

Fuzzy Random Variables—**II. Algorithms and Examples for the Discrete Case**

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ABSTRACT

The results obtained in part I of the paper are specialized to the case of discrete fuzzy random variables. A more intuitive interpretation is given of the notion of fuzzy random variables. Algorithms are derived for determining expectations, fuzzy probabilities, fuzzy conditional expectations and fuzzy conditional probabilities related to discrete fuzzy random variables. These algorithms are applied to illustrative examples. A sample application to a medical diagnosis problem is briefly discussed.

1. INTRODUCTION

In part I of this paper [1], a rather abstract definition of fuzzy random variables was given, and a number of results were proved. In the present paper, the definition and results will be made more concrete for discrete fuzzy random variables. Algorithms will be given for the determination of expectations and fuzzy probabilities connected to fuzzy random variables.

Fuzzy random variables are random variables whose values are not real numbers, as usually is the case, but fuzzy numbers. Fuzzy numbers are numbers whose values are only vaguely defined. A fuzzy number may assume different real values, with each of which a degree of acceptability is associated. These degrees of acceptability are considered as truth values, and are handled according to the rules of fuzzy logic, as explained in part I of the paper.

As an example of a (discrete) fuzzy random variable, let us consider the results of the opinion poll discussed in Sec. 1 of part I of the paper. The response to a question concerning the opinion of the person interviewed about the weather in Europe during a particular summer is summarized in Table 1. It is assumed that the three responses "very warm," "warm," and "no opinion"

TABLE I
Results of an Opinion Poll

Fraction of respondents	Response
0.4	Very warm
0.5	Warm
0.1	No opinion

may be characterized as fuzzy numbers with membership functions as sketched in Fig. 1.

The choice of these membership functions is a problem by itself and will not be discussed here. It is only noted that the membership function for "no opinion" was obtained from the other two membership functions as follows. The left side is the left side of the membership function for "warm," and the right side that of the membership function for "very warm," while the intermediate section was obtained by assuming all intermediate values to be fully plausible. This appears to lead to an unprejudiced choice of the membership function for "no opinion."

We shall use this example to illustrate various notions introduced in part I of the paper.

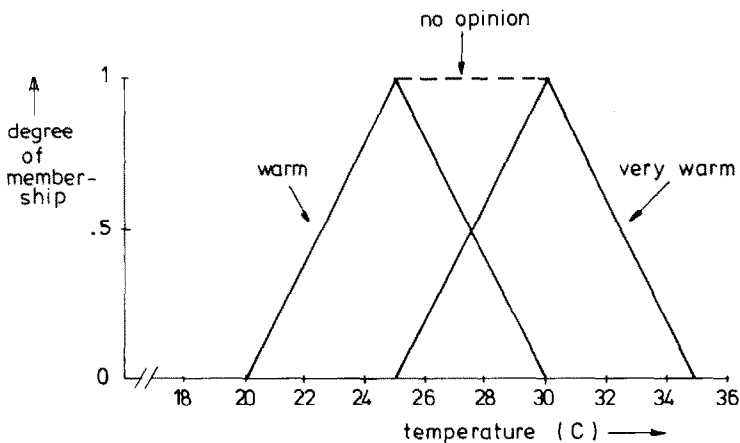


Fig. 1. Membership functions of "warm," "very warm," and "no opinion."

2. DISCRETE FUZZY RANDOM VARIABLES

A fuzzy random variable X defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is characterized by a map $X: \Omega \rightarrow S$ such that

$$\omega \mapsto X_\omega,$$

where S is the space of all piecewise continuous functions $R \rightarrow [0, 1]$. Each element of the space S is the membership function of a fuzzy number. The map X has to satisfy certain measurability conditions specified in part I of the paper.

The fuzzy random variable X is said to be discrete if Ω is a countable set. When we are dealing with a single discrete fuzzy random variable X we may as well take $\Omega = N$, with N the set of natural numbers, and \mathcal{F} the sigma algebra of subsets of N . We shall denote $\mathcal{P}(\{i\}) = p_i, i \in N$, and

$$i \mapsto X^i \quad \text{for all } i \in N.$$

When we deal with two discrete fuzzy random variables X and Y , both defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, we take $\Omega = N \times N$ such that

$$(i, j) \mapsto X^i \quad \text{and} \quad (i, j) \mapsto Y^j,$$

and denote $\mathcal{P}(\{(i, j)\}) = p_{ij}$ for all $(i, j) \in N \times N$.

This means that a single discrete fuzzy random variable X is essentially characterized by the set of pairs $(p_i, X^i), i = 1, 2, \dots$, where $p_i, i = 1, 2, \dots$, are probabilities, adding up to one, and $X^i, i = 1, 2, \dots$, are membership functions, characterizing the fuzzy values assumed by the fuzzy random variable. Similarly, two jointly discrete fuzzy random variables X and Y are characterized by the triples $(p_{ij}, X^i, Y^j), i = 1, 2, \dots, j = 1, 2, \dots$, with the p_{ij} probabilities and the X^i and Y^j membership functions characterizing fuzzy values.

The results of the opinion poll described in the preceding section may be considered as an example of a discrete fuzzy random variable. The probabilities are $p_1 = 0.4, p_2 = 0.5, p_3 = 0.1, p_i = 0$ for $i \geq 4$, while X^1, X^2 and X^3 are the three membership functions for respectively "very warm," "warm" and "no opinion" depicted in Fig. 1.

In Part I of the paper a fuzzy random variable was eventually defined as a fuzzy set $X = (\tilde{X}, X)$, where \tilde{X} is the set of "originals" of the fuzzy random variable. An original is an ordinary random variable \tilde{U} , defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{P}}) = (\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathcal{P} \otimes \mathcal{P}')$, measurable with respect to $\sigma(X) \otimes \mathcal{F}'$. Here $(\Omega', \mathcal{F}', \mathcal{P}')$ is an auxiliary probability space, while $\sigma(X)$ is the sigma algebra generated by X , as explained in part I.

In the case of a single discrete fuzzy random variable \mathbf{X} , an original of \mathbf{X} is a $\sigma(X) \otimes \mathcal{F}'$ -measurable random variable \tilde{U} defined on $N \times \Omega'$. We shall denote the value assumed by \tilde{U} at the point $(i, \omega') \in N \times \Omega'$ as $\tilde{U}_i(\omega')$. The degree of membership of \tilde{U} in the fuzzy set $\mathbf{X} = (\tilde{\mathcal{X}}, X)$ is given by

$$\inf_{i \in N} \inf_{\omega' \in \Omega'} X^i(\tilde{U}_i(\omega')). \tag{2.1}$$

We now give the following interpretation of the introduction of the auxiliary probability space $(\Omega', \mathcal{F}', \mathcal{P}')$. We consider the original sample space N as representative of a population divided into disjoint groups, successively numbered $1, 2, \dots$. The (fuzzy) value X^i assumed by the fuzzy random variable corresponding to the i th group represents the (fuzzy) opinion of the group regarding the issue at hand. Thus, the acceptability that the i th group endorses a real number u as the opinion of the group (in the example: concerning the temperature) is $X^i(u)$.

The introduction of the auxiliary probability space $(\Omega', \mathcal{F}', \mathcal{P}')$ allows for a divided opinion within each group. Thus, for fixed i , the random variable $\tilde{U}_i(\omega')$, $\omega' \in \Omega'$, represents a certain division of opinions over the i th group. The acceptability that $\tilde{U}_i(\omega')$, $\omega' \in \Omega'$, actually represents the opinion of the i th group is $\inf_{\omega' \in \Omega'} X^i(\tilde{U}_i(\omega'))$. The acceptability that \tilde{U} represents the distribution of opinions over all groups hence is given by (2.1).

