$ *is a log-convex function of* x.

Choosing $h(t) = \log(1 + t)$, a concave increasing function with $h(0) = 0$, (i) above gives Theorem 1, while (ii) states that $B(x, a)^{-1}$ is a log-convex function of x, the latter result being well known [7].

3. Proof of Theorems 2 and 3

Let $h(t) = \log(1 + t)$. Then, in particular, $f(x, a) = m(x, a)^{-1}$ by (1), (2) and (5). Our problem is to determine the number of solutions (x, a), with $x \geq 0$ and $a > 0$, of the two simultaneous equations

$$\mu = m(x, a), \quad \zeta = z(x, a). \tag{9}$$

We observe that $f(0, a) = a^{-1}$ and $f(x, a) \uparrow \infty$ as $x \to \infty$. Accordingly, $m(0, a) = a$ and $m(x, a) \downarrow 0$ as $x \to \infty$. Also $z(0, a) = 1$.

Lemma 2. *Let* $x > 0$ *and* $a > 0$. *Then,*

$$\max\{0, a - x\} < m(x, a) < a \tag{10}$$

and

$$1 < m(x, a) + z(x, a) < a + 1. \tag{11}$$

Proof. Clearly, $0 < m(x, a) < a = m(0, a)$ for $x > 0$. Suppose $a > x > 0$. Then $m(x, a)^{-1} = f(x, a) = \int_0^\infty \exp(-(a-x)t)\exp(-x(t - h(t)))\mathrm{d}t < f(a - x, 0) = (a - x)^{-1}$, since $t > h(t)$ for $t > 0$. This establishes (10). But also $x + 1 + m(x, a) - a > 1$, so that (11) follows from (3). □

From Lemma 2 we conclude that (9) has no solution if $\mu \leq 0$ or $\mu + \zeta \leq 1$. So from now on we suppose that $\mu > 0$ and $\mu + \zeta > 1$. Note also that if (x, a), $x \geq 0$ and $a > 0$, is a solution of (9), then $a \geq \mu$ and $a + 1 \geq \mu + \zeta$ with equality if and only if $x = 0$.

We put $p = (\mu + \zeta)/(\mu + \zeta - 1)$, $q = (\mu + 1)/p$ and observe that $p > 1$ and

$$\zeta \gtreqless 1 \Leftrightarrow q \gtreqless \mu. \tag{12}$$

Solving for x in $\zeta = z(x, a)$ we get $x = ap - m - 1 = p(a - q)$, by virtue of (3). So we can reformulate our problem as that of determining the number of solutions a, with $a \geq q$, of the single equation

$$F(a) = \mu^{-1}, \tag{13}$$

where $F(a) = f(p(a - q), a)$.

Lemma 3. $F(q) = q^{-1}$, *F is convex on* $[q, \infty)$ *and* $F(a) \to \infty$ *as* $a \to \infty$.

Proof. $F(q) = f(0, q) = q^{-1}$. Furthermore, $F''(a) = \int_0^\infty (-t + ph(t))^2 \ \exp(-at + p(a - q)h(t))\mathrm{d}t > 0$ and $F(a) > \int_0^{t_0} \exp(a(ph(t) - t)) \ \exp(-pqh(t))\mathrm{d}t$, where $t_0 > 0$ is chosen such that $ph(t) - t > 0$ for $0 < t < t_0$, which is possible since $p > 1$. Obviously, this last integral tends to ∞ as $a \to \infty$. □

If $\zeta > 1$, then, in view of (12), $\mu^{-1} > q^{-1} = F(q)$. Hence (13) has at most one solution $a \geq q$, since F is convex on $[q, \infty)$, and at least one, since $F(a) \to \infty$ as $a \to \infty$. This establishes Theorem 2 apart from the case $\zeta = 1$.

To prove Theorem 3 and the remainder of Theorem 2 we observe the following.

Lemma 4. *If* $\zeta \leq 1$, *then* $F'(q) > 0$.

Proof. By integration by parts and [1, (5.1.19)] we have $q^2F'(q) = q^2\int_0^\infty(-t + ph(t))\exp(-qt)\mathrm{d}t = -1 + pq\int_0^\infty h'(t)\exp(-qt)\,\mathrm{d}t = -1 + pq\mathrm{e}^q\int_1^\infty t^{-1}\exp(-qt)\mathrm{d}t > -1 + pq/(q+1) = (m-q)/(q+1) > 0$, as required, in view of (12). □

It now follows with (12) that if $\zeta = 1$, then $\mu^{-1} = q^{-1} = F(q) < F(a)$ for all $a > q$, so $a = \mu$ is the only solution to (13) in $[q, \infty)$, which completes the proof of Theorem 2.

From (12) and Lemma 4 we also see that if $\zeta < 1$, then $\mu^{-1} < q^{-1} = F(q) \leq F(a)$ for all $a \geq q$, so there is no solution to (13) in $[q, \infty)$. This, together with Theorem 2 and the observation that $x = 0$ if $z(x, a) = 1$ and $m(x, a) > 0$, establishes the proof of Theorem 3.

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