A Testing Procedure for Triaxial Tests and a Numerical Method for the Calculation of Powder Flow Properties

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SUMMARY

A testing procedure for measuring flow properties of powders is developed which makes it possible to use results from triaxial tests in the Jenike bin theory. For the elaboration of the results a numerical method is used, based upon the Warren Spring equation $(\tau/C)^2 = (\sigma + T)/T$. In this equation σ and τ are the normal and shear stresses respectively, C and T the cohesion and tension, and N a curvature parameter. A constant $K = C/T$ and a constant curvature parameter N are assumed for a material. The algorithm used is a least-squares method using Newton's zero finding technique. The solutions obtained by this procedure are not influenced by the initial estimates.

INTRODUCTION

In describing the behaviour of a powder in bins and other types of bulk storage installations, interest has been concentrated on predicting the relations between normal and shearing forces of the powder during incipient failure and steady state flow.

The best-known theory for designing bins is that of Jenike [1] for which σ - τ values of a powder are obtained by measurements with the flow-factor tester. This gives, via so-called yield loci, the necessary information for bin design. Although the Jenike flow-factor tester is commonly used for measuring the flow properties, there is at least some uncertainty in the interpretation of the test results. By using the triaxial apparatus, principal stresses are measured directly, and there is no need for assumptions on the direction of the failure surfaces. Furthermore, using the internal

pressure method as described in this article, the tests are as easy as in the case of the Jenike tester. Elaboration of the results from triaxial or Jenike tests by a graphical method is greatly influenced by the judgement of the executor, and gives scatter in the results.

Using the Warren Spring equation published by Ashton *et al.* [2] and adopting the constant ratio between cohesion and tension suggested by Farley and Valentin [3], a numerical solution for triaxial tests is given, based on an article published previously [4].

THE TRIAXIAL APPARATUS

Principles of testing

The triaxial test is in principle a very simple apparatus; a sample of the powder is enclosed in a thin cylindrical membrane (Fig. 1a), and at the top and the bottom the membrane is closed by metal covers and sealed by O-rings. The following two possibilities are available for loading the specimen:

(a) Loading by external pressure

This is the procedure commonly used in soil mechanics. A pressure cylinder filled with water (*casu quo* air) is placed over the membrane with the sample. By bringing the water under pressure, the sample is loaded by a hydrostatic pressure. An additional vertical pressure can be forced upon the sample by means of a die on the upper cover, which can be increased until the sample fails.

This loading method is rather complicated, and for this reason Jenike has rejected triaxial measurements. In most cases the membrane must be filled in a mould which can only be removed after supporting the sample by hydrostatic pressure. This means that the mould

Fig. 1.(a) Principle of triaxial tester. (b) Triaxial apparatus with internal underpressure.

must be manipulated in the pressure cylinder. A further disadvantage is the friction between the die and the transit of the die in the pressure cylinder. This can partly be improved by using a rotating die or, better, placing the load transducer in the pressure cylinder. An advantage of this loading procedure is that volume changes of the sample can be measured easily by the volume of water in the cylinder.

(b) Loading by internal underpressure (Fig. 1b)

With this approach, an underpressure is created inside the sample by a vacuum pump. The pressure cylinder can be eliminated, and the atmosphere serves as a hydrostatic pressure. Increasing the vertical pressure (as described above) causes the sample to fail. By this procedure the sample is completely accessible. The fact that the maximum hydrostatic pressure which can be obtained is 1 bar is not a severe limitation in measurements with powders (although it would be in soil mechanics). Volume changes of the sample are not directly measured, but changes of diameter can be established easily at any axial position. Powders tested this way must be sufficiently permeable to attain, within a reasonable time, the same negative pressure throughout the sample. Furthermore, the powders must be rather dry, for a portion of the water in the sample can also be removed when the sample is evacuated.

The measurements used in this article are all made by internal underpressure. An apparatus using external pressure, but without the difficulties mentioned above, has been developed, and will be described in a future publication.