To continue this discussion, let us consider the definition of the expectation of a fuzzy discrete random variable \mathbf{X} . Suppose that the random variable \tilde{U} as described above is a possible description for the distribution of opinions. The acceptability that \tilde{U} represents the opinions of the groups is given by (2.1). The corresponding average opinion is $E\tilde{U}$. Since there are many distributions of opinions \tilde{U} leading to a given average opinion x , the acceptability that the average opinion is x is given by

$$\sup_{\tilde{U} \in \tilde{\mathcal{X}}: E\tilde{U} = x} \inf_{i \in N} \inf_{\omega' \in \Omega'} X^i(\tilde{U}_i(\omega')). \tag{2.2}$$

If we define the fuzzy number EX as the fuzzy set $(R, (EX), s)$, with $s(x)$ the statement “the average opinion is x ,” (2.2) gives the degree of membership $t(s(x)) = (EX)(x)$. EX simply is the image of the fuzzy set $\mathbf{X} = (\tilde{\mathcal{X}}, X)$ in R under the map $E: \tilde{\mathcal{X}} \rightarrow R$. This is the way EX was defined in part I of the paper.

Other concepts relating to fuzzy random variables, such as probabilities and conditional expectations, are defined completely analogously. The precise definitions are given in part I.

Throughout this paper, we assume that each discrete fuzzy random variable \mathbf{X} we consider satisfies the following conditions, already stipulated in part I.

We require that for each $\mu \in (0, 1]$ and each $i \in N$,

$$u_i^*(\mu) \stackrel{\text{df}}{=} \inf \{ x \in R \mid X^i(x) \geq \mu \} > -\infty, \quad (2.3)$$

$$u_i^{**}(\mu) \stackrel{\text{df}}{=} \sup \{ x \in R \mid X^i(x) \geq \mu \} < +\infty, \quad (2.4)$$

while

$$X^i(u_i^*(\mu)) \geq \mu, \quad X^i(u_i^{**}(\mu)) \geq \mu. \quad (2.5)$$

The first two conditions imply that for each i the function X^i must have finite support. The third condition is satisfied if for each i the function X^i is upper semicontinuous, i.e., for each $\mu \in (0, 1]$ the set $\{x \in R \mid X^i(x) \geq \mu\}$ is closed. The requirement of a finite support somewhat restricts the generality of the theory, although in concrete cases the support can probably always be chosen large enough to suit all purposes. The semicontinuity condition does not appear to impose any real restriction.

In addition to these conditions, we require that each random variable X we consider should be normal. This means that for each i there exists an $x_i \in R$ such that $X^i(x_i) = 1$.

3. EXPECTATION OF A DISCRETE FUZZY RANDOM VARIABLE

In this section we shall give algorithms for determining the (fuzzy) expectation of a fuzzy random variable. The clue to these algorithms is to find the family of level sets corresponding to a given fuzzy set. If these level sets are known, the fuzzy set is completely characterized. Suppose that (M, m) is a fuzzy set, defined on the basic space M , with $m: M \rightarrow [0, 1]$ its membership function. Then for each $\mu \in [0, 1]$, the set

$$C_\mu = \{ a \in M \mid m(a) \geq \mu \} \quad (3.1)$$

is called a level set. Given the family of level sets C_μ , the membership function m may be recovered with the aid of the formula

$$m(a) = \sup \{ \mu \in [0, 1] \mid a \in C_\mu \}, \quad a \in M. \quad (3.2)$$

In general (though not always) the algorithms that will be presented only allow the evaluation of the level sets C_μ at discrete values of μ in the interval $[0, 1]$. Although these level sets may be determined exactly, the inherent discretization of the variable μ will only allow an approximate evaluation of the membership function m with the aid of (3.2).

We now consider the determination of the level sets of the expectation EX of a discrete fuzzy random variable X characterized by the pairs (p_i, X^i) , $i = 1, 2, \dots$.

ALGORITHM 3.1 (Level sets of EX).

Step 1. Choose a level $\mu \in [0, 1]$.

Step 2. Determine for each $i = 1, 2, \dots$ the numbers

$$u_i^*(\mu) = \inf \{ x \in R \mid X^i(x) \geq \mu \},$$

$$u_i^{**}(\mu) = \sup \{ x \in R \mid X^i(x) \geq \mu \}.$$

Step 3. Determine the level set $C_\mu = \{ x \in R \mid (EX)(x) \geq \mu \}$ as

$$C_\mu = \left[\sum_i p_i u_i^*(\mu), \sum_i p_i u_i^{**}(\mu) \right].$$

Step 4. Repeat steps 1 through 3 for a sufficient number of values of μ .

Step 5. Determine the level set $D_0 = \{ x \in R \mid (EX)(x) > 0 \}$ as

$$D_0 = \left[\sum_i p_i u_i^0, \sum_i p_i u_i^{00} \right],$$

where

$$u_i^0 = \inf \{ x \in R \mid X^i(x) > 0 \},$$

$$u_i^{00} = \sup \{ x \in R \mid X^i(x) > 0 \}.$$

Proof. The representation of the level set C_μ given in Step 3 is demonstrated in the proof of Theorem 5.2 of part I of the paper. The representation of the level set D_0 given in Step 5 follows similarly. ■

Step 5 serves to establish the support D_0 of the membership function (EX) . The shape of (EX) may be traced from the level sets obtained in Step 3 with the aid of (3.2).

EXAMPLE 3.1. By way of example we consider the expectation of the fuzzy random variable defined by Table 1 and the membership functions of Fig. 1. For different values of μ , the level sets of EX are found by easy calculations following Algorithm 3.1. Using the formula (3.2), the membership function of EX may easily be sketched as in Fig. 2. We observe that average temperatures between 27 and 27.5°C are fully plausible, while the plausibility decreases to zero as the temperatures decrease to 22°C or increase to 32.5°C.

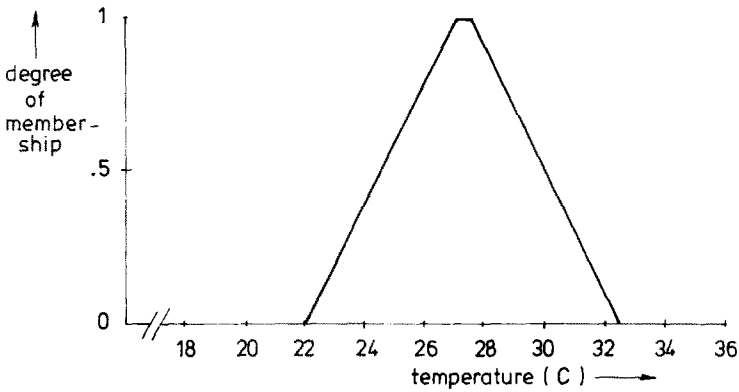


Fig. 2. Membership function of EX for Example 3.1.

4. FUZZY PROBABILITIES

In this section we consider an algorithm to determine the fuzzy probability $\Pr(\mathbf{X} \in A)$, with A a Borel set in R , of the discrete fuzzy random variable \mathbf{X} , characterized by the set of pairs (p_i, X^i) , $i = 1, 2, \dots$.

ALGORITHM 4.1 (for $\Pr(\mathbf{X} \in A)$).

Step 1. Determine for each $i = 1, 2, \dots$, the numbers

$$r'_i = \sup_{x \in A} X^i(x), \quad r''_i = \sup_{x \in A^c} X^i(x),$$

with the superscript c denoting the complement.

Step 2. Determine the levels μ_k , $k = 0, 1, 2, \dots$, assumed by $\Pr(\mathbf{X} \in A)$, with $\mu_0 = 1$, and μ_k , $k = 1, 2, \dots$, the distinct values different from 1 assumed by $\min(r'_i, r''_i)$, $i = 1, 2, \dots$. It is useful to order the μ_k so that $1 = \mu_0 > \mu_1 > \mu_2 > \dots$.

Step 3. Determine for $k = 0, 1, 2, \dots$, and $q \in [0, 1]$ the function

$$\chi_k(q) = \begin{cases} 1 & \text{for } \sum_{i: r'_i < \mu_k} p_i \leq q \leq 1 - \sum_{i: r''_i < \mu_k} p_i, \\ 0 & \text{otherwise.} \end{cases}$$

Step 4. Finally, determine for $q \in [0, 1]$

$$(\Pr(\mathbf{X} \in A))(q) = \sup_k \chi_k(q) \mu_k.$$

Proof. According to Theorem 7.3 of part I of the paper, $\Pr(\mathbf{X} \in A) = E\mathbf{I}^{\mathbf{X} \in A}$, where $\mathbf{I}^{\mathbf{X} \in A}$ is the indicator function of the event $\mathbf{X} \in A$. The indicator function is a reduced fuzzy random variable with membership function as given in Theorem 7.2 of part I. Transposed to the present context, $\mathbf{I}^{\mathbf{X} \in A}$ is a discrete fuzzy random variable, characterized by the pairs (p_i, I^i) , $i = 1, 2, \dots$, where for each i the membership function I^i is given by

$$I^i(\pi) = \begin{cases} r_i'' & \text{for } \pi = 0, \\ \min(r_i', r_i'') & \text{for } 0 < \pi < 1, \\ r_i' & \text{for } \pi = 1, \\ 0 & \text{elsewhere.} \end{cases} \quad (4.1)$$

The numbers r_i' and r_i'' , $i = 1, 2, \dots$, are as obtained in step 1 of the present algorithm. We now determine $E\mathbf{I}^{\mathbf{X} \in A}$ according to Algorithm 3.1. It follows from (4.1) that the quantities $u_i^*(\mu)$ and $u_i^{**}(\mu)$ occurring in Step 2 of Algorithm 3.1 are given by

$$u_i^*(\mu) = \begin{cases} 0 & \text{if } \mu < r_i'', \\ 1 & \text{if } \mu > r_i'', \end{cases} \quad (4.2)$$

$$u_i^{**}(\mu) = \begin{cases} 0 & \text{if } \mu > r_i', \\ 1 & \text{if } \mu < r_i'. \end{cases}$$

Note that in establishing these relationships we need the assumption that \mathbf{X} is normal, which implies that for each i either $r_i' = 1$ or $r_i'' = 1$. Using this, it immediately follows from Step 3 of Algorithm 3.1 that the left- and right-hand limits of the level set $C_\mu = \{q \in [0, 1] \mid (\Pr(\mathbf{X} \in A))(q) \geq \mu\}$ are respectively given by

$$\sum_{i: r_i' < \mu} p_i \quad \text{and} \quad \sum_{i: r_i' > \mu} p_i = 1 - \sum_{i: r_i'' < \mu} p_i. \quad (4.3)$$

These expressions show that the number of different levels assumed by $\Pr(\mathbf{X} \in A)$ is at most denumerable; the possible values of the levels are the numbers μ_k determined in Step 2 of the present algorithm. This leads with the aid of (3.2) to the result of Step 4 of the present algorithm. ■

EXAMPLE 4.1. We shall compute the fuzzy probability $\Pr(\mathbf{X} \geq z)$, with z a real number, for the fuzzy random variable introduced in Sec. 1 and also discussed in Example 3.1. We first discuss the interpretation of the fuzzy probability $\Pr(\mathbf{X} \geq z)$. For given z and q , the number $(\Pr(\mathbf{X} \geq z))(q)$ is the acceptability that a fraction q of the respondents agree with the proposition

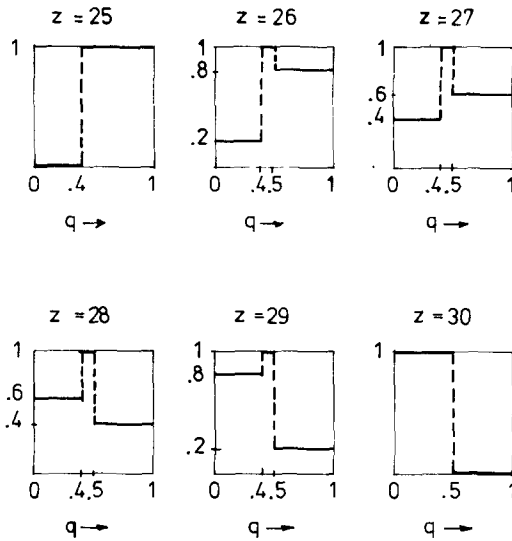


Fig. 3. Fuzzy probabilities $\Pr(X > z)$ for various values of z for Example 4.1.

that the temperature was over $z^\circ\text{C}$. Using Algorithm 4.1, $\Pr(X \geq z)$ may easily be determined for any given value of z . Figure 3 depicts the appearance of $\Pr(X \geq z)$ for $z = 25, 26, 27, 28, 29$ and 30°C . It is seen that the fuzzy probabilities gradually shift from $\Pr(X \geq 25) = \text{“at least .4”}$ to $\Pr(X \geq 30) = \text{“not more than .5”}$, with $\Pr(X \geq 27)$ and $\Pr(X \geq 28)$ both not very different from “more or less .5”.

5. FUZZY CONDITIONAL EXPECTATION

In this section we deal with two jointly discrete fuzzy random variables X and Y . These fuzzy random variables are defined by the set of joint probabilities $p_{ij}, i = 1, 2, \dots, j = 1, 2, \dots$, the set of fuzzy values of X characterized by the membership functions $X^i, i = 1, 2, \dots$, and the set of fuzzy values of Y characterized by the membership functions $Y^j, j = 1, 2, \dots$. We shall state and prove an algorithm to determine the fuzzy conditional expectation $E(X|Y \in B)$, with B a Borel set in R . This conditional expectation is described and discussed in Sec. 8 of part I of this paper.

ALGORITHM 5.1 [Level sets of $E(\mathbf{X}|\mathbf{Y} \in B)$].

Step 1. Determine for $j=1,2,\dots$ the numbers

$$s'_j = \sup_{y \in B} Y^j(y), \quad s''_j = \sup_{y \in B^c} Y^j(y).$$

Step 2. Determine the levels μ_k , $k=0,1,2,\dots$, as the distinct values assumed by 1 and $\min(s'_k, s''_k)$, $k=1,2,\dots$. It is useful to order the μ_k so that $1 = \mu_0 > \mu_1 > \mu_2 > \dots$.

Step 3. Choose a value $\mu \in [0,1]$. It is best to take μ first equal to μ_k , $k=0,1,\dots$, then $\mu=0$, and finally intermediate values.

Step 4. Determine the sets

$$J'_\mu = \{j \in N | s'_j < \mu\}, \quad J''_\mu = \{j \in N | s''_j < \mu\},$$

$$\bar{J}'_\mu = \{j \in N | s'_j \leq \mu\}, \quad \bar{J}''_\mu = \{j \in N | s''_j \leq \mu\}.$$

Step 5. For $i=1,2,\dots$ determine the numbers

$$u_i^*(\mu) = \inf\{x \in R | X^i(x) \geq \mu\},$$

$$u_i^{**}(\mu) = \sup\{x \in R | X^i(x) \geq \mu\}.$$

Step 6. Determine the numbers

$$\alpha_\mu = \min \left\{ \frac{\sum_{i,j} p_{ij} \pi_j u_i^*(\mu)}{\sum_{i,j} p_{ij} \pi_j} \mid (\forall j \in N) \pi_j \in [0,1], \pi_j = 0 \text{ if } j \in J'_\mu, \pi_j = 1 \text{ if } j \in J''_\mu \right\},$$

$$\beta_\mu = \max \left\{ \frac{\sum_{i,j} p_{ij} \pi_j u_i^{**}(\mu)}{\sum_{i,j} p_{ij} \pi_j} \mid (\forall j \in N) \pi_j \in [0,1], \pi_j = 0 \text{ if } j \in \bar{J}'_\mu, \pi_j = 1 \text{ if } j \in \bar{J}''_\mu \right\},$$

$$\alpha_\mu = \min \left\{ \frac{\sum_{i,j} p_{ij} \pi_j u_i^*(\mu)}{\sum_{i,j} p_{ij} \pi_j} \mid (\forall j \in N) \pi_j \in [0,1], \pi_j = 0 \text{ if } j \in \bar{J}'_\mu, \pi_j = 1 \text{ if } j \in \bar{J}''_\mu \right\},$$

$$\beta_\mu = \max \left\{ \frac{\sum_{i,j} p_{ij} \pi_j u_i^{**}(\mu)}{\sum_{i,j} p_{ij} \pi_j} \mid (\forall j \in N) \pi_j \in [0,1], \pi_j = 0 \text{ if } j \in \bar{J}'_\mu, \pi_j = 1 \text{ if } j \in \bar{J}''_\mu \right\}.$$