Testing procedure

Unlike the Jenike tester, a predicted procedure for measurements with triaxial apparatus for purposes of bunker design does not exist. For this reason a testing procedure is developed which makes it possible to compare results from triaxial apparatus with the results from Jenike shear tests. The testing procedure consists of the following stages:

(a) Preparation of the sample

A sufficient amount of the powder is sieved out (to break possible contaminated parts) and layered into the mould with the membrane.

Fig. 2. Possible curves during the consolidation state.

After each layer the sample is lightly pressed with a stamper.

(b) Preconsolidation

A hydrostatic pressure σ_{pr} is created with the vacuum pump (*casu quo* the press cylinder) and the mould is removed. This hydrostatic pressure σ_{pr} is higher than the pressures used in the next stages, and is meant to hydrostatically preconsolidate the sample to a uniform density near the critical density of the respective yield locus. The magnitude of σ_{pr} is not known *a priori*, but must be established in some previous tests.

(c) Consolidation

The pressure σ_{pr} is reduced to σ_{co} and the vertical pressure is enlarged by displacement of the sample against the upper die. The extra vertical force is measured by a force-transducer fixed to the die. Displacement is continued until the sample reaches a steady-state flow situation, which is characterized by a constant value of the maximum vertical pressure. In this stage, the sample is (at least in the neighbourhood of the later failure area) in a critical density.

When the extra vertical pressure $\Delta\sigma_{co}$ (*casu quo* force) is plotted against the vertical displacement of the sample, the three curves of Fig. 2 are possible. For curve "a", the vertical pressure goes over a maximum and the sample is weakened. In this case, the sample has been overconsolidated during the preconsolidation stage. Curve "b" indicates a rather long continuing consolidation, together with a relatively big vertical displacement and geometrical deformation of the sample before the steady state is reached. This means that the preconsolidation pressure σ_{pr} has been too low. For curve "c", a rapid consolidation occurs with very low geometrical deformation of the sample. This is the desired form of the test, and can be attained by the right adjust-

ment of the hydrostatic pressure during the preconsolidation stage.

(d) Shear

The extra vertical pressure is removed, and the hydrostatic pressure reduced to the desired value $\sigma_i < \sigma_{co}$. Displacement of the sample is now continued until the additional vertical pressure goes through a maximum. In this shear stage, a curve type "a" is always found, for the sample is overconsolidated.

It is obvious that the testing procedure outlined above has the merit of establishing the consolidating stresses in a steady-state flow situation. It is just the main consolidating stress in this state that characterizes the parameter of the yield locus to be measured.

COMPARISON WITH THE JENIKE FLOW-FACTOR TESTER

Although the Jenike flow-factor tester gives quick and reproducible results when used for measuring flow properties of powders and granular material, there is some uncertainty in the interpretation of the test results. The procedure has the following principal deficiencies:

(a) Jenike assumes that the measured values of σ and τ are points of the yield locus, which implies that the horizontal plane must coincide with a yield plane in the material. The correctness of this assumption cannot be verified experimentally. If the upper ring, filled with powder, is removed from the base ring, an inspection of the exposed surface reveals a layered surface. The layers of this surface are not horizontal.

(b) The plastic failure of the material takes place in a lens-shaped area around the horizontal plane. The size of this area cannot be measured easily and varies in different tests. Measuring the displacement of the lid gives little indication of the variation of density in the failure zone; only a tendency can be found.

(c) By preconsolidation (twisting), a displacement is imposed on the material in the cell, which varies over the height of the sample. This implies that the density can also vary over the height.

(d) Depending on the ratio of diameter to height of the cell, it is reasonable to assume that the normal stress is constant over the section of the cell. Owing to the introduction

Fig. 3. Measured Mohr circles and yield locus.

of the shear force, the distribution of the shear stress over the section is not so well defined. This situation can be improved, by dividing the shear force over the lid and the upper ring, but the improvement achieved by this procedure cannot be established.

Most of the difficulties can be avoided by making use of the triaxial apparatus. Here principal stresses are directly measured and the Mohr circles can be drawn immediately, so there is no need for assumptions on the direction of the failure surfaces. The biggest Mohr circle of the yield locus is defined by the stresses in the consolidation stage when the vertical pressure reaches the steady state. Further circles touching the yield locus are defined during the shear stage when the vertical pressure reaches its maximum value. The yield locus can now be constructed as the envelope of the Mohr circles, ending at the Mohr circle from the consolidating stage (Fig. 3).

THE ELABORATION USING THE WARREN SPRING EQUATION

The calculation of the Warren Spring parameters C_i , K and N

In the σ - τ plane the semicircles obtained from measurements with the triaxial apparatus can be described by (Fig. 4)

$$\left\{ \sigma - \left(\sigma_{ij} + \frac{d_{ij}}{2} \right) \right\}^2 + \tau^2 = \left(\frac{d_{ij}}{2} \right)^2 \quad (1)$$

$$(i = 1, 2, \dots, n; j = 1, 2, \dots, m_i)$$

in which the values σ_{ij} and d_{ij} , with a certain scatter, correspond with the j th circle belonging to the i th locus. They are now treated as simple numerical quantities.

In Fig. 4, YL denotes a yield locus and ρ_{ij} the distance between A and B, with MB $\perp$ YL; then a least-squares solution of the

Fig. 4. σ - τ plane with one measurement.

Warren Spring equation is obtained by minimizing

$$\sum_i \sum_j w_{ij} \rho_{ij} \quad (2)$$

with w_{ij} being a weight factor.

However, it proved that, with some minimization methods [5], in no case was convergence achieved by straightforward filling in the value ρ_{ij} . Another method was then adopted which, though rather cumbersome, has up to now always achieved convergence.

Consider the measured values σ_{ij} and $d_{ij}/2$ (in abridged notation σ_1 and r respectively), see Fig. 5. Now let σ_0 and τ_0 be estimates for A with

$$\sigma_0 = \sigma_1 + \frac{r}{2} \quad (3)$$

and

$$\tau_0 = \frac{1}{2} r \sqrt{3} \quad (4)$$

then the Warren Spring equation

$$\left(\frac{\tau}{C_i} \right)^N = \frac{\sigma + T_i}{T_i} \quad (5)$$

with $K = C_i/T_i$ can be computed via a method described in [4].

From these parameters, the coordinates (σ_0, τ_0) are known. The line perpendicular to the yield locus, through $(\sigma^*, 0)$ and (σ_0, τ_0) , now becomes

$$\begin{aligned} -\frac{1}{\left(\frac{\partial \tau}{\partial \sigma} \right)_{\substack{\sigma=\sigma_0 \\ \tau=\tau_0}}} &= -\frac{1}{\left\{ \frac{\partial}{\partial \sigma} \left(\frac{\sigma + T}{T} \right) \frac{1}{N} \right\}_{\substack{\sigma=\sigma_0 \\ \tau=\tau_0}}} = \\ &= \frac{N(\sigma_0 + T)}{\tau_0'} \end{aligned} \quad (6)$$

The line through $(\sigma_M, 0)$, the centre of the Mohr circle, and parallel to the line through $(\sigma^*, 0)$ and (σ_0, τ_0) , is defined by

$$\tau = \frac{N(\sigma_0 + T)}{\tau_0'} (\sigma - \sigma_M) \quad (7)$$

Fig. 5. Elaboration using the Warren Spring equation.

This line intersects the semicircle in (σ_N, τ_N) . The coordinates of (σ_N, τ_N) then become

$$\sigma_N = \sigma_M - \frac{\tau_0'}{\sqrt{(\tau_0')^2 + \{N(\sigma_0 + T)\}^2}} \quad (8)$$

$$\tau_N = \sqrt{r^2 - (\sigma_M - \sigma_N)^2} \quad (9)$$

These values serve as new values and are used to recompute a new set of parameters.

The process is terminated if

$$\sum_{ij} (\sigma_N - \sigma_0)^2 < \epsilon \quad (10)$$

with ϵ being a small number.

This iteration fulfils two requirements. First, the computed parameters are correct in a least-squares sense [4], and second they satisfy the requirement that the distances ρ_{ij} must be measured perpendicular to the yield locus, as the yield locus must be the envelope of the Mohr semicircles. Further, a weight factor may be introduced which is used by the algorithm to find new parameter values.