Step 7. Define

$$C_\mu = \{z \in R | (E(X|Y \in B))(z) \geq \mu\}$$

and

$$D_\mu = \{z \in R | (E(X|Y \in B))(z) > \mu\},$$

and set

$$C_\mu = [a_\mu, b_\mu], \quad D_\mu = (\alpha_\mu, \beta_\mu).$$

Step 8. Return to Step 3.

Before we prove this algorithm, we comment on it as follows. For $\mu \neq \mu_k$, $k = 1, 2, \dots$, we have $J'_\mu = \bar{J}'_\mu$ and $J''_\mu = \bar{J}''_\mu$, and hence $a_\mu = \alpha_\mu$ and $b_\mu = \beta_\mu$. As we shall see in the example at the end of the section, the membership function of $E(X|Y \in B)$ has a terraced shape. The determination of both the sets C_μ and D_μ at $\mu = \mu_k$ serves to determine the extent of the terrace as well as the points where the membership function rises away from the terrace. The determination of a_μ , b_μ , α_μ and β_μ as given in Step 6 requires the solution of a fractional interval programming problem of a special type, for which an algorithm is given in the Appendix. Finally, once the level sets of $E(X|Y \in B)$ have been obtained at a sufficient number of levels, the membership function may be recovered (approximately) with the aid of the decomposition formula (3.2).

Proof of Algorithm 5.1. According to Theorem 8.3 of part I of the paper,

$$(E(X|Y \in B))(z) = \sup \left\{ \inf_{i \in N, j \in N} \min [\bar{X}^i(u_i), I^j(\pi_j)] \mid (\forall i \in N) u_i \in R, \right. \\ \left. (\forall j \in N) 0 \leq \pi_j \leq 1, \frac{\sum_{i,j} p_{ij} \pi_j u_i}{\sum_{i,j} p_{ij} \pi_j} = z \right\}, \quad (5.1)$$

where \bar{X}^i is the unimodalized version of X^i (see Sec. 5 of part I), and where I is the characteristic function of $Y \in B$. It immediately follows that C_μ as defined

in Step 7 is given by $C_\mu = [a_\mu, b_\mu]$, where

$$a_\mu = \inf \left\{ \frac{\sum_{i,j} p_{ij} \pi_j u_i^*(\mu)}{\sum_{i,j} p_{ij} \pi_j} \mid (\forall j \in N) 0 \leq \pi_j < 1, I^j(\pi_j) \geq \mu \right\}, \quad (5.2)$$

$$b_\mu = \sup \left\{ \frac{\sum_{i,j} p_{ij} \pi_j u_i^{**}(\mu)}{\sum_{i,j} p_{ij} \pi_j} \mid (\forall j \in N) 0 \leq \pi_j \leq 1, I^j(\pi_j) \geq \mu \right\}, \quad (5.3)$$

with $u_i^*(\mu)$ and $u_i^{**}(\mu)$ as defined in Step 4. It is easily verified that the condition $I^j(\pi_j) \geq \mu$ is equivalent to $\pi_j = 0$ if $j \in J'_\mu$, $\pi_j = 1$ if $j \in J''_\mu$, and $0 \leq \pi_j \leq 1$ otherwise. This yields the expressions for a_μ and b_μ as given in Step 6. The representation of D_μ is proved similarly. ■

EXAMPLE 5.1. Suppose that the individuals of the population that was polled in Sec. 1 about the weather in Europe are also asked whether they had a good vacation during that season. We assume that the possible answers are “good,” “fair,” and “no opinion.” The membership functions representing these responses are defined on an assumed scale ranging from zero (absolutely no satisfaction) to one (complete satisfaction). Figure 4 depicts the assumed membership functions.

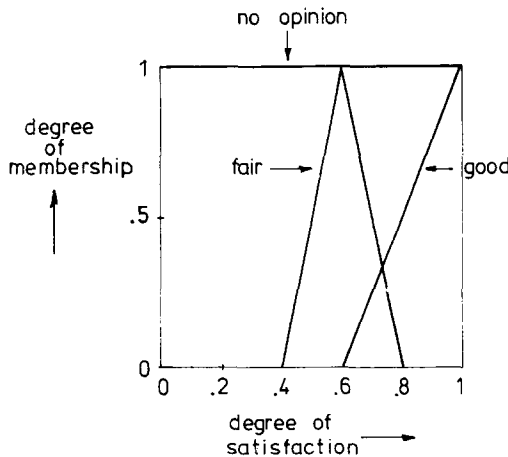


Fig. 4. Membership functions for “good,” “fair” and “no opinion” in Example 5.1.

TABLE 2 Probabilities of Occurrence p_{ij} of the Joint Responses X^i and Y^j

	Y^1 Good vacation	Y^2 Fair vacation	Y^3 No opinion
X^1 Very warm	.3	.1	0
X^2 Warm	.2	.25	.05
X^3 No opinion	0	.05	.05

In Table 2 we summarize the responses obtained to the two questions. The table defines two jointly fuzzy discrete random variables X and Y , with X referring to the question concerning the weather, and Y to the vacations. By way of example, the conditional expectation $E(X|Y \in B)$ has been calculated with $B = [.8, 1]$.

In words, we wish to determine the conditional expectation of an individual's perception of the temperature during this summer, given that his satisfaction with his vacation is between .8 and 1. Using the algorithm, it is not difficult to find that $\mu_0 = 1$, $\mu_1 = .5$, $\mu_2 = 0$, while $C_1 = [27.5, 28]$, $D_{.5} = (25, 30.5)$, $C_{.5} = [22.5, 30.5]$, and $D_0 = (20, 33)$. The determination of C_μ for some values of μ between .5 and 1 and between 0 and .5 completes the picture of $E(X|Y \in B)$, which is sketched in Fig. 5.

It is of some interest to take note of the conditional expectation $E(X|Y \in B)$ for $B = [0, 1]$, i.e., each degree of satisfaction on which the individual's satisfaction with his vacation is conditioned is equally and fully acceptable. We find here $\mu_0 = 1$ and $\mu_1 = 0$, while $C_1 = [25, 28]$ and $D_0 = (20, 32.5)$. At the intermediate value $\mu = .5$ we have $C_{.5} = [22.5, 30.5]$, $D_{.5} = (22.5, 30.5)$. The membership function of $E(X|Y \in B)$ is also sketched in Fig. 5. Comparing with the membership function of EX as sketched in Fig. 2, we observe a remarkable difference.

6. FUZZY CONDITIONAL PROBABILITY

We continue in this section with an algorithm for the determination of the fuzzy conditional probability $\Pr(X \in A | Y \in B)$, with A and B Borel sets contained in R , and X and Y jointly discrete fuzzy random variables, characterized by the joint probabilities p_{ij} , $i \in N$, $j \in N$, and the membership functions X^i , $i \in N$, and Y^j , $j \in N$.