The absolute and relative error $\Delta\tau$ and $\rho\tau$ respectively are computed by means of the following relation:

$$\Delta\tau = \tau_{YL} - \tau_N \quad (11)$$

$$\rho\tau = \frac{\Delta\tau}{\tau_{YL}} \quad (12)$$

in which

$$\tau_{YL} = C \left(\frac{\sigma_N + T}{T} \right)^{1/N} \quad (13)$$

The calculation of the relevant quantities for design (e.g. bins) is similar to [4].

The procedure outlined above has been coded in Algol 60 [6] and runs on a DEC 1070 computer.

TABLE 1

Computed absolute and relative errors for bentonite

i	j	σ_{ij} (kN/m ²)	d_{ij} (kN/m ²)	τ_{yLij} (kN/m ²)	$\Delta\tau_{ij}$ (kN/m ²)	$\rho\tau_{ij}$ (—)
1	1	0.30	4.70	1.78	-0.20	-0.11
1	2	0.30	5.55	1.91	-0.43	-0.22
1	3	0.60	5.45	2.09	-0.21	-0.10
1	4	0.60	5.35	2.08	-0.18	-0.09
1	5	1.20	6.30	2.62	-0.07	-0.03
1	6	1.20	5.80	2.54	-0.07	-0.03
1	7	1.80	7.50	3.18	-0.05	-0.02
1	8	1.80	7.05	3.11	0.08	0.03
1	9	2.40	7.40	3.53	0.34	0.10
1	10	2.40	7.75	3.58	0.24	0.07
1	11	2.80	8.75	3.97	0.18	0.05
2	1	0.30	6.60	2.70	-0.05	-0.02
2	2	0.30	7.55	2.85	-0.30	-0.11
2	3	0.60	7.70	3.08	-0.15	-0.05
2	4	0.60	7.70	3.08	-0.15	-0.05
2	5	1.20	8.80	3.65	-0.06	-0.02
2	6	1.20	9.30	3.73	-0.20	-0.05
2	7	2.00	10.25	4.40	0.04	0.01
2	8	2.00	10.00	4.36	0.11	0.02
2	9	2.60	10.65	4.84	0.29	0.06
2	10	2.60	11.45	4.96	0.07	0.01
2	11	3.20	12.35	5.48	0.17	0.03
2	12	3.20	12.80	5.54	0.04	0.01
2	13	3.60	13.65	5.91	0.04	0.01
3	1	0.40	8.45	3.60	0.09	0.03
3	2	0.40	8.90	3.67	-0.03	-0.01
3	3	0.80	10.85	4.24	-0.29	-0.07
3	4	0.80	9.05	3.98	0.20	0.05
3	5	1.80	12.10	5.12	0.01	0.00
3	6	1.80	11.05	4.96	0.30	0.06
3	7	2.60	14.10	5.95	-0.05	-0.01
3	8	2.60	13.45	5.85	0.13	0.02
3	9	3.40	16.00	6.75	-0.09	-0.01
3	10	3.40	16.75	6.87	-0.30	-0.04
3	11	4.20	17.05	7.41	0.09	0.01
3	12	4.20	16.45	7.32	0.26	0.04
3	13	5.00	19.80	8.32	-0.22	-0.03
4	1	0.30	11.05	4.67	0.09	0.02
4	2	0.30	13.05	4.99	-0.45	-0.09
4	3	0.60	13.15	5.21	-0.27	-0.05
4	4	0.60	12.55	5.12	-0.11	-0.02
4	5	1.20	13.65	5.70	-0.01	-0.00
4	6	1.20	13.60	5.70	0.00	0.00
4	7	2.40	13.85	6.54	0.71	0.11
4	8	2.40	17.25	7.07	-0.23	-0.03
4	9	3.60	18.85	8.19	0.08	0.01
4	10	3.60	17.30	7.86	0.51	0.07
4	11	4.80	22.55	9.43	-0.23	-0.02
4	12	4.80	21.85	9.32	-0.03	-0.00
4	13	6.00	23.25	10.28	0.30	0.03
4	14	6.00	24.85	10.53	-0.16	-0.02
4	15	7.00	26.95	11.46	-0.17	-0.02
5	1	0.30	15.85	6.95	0.40	0.06
5	2	0.30	17.50	7.21	-0.04	-0.01
5	3	0.80	19.75	7.92	-0.31	-0.04
5	4	0.80	17.90	7.63	0.19	0.02