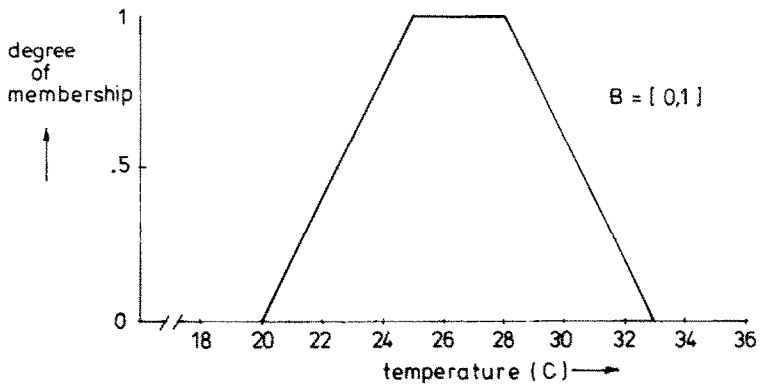
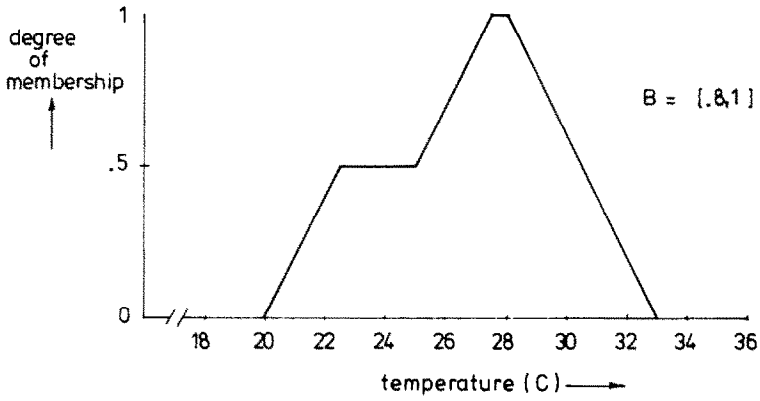


Fig. 5. Membership functions of $E(X|Y \in B)$ for two different sets B for Example 5.1.

ALGORITHM 6.1 [for $\Pr(X \in A | Y \in B)$].

Step 1. Determine for each $i \in N$ and $j \in N$ the numbers

$$r'_i = \sup_{x \in A} X^i(x), \quad r''_i = \sup_{x \in A^c} X^i(x),$$

$$s'_j = \sup_{y \in B} Y^j(y), \quad s''_j = \sup_{y \in B^c} Y^j(y).$$

Step 2. Determine the levels $\mu_k, k=0, 1, \dots$, assumed by $(\Pr(\mathbf{X} \in A | \mathbf{Y} \in B))$ as the distinct values assumed by $1, \min(r'_i, r''_i), i \in N$, and $\min(s'_j, s''_j), j \in N$. It is useful to order the μ_k so that $1 = \mu_0 > \mu_1 > \mu_2 > \dots$.

Step 3. Determine for $k=0, 1, \dots$ the integer sets

$$I'_k = \{i \in N | r'_i < \mu_k\}, \quad I''_k = \{i \in N | r''_i < \mu_k\},$$

$$J'_k = \{j \in N | s'_j < \mu_k\}, \quad J''_k = \{j \in N | s''_j < \mu_k\}.$$

Step 4. Determine for $k=0, 1, \dots$ the function

$$\chi_k(q) = \begin{cases} 1 & \text{for } q \in [q_k^*, q_k^{**}], \\ 0 & \text{otherwise,} \end{cases}$$

where

$$q_k^* = \min \left\{ \frac{\sum_{j \notin J'_k \cup J''_k} \pi_j \sum_{i \in I'_k} p_{ij} + \sum_{j \in J''_k} \sum_{i \in I''_k} p_{ij}}{\sum_{j \notin J'_k \cup J''_k} \pi_j \sum_i p_{ij} + \sum_{j \in J''_k} \sum_i p_{ij}} \mid (\forall j \in N) \pi_j \in [0, 1] \right\}$$

$$q_k^{**} = \max \left\{ \frac{\sum_{j \notin J'_k \cup J''_k} \pi_j \sum_{i \notin I'_k} p_{ij} + \sum_{j \in J''_k} \sum_{i \notin I''_k} p_{ij}}{\sum_{j \notin J'_k \cup J''_k} \pi_j \sum_i p_{ij} + \sum_{j \in J''_k} \sum_i p_{ij}} \mid (\forall j \in N) \pi_j \in [0, 1] \right\}.$$

Step 5. Finally, determine for $q \in [0, 1]$

$$(\Pr(\mathbf{X} \in A | \mathbf{Y} \in B))(q) = \sup_k \chi_k(q) \mu_k.$$

Proof. According to (8.26) of part I of the paper, we may write in the present context for $q \in [0, 1]$

$$(\Pr(\mathbf{X} \in A | \mathbf{Y} \in B))(q) = \sup \left\{ \inf_{i \in N, j \in N} \min [I^i(\psi_i), J^j(\pi_j)] \mid (\forall i \in N, \forall j \in N) \right.$$

$$\left. 0 \leq \psi_i < 1, 0 \leq \pi_j < 1, \frac{\sum_{i,j} p_{ij} \psi_i \pi_j}{\sum_{i,j} p_{ij} \pi_j} = q \right\}. \quad (6.1)$$

Here $I^i, i \in N$, are the membership functions of the characteristic function $\mathbf{I}^{\mathbf{X} \in A}$, and $J^j, j \in N$, are those of $\mathbf{I}^{\mathbf{Y} \in B}$. We now define the level set $C_\mu = \{q \in [0, 1] | (\Pr(\mathbf{X} \in A | \mathbf{Y} \in B))(q) \geq \mu\}$. Then for $\mu \in [0, 1]$ we have $C_\mu = [a_\mu, b_\mu]$, where

$$a_\mu = \inf \left\{ \frac{\sum_{i,j} p_{ij} \pi_j \psi_i^*(\mu)}{\sum_{i,j} p_{ij} \pi_j} \mid (\forall j \in N) 0 \leq \pi_j \leq 1, J^j(\pi_j) \geq \mu \right\}, \quad (6.2)$$

$$b_\mu = \sup \left\{ \frac{\sum_{i,j} p_{ij} \pi_j \psi_i^{**}(\mu)}{\sum_{i,j} p_{ij} \pi_j} \mid (\forall j \in N) 0 \leq \pi_j \leq 1, J^j(\pi_j) \geq \mu \right\}, \quad (6.3)$$

with

$$\psi_i^*(\mu) = \inf \{ \psi \in [0, 1] | I^i(\psi) \geq \mu \}, \quad (6.4)$$

$$\psi_i^{**}(\mu) = \sup \{ \psi \in [0, 1] | I^i(\psi) \geq \mu \}. \quad (6.5)$$

With the aid of (4.1) it is easily found that

$$\psi_i^*(\mu) = \begin{cases} 0 & \text{if } r_i'' > \mu, \\ 1 & \text{if } r_i'' < \mu, \end{cases} \quad (6.6)$$

$$\psi_i^{**}(\mu) = \begin{cases} 0 & \text{if } r_i' < \mu, \\ 1 & \text{if } r_i' \geq \mu. \end{cases}$$

Furthermore the condition $J^j(\pi_j) \geq \mu$ implies

$$\psi_j = \begin{cases} 0 & \text{if } s_j' < \mu, \\ 1 & \text{if } s_j'' < \mu, \\ \text{arbitrary} & \text{otherwise.} \end{cases} \quad (6.7)$$

We conclude from the expressions given that as a function of μ the level set C_μ only changes at the levels $\mu_k, k=0, 1, 2, \dots$, indicated in Step 2 of the algorithm. Defining $a_{\mu_k} = q_k^*$ and $b_{\mu_k} = q_k^{**}$, it is straightforward to obtain the two expressions given in Step 4, using the integer sets introduced in Step 3. The final expression given in Step 5 for $(\Pr(\mathbf{X} \in A | \mathbf{Y} \in B))$ follows from the general expression (3.2). ■

The determination of q_k^* and q_k^{**} in Step 4 requires the solution of a fractional interval programming problem, for which an algorithm is given in the Appendix.

EXAMPLE 6.1. We consider the jointly distributed fuzzy random variables X and Y of Example 5.1, and compute the fuzzy conditional probability $\Pr(X \in A|Y \in B)$, with $A=[z, \infty)$ and $B=[.8, 1]$. Thus we compute the fuzzy conditional probability that an individual perceives the temperature as having been greater than z , given that his satisfaction with his vacation was between .8 and 1. Application of the algorithm given in this section is straightforward. For $z = 27^\circ\text{C}$, for example, we find $\mu_0 = 1, \mu_1 = .6, \mu_2 = .5, \mu_3 = .4, \mu_4 = 0$. The integer sets are $I'_0 = \{2\}, I''_0 = \{1\}, J'_0 = \{2\}, J''_0 = \{1\}, I'_1 = \emptyset, I''_1 = \{1\}, J'_1 = \{2\}, J''_1 = \{1\}, I'_2 = \emptyset, I''_2 = \{1\}, J'_2 = \{2\}, J''_2 = \emptyset, I'_3 = \emptyset, I''_3 = \emptyset, J'_3 = \{2\}, J''_3 = \emptyset, I'_4 = I''_4 = J'_4 = J''_4 = \emptyset$. From this we obtain $C_1 = [.5, .6], C_6 = [.5, 1], C_5 = C_4 = C_0 = [0, 1]$.

The results of the computation for a number of values of z are given in Fig. 6. Comparison with Fig. 3 shows that conditioning on $Y \in [.8, 1]$ does not greatly influence the various probabilities.

Before passing on to the next section, we generalize the results of the present section by extending the notion of fuzzy conditional probability to the case where the sets A and B are themselves fuzzy. Thus, let A denote a fuzzy set defined on the space \mathfrak{B} of Borel sets in R , such that the degree of membership of $A \in \mathfrak{B}$ in A is given by $\alpha(A)$, where $\alpha : \mathfrak{B} \rightarrow [0, 1]$. Similarly, let

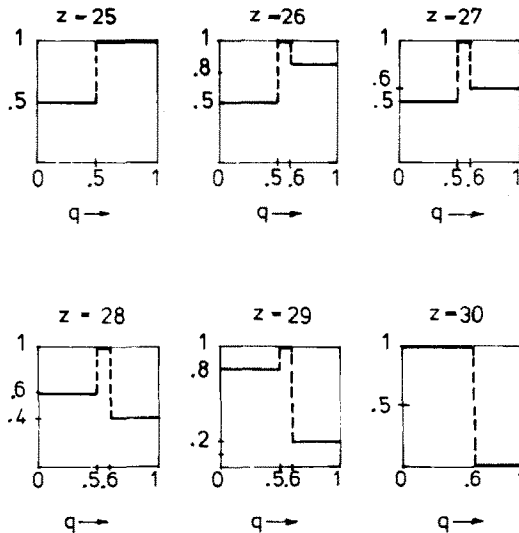


Fig. 6. Fuzzy conditional probabilities for various values of z .

\mathbf{B} denote a fuzzy set defined on \mathfrak{B} such that $B \in \mathfrak{B}$ has degree of membership $\beta(B)$ in \mathbf{B} , where $\beta: \mathfrak{B} \rightarrow [0, 1]$. Then we define $\Pr(\mathbf{X} \in \mathbf{A} | \mathbf{Y} \in \mathbf{B})$ as a fuzzy number with membership function

$$(\Pr(\mathbf{X} \in \mathbf{A} | \mathbf{Y} \in \mathbf{B}))(\cdot) = \sup_{A, B \in \mathfrak{B}} \min[(\Pr(\mathbf{X} \in A | \mathbf{Y} \in B))(\cdot), \alpha(A), \beta(B)]. \quad (6.8)$$

ALGORITHM 6.2 (for $\Pr(\mathbf{X} \in \mathbf{A} | \mathbf{Y} \in \mathbf{B})$).

Step 1. Determine for each $i \in N$ and $j \in N$ the numbers

$$r'_i = \sup_{A \in \mathfrak{B}} \min \left[\alpha(A), \sup_{x \in A} X^i(x) \right],$$

$$r_i = \sup_{A \in \mathfrak{B}} \min \left[\alpha(A), \sup_{x \in A^c} X^i(x) \right],$$

$$s'_j = \sup_{B \in \mathfrak{B}} \min \left[\beta(B), \sup_{y \in B} Y^j(y) \right],$$

$$s''_j = \sup_{B \in \mathfrak{B}} \min \left[\beta(B), \sup_{y \in B^c} Y^j(y) \right].$$

Steps 2 through 4. As in Algorithm 6.1.

Step 5. Finally, determine for $q \in [0, 1]$

$$(\Pr(\mathbf{X} \in \mathbf{A} | \mathbf{Y} \in \mathbf{B}))(q) = \sup_k X_k(q) \mu_k.$$

Thus, Algorithm 6.2 is practically the same as Algorithm 6.1, except that the expressions for r'_i , r''_i , s'_j and s''_j in Step 1 are modified. The proof of this is not difficult.

7. APPLICATION TO A SIMPLE DECISION PROBLEM

In this section we discuss a hypothetical application to a simple decision problem, related to medical diagnosis. Suppose that a certain medical test is applied to determine whether a given individual suffers from a certain disease. By studying a large number of cases, statistical information is available relating the result of the test to the presence of the disease.

Imprecision arises because the result of the test is difficult to analyze, and cannot be expressed by a single number. We assume that the results of the test are classified as positive, inconclusive and negative, which are represented by membership functions on an underlying scale ranging from 0 to 1 as depicted in Fig. 7. The reason that the membership functions partially overlap is the possible presence of ambiguities. In a more realistic application, presumably a larger assortment of possible classifications (strongly positive, moderately positive, weakly positive, strongly negative, etc.) would be admitted.

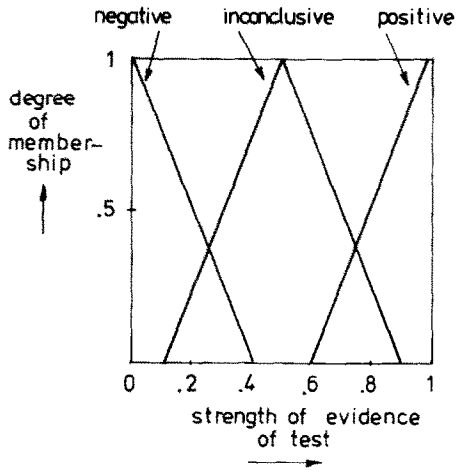


Fig. 7. Membership functions for the result of a test.

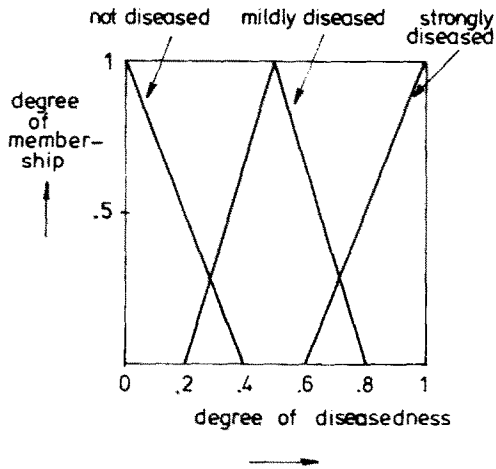


Fig. 8. Membership functions for the degree of diseasedness.

Similarly, there is impreciseness in the characterization of the extent to which a patient has contracted the disease. We assume that the three allowed characterizations—strongly diseased, mildly diseased and not diseased—may be represented by membership functions as given in Fig. 8.