(continued on facing page)

TABLE 1 (continued)

i	j	σ_{ij} (kN/m ²)	d_{ij} (kN/m ²)	$\tau_{YL_{ij}}$ (kN/m ²)	$\Delta\tau_{ij}$ (kN/m ²)	$\rho\tau_{ij}$ (—)
5	5	1.80	22.55	9.04	-0.41	-0.05
5	6	1.80	21.35	8.86	-0.08	-0.01
5	7	3.60	26.50	10.81	-0.23	-0.02
5	8	3.60	25.50	10.71	-0.06	-0.01
5	9	5.40	30.25	12.62	-0.24	-0.02
5	10	5.40	28.00	12.28	0.38	0.03
5	11	7.20	33.65	14.29	-0.09	-0.01
5	12	7.20	31.15	13.91	0.61	0.04
5	13	10.00	39.60	16.94	-0.11	-0.01

TABLE 2

Values of Warren Spring parameters and relevant bin design quantities

Locus number	C	K	N	σ_{co} (kN/m ²)	σ_p (kN/m ²)	ϕ_e (°)
1	0.839	0.844	1.155	9.86	3.23	41.13
2	1.469	0.844	1.155	16.18	5.66	41.98
3	2.019	0.844	1.155	24.61	7.77	41.75
4	2.758	0.844	1.155	33.80	10.62	41.21
5	4.303	0.844	1.155	49.51	16.57	41.66

Fig. 6. Results for bentonite.

RESULTS

Several cohesive powders were tested in correspondence with the proposed procedure. The diameter and the height of the sample used in the triaxial apparatus are 50 mm and 150 mm respectively.

As an example, the results for bentonite are given. Table 1 represents the measured and

calculated values of 5 yield loci with the absolute and relative errors.

Table 2 shows the Warren Spring parameters C_i , K , N together with the consolidation pressure σ_{co} , the unconfined yield strength σ_p and the effective angle of friction ϕ_e . The latter three values can be used for bunker design.

Figure 6 gives a graphical representation of the results.

A comparison with results from the Jenike shear cell indicates that the triaxial test gives up to 20% higher numerical values for the unconfined yield strength. The values of ϕ_e were in good agreement. In a further publication the differences between these two methods will be discussed in detail.

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LIST OF SYMBOLS

	eqn.	
d_{ij}	(1)	diameter of the Mohr circle of the j th point of the i th yield locus (N/m^2)
C_i	(4)	cohesion of the i th yield locus (N/m^2)
ϵ	(10)	small number (—)
K	(4a)	constant (—)
m_i	(1)	number of experiments of the i th yield locus (—)
n	(1)	number of yield loci (—)
N	(4)	curvature parameter $d_{ij}/2$
r	(3)	$d_{ij}/2$
T_i	(4)	tension of the i th yield locus (—)

w_{ij}	(2)	weight factor
ϕ_e		effective angle of friction (—)
σ_{co}		major consolidation pressure (N/m^2)
σ_{pr}		hydrostatic pressure (N/m^2)
σ_p		unconfined yield strength (N/m^2)
σ_{ij}	(1)	initial stress value of the j th point of the i th yield locus (N/m^2)
σ_M	(7)	normal stress corresponding with average stress during failure = $(\sigma_1 + \sigma_2)/2$ (N/m^2)
σ_N	(8)	corrected normal stress (N/m^2)
σ_0	(3)	estimated normal stress (N/m^2)
τ_N	(9)	corrected shear stress lying on circle corresponding with σ_N (N/m^2)
τ_0	(4)	estimated shear stress lying on Mohr circle, corresponding with σ_0 (N/m^2)
τ'_0	(7)	estimated shear stress lying on yield locus, corresponding with σ_0 (N/m^2)
$\Delta\tau$	(11)	absolute error of shear stress (N/m^2)
$\rho\tau$	(12)	relative error of shear stress (—)

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