The statistical evidence, relating the presence of disease to the results of the test, is collected in Table 3. As the table shows, the degree of diseasedness is represented by the fuzzy random variable X , and the result of the test by the fuzzy random variable Y .

TABLE 3
Relation between Results of Test and Diseasedness

	Result of test		
	Y^1 Negative	Y^2 Inconclusive	Y^3 Positive
X^1 Not diseased	.78	.07	.05
X^2 Mildly diseased	.02	.02	.01
X^3 Strongly diseased	0	.01	.04

Suppose now that the result of the test applied to a given patient is diagnosed as negative to inconclusive. This is taken to mean that the strength of evidence of the result of the test lies in a fuzzy Borel set B , with membership function characterized by $\beta(B) = \sigma(a)$ if $B = [0, a]$, and $\beta(B) = 0$ otherwise. The membership function σ is sketched in Fig. 9.

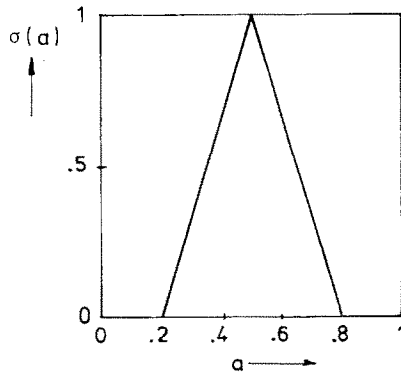


Fig. 9. The membership function σ .

We now compute the fuzzy probabilities $\Pr(X \in A_i | Y \in B)$, $i = 1, 2, 3$, with

$$A_1 = [0, .3), \quad A_2 = [.3, .7), \quad A_3 = [.7, 1].$$

We classify the event $X \in A_1$ as "not seriously diseased," $X \in A_2$ as "moderately diseased," and $X \in A_3$ as "seriously diseased." The three membership functions may be obtained with the aid of Algorithm 6.2. They are pictured in Fig. 10. The graphs show that the conditional probability that the patient is not seriously diseased, given the results of his test, is between .944 and .975 with full plausibility; other values have plausibility $\frac{1}{3}$ or less. On the other hand, the conditional probability that the individual is seriously diseased, given the result of the test, is with full plausibility between .0111 and .0556, while other probabilities have a plausibility of $\frac{1}{3}$ or less. These results indicate fairly conclusively that the individual has not contracted the disease with a high probability.

It is interesting to note that the observed result of the test in this particular example, "negative to inconclusive," is not one of the test results used in compiling the statistics ("negative," "inconclusive" and "positive"). Nevertheless we are able to derive (vague) conclusions. This is done by attributing the result of the test partly to the outcome "negative" and partly to the outcome "inconclusive," following the rules of fuzzy logic.

8. CONCLUSIONS

In this paper the mathematical theory of fuzzy random variables has been further developed. In particular we have given algorithms for the determination of expectations and probabilities connected with discrete fuzzy random variables. The technique of developing these and other algorithms in fuzzy set theory consists of delimiting the level sets associated with any fuzzy set to be determined. The numerical efforts required in applying the algorithms that were obtained are modest. For not too extensive problems a pocket calculator is more than sufficient.

Fuzzy random variables may be used to describe and characterize situations where we have to deal with statistical data that are imprecise. By using ideas from fuzzy set theory, this impreciseness is described in a "possibilistic" sense [2]. This means that all possible alternatives are kept track of, while associating with each possibility a degree of acceptability. The degree of acceptability of combined possibilities is obtained following the rules of fuzzy logic.

The results of this paper (parts I and II) show that this approach leads to manageable and consistent results. Well-known results from the theory of random variables generalize in plausible way. Many other results from the

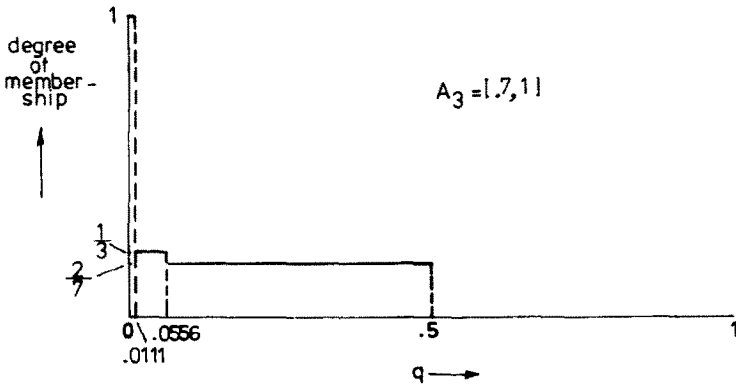
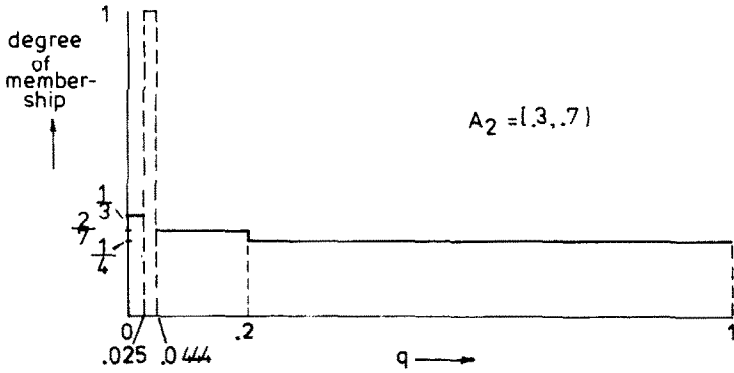
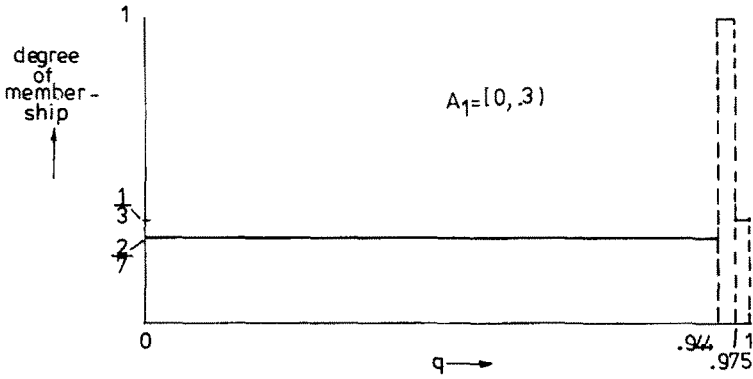


Fig. 10. The membership functions for $\Pr(X \in A_i | Y \in B)$, $i = 1, 2, 3$.

theory of random variables and statistics may no doubt be generalized equally plausibly.

In the present paper numerical examples have been given illustrating the theory and the algorithms. The outcomes of these examples are sometimes slightly surprising but never counterintuitive. One somewhat surprising result was noted in Sec. 7, where it was concluded that it is possible to define and determine conditional probabilities given events that are not among the original observations. The solution to this apparent contradiction is that the event on which the conditioning takes place is still allowed as a possibility by the original observations.

Although in the present context the fuzzy approach leads to a consistent theory, it is not certain that this theory is a good model for certain aspects of reality. Here much work remains to be done, possibly extending to the realm of psychology.

Verifying the plausibility of the axioms of a theory is one approach to validating this theory. Another approach is to study applications of the theory, and to see whether the results stand up to the tests of plausibility, intuitiveness and consistency. In this respect the author feels encouraged by the results of the present paper and some previous work in the area of fuzzy sets [3].

The sample application discussed in Sec. 7 of the paper relates to medical diagnosis. Another area of application might be found in the statistical evaluation of the results of opinion polls admitting partly or totally uncommitted or ambiguous responses.

APPENDIX—TWO ALGORITHMS FOR FRACTIONAL INTERVAL PROGRAMMING PROBLEMS

In this appendix we formulate algorithms for four different fractional-interval programming problems that occur in the determination of fuzzy expectations and conditional probabilities. The problem is to determine

$$\lambda^* = \min_{\substack{\pi_j \in [0, 1], \\ j=1, 2, \dots, n}} \frac{\sum_{j=1}^n a_j \pi_j + a_0}{\sum_{j=1}^n b_j \pi_j + b_0},$$

$$\mu^* = \max_{\substack{\pi_j \in [0, 1], \\ j=1, 2, \dots, n}} \frac{\sum_{j=1}^n a_j \pi_j + a_0}{\sum_{j=1}^n b_j \pi_j + b_0}, \quad (\text{A.1})$$

where we distinguish the two following sets of assumptions:

- (1) $b_0 > 0$ and $b_j \geq 0, j = 1, 2, \dots, n;$
- (2) $a_0 = b_0 = 0$ and $b_j > 0, j = 1, 2, \dots, n.$

We state two algorithms covering these cases. More general fractional interval programming problems are treated by Charnes, Granot and Granot [4].

ALGORITHM A.1 (for λ^* and μ^* with $b_0 > 0$ and $b_j \geq 0, j = 1, 2, \dots, n).$

Step 1. Without loss of generality, renumber the variables so that

$$\frac{a_1}{b_1} \leq \frac{a_2}{b_2} \leq \dots \leq \frac{a_n}{b_n}.$$

Step 2. Compute for $k=0, 1, \dots, n$ the numbers

$$\lambda_k = \frac{\sum_{j=1}^k a_j + a_0}{\sum_{j=1}^k b_j + b_0}, \quad \mu_k = \frac{\sum_{j=k+1}^n a_j + a_0}{\sum_{j=k+1}^n b_j + b_0}.$$

We adopt the convention that a summation vanishes if the lower limit exceeds the upper limit.

Step 3. Determine

$$\lambda^* = \min_k \lambda_k \quad \text{and} \quad \mu^* = \max_k \mu_k.$$

Proof. We only consider the proof for the determination of the minimum; the proof of the algorithm for the maximum is similar. Since the region over which the minimum is sought is finite and closed, there always exists a point where the minimum (which by the assumptions $b_0 > 0, b_j \geq 0, j = 1, 2, \dots, n,$ is finite) is assumed. Denoting such a point as $\pi_j^0, j = 1, 2, \dots, n,$ we have

$$\begin{aligned} & \frac{\sum_{j=1}^n \pi_j a_j + a_0}{\sum_{j=1}^n \pi_j b_j + b_0} - \frac{\sum_{j=1}^n \pi_j^0 a_j + a_0}{\sum_{j=1}^n \pi_j^0 b_j + b_0} \\ &= \frac{\left(\sum_j \pi_j^0 b_j + b_0\right) \sum_j (\pi_j - \pi_j^0) a_j - \left(\sum_j \pi_j^0 a_j + a_0\right) \sum_j (\pi_j - \pi_j^0) b_j}{\left(\sum_j \pi_j^0 b_j + b_0\right) \left(\sum_j \pi_j b_j + b_0\right)} \geq 0 \end{aligned} \tag{A.2}$$

for all $\pi_j \in [0, 1]$, $j = 1, 2, \dots, n$. Equivalently this condition may be written in the form

$$\sum_j (\pi_j - \pi_j^0) \left[a_j - \frac{\sum_{j=0}^n \pi_j^0 a_j + a_0}{\sum_{j=0}^n \pi_j^0 b_j + b_0} b_j \right] \geq 0, \quad (\text{A.3})$$

or

$$\sum_j (\pi_j - \pi_j^0)(a_j - \lambda^* b_j) \geq 0. \quad (\text{A.4})$$

Clearly, a necessary and sufficient condition for π_j^0 , $j = 1, 2, \dots, n$, to be optimal is that

$$\begin{aligned} \frac{a_j}{b_j} &\geq \lambda^* && \text{if } \pi_j^0 = 0, \\ \frac{a_j}{b_j} &< \lambda^* && \text{if } \pi_j^0 = 1, \\ \frac{a_j}{b_j} &= 0 && \text{if } \pi_j^0 \in (0, 1). \end{aligned} \quad (\text{A.5})$$

It will be shown that the latter possibility can be ruled out, i.e., there always exists an optimizing solution such that π_j^0 is either 0 or 1 for each j . To prove this, let π_j^0 , $j = 1, 2, \dots, n$, be an optimizing solution such that $\pi_k^0 \in (0, 1)$ for some k . Consider the effect of varying π_k while keeping $\pi_j = \pi_j^0$ for $j \neq k$. Since $a_k/b_k = \lambda^*$, it easily follows by substituting $\sum_j \pi_j^0 a_j + a_0 = (a_k/b_k)(\sum_j \pi_j^0 b_j + b_0)$ that for all $\mu_k \in [0, 1]$

$$\begin{aligned} \frac{\sum_{j \neq k} \pi_j^0 a_j + \pi_k a_k + a_0}{\sum_{j \neq k} \pi_j^0 b_j + \pi_k b_k + b_0} &= \frac{\left(\sum_j \pi_j^0 a_j + a_0 \right) + (\pi_k - \pi_k^0) a_k}{\left(\sum_j \pi_j^0 b_j + b_0 \right) + (\pi_k - \pi_k^0) b_k} \\ &= \frac{a_k}{b_k} = \lambda^*. \end{aligned} \quad (\text{A.6})$$

This proves that the function to be minimized does not depend on π_k , and hence that we may as well choose $\pi_k^0 = 0$ or $\pi_k^0 = 1$. Thus necessary and sufficient conditions for $\pi_j^0 \in \{0, 1\}$, $j = 1, 2, \dots, n$, to be an optimizing solution

are

$$\pi_j^0 = \begin{cases} 1 & \text{if } a_j/b_j \leq \lambda^*, \\ 0 & \text{if } a_j/b_j \geq \lambda^*. \end{cases} \quad (\text{A.7})$$

Assuming that the numbers a_j/b_j are ordered as in Step 1 of the algorithm, it is clear that there must exist a $k \in \{0, 1, \dots, n\}$ such that

$$\pi_j^0 = \begin{cases} 1 & \text{for } j=1, 2, \dots, k, \\ 0 & \text{for } j=k+1, k+2, \dots, n, \end{cases} \quad (\text{A.8})$$

and consequently, $\lambda^* = \lambda_k$, with λ_k as defined in Step 2. The correct value of k is most easily found by minimizing λ_k with respect to k . ■

The second algorithm applies to a slightly different situation, and is even simpler.

ALGORITHM A.2 (for λ^* and μ^* with $a_0 = b_0 = 0$ and $b_j > 0$, $j = 1, 2, \dots, n$). Determine

$$\lambda^* = \min_{j \neq 0} \frac{a_j}{b_j}, \quad \mu^* = \max_{j \neq 0} \frac{a_j}{b_j}.$$

Proof. Suppose that $\min_j(a_j/b_j)$ is assumed for $j = k$. Then

$$\frac{\sum_j a_j \pi_j}{\sum_j b_j \pi_j} - \frac{a_k}{b_k} = \frac{\sum_j \pi_j b_j \left(\frac{a_j}{b_j} - \frac{a_k}{b_k} \right)}{\sum_j b_j \pi_j} \geq 0,$$

which proves that $\lambda^* = \min_{j \neq 0}(a_j/b_j)$. The result for μ^* is proved similarly. ■

REFERENCES

1. H. Kwakernaak, Fuzzy random variables. Part I: Definitions and theorems, *Information Sci.* 15, 1-15 (1978).
2. L. A. Zadeh, Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems* 1(1): 3-28 (1978).
3. S. M. Baas, H. Kwakernaak, Rating and ranking of multiple-aspect alternatives using fuzzy sets, *Automatica* 13(1): 47-58 (Jan. 1977).
4. A. Charnes, D. Granot and F. Granot, An algorithm for solving general fractional interval programming problems, *Naval Res. Logist. Quart.* 23(1): 53-65 (1976).

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