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Note

Long cycles in graphs containing a 2-factor with many odd components

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Abstract

We prove a result on the length of a longest cycle in a graph on n vertices that contains a 2-factor and satisfies $d(u)+d(v)+d(w)\geqslant n+2$ for every triple u, v, w of independent vertices. As a corollary we obtain the following improvement of a conjecture of Häggkvist (1992): Let G be a 2-connected graph on n vertices where every pair of nonadjacent vertices has degree sum at least $n-k$ and assume G has a 2-factor with at least $k+1$ odd components. Then G is hamiltonian.

Keywords: (Long, Hamilton) cycle; 2-factor; Degree sum

1. Results

We use [4] for terminology and notation not defined here and consider finite, simple graphs only.

The following three conjectures, among many others, appear in [6].

Conjecture 1. Let G be a 2-connected graph on n vertices where every pair of nonadjacent vertices has degree sum at least $n-k$ and assume furthermore that G has a k-factor. Then G is hamiltonian.

Conjecture 2. Let G be a 2-connected graph on n vertices where every pair of nonadjacent vertices has degree sum at least $n-k$ and assume that G has a 2-factor where every component is of order more than k. Then G is hamiltonian.

Conjecture 3. Let G be a 2-connected graph on n vertices where every pair of nonadjacent vertices has degree sum at least $n-k$ and assume that G has a 2-factor with at least $2k$ odd components. Then G is hamiltonian.

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A generalization of Conjecture 1 is proved in [5]. The Petersen graph is a counterexample to Conjecture 2, but it is well possible that there are no other counterexamples.

The main goal of this paper is to prove a result which implies that Conjecture 3 is true. In fact, our result implies that the bound $2k$ in Conjecture 3 can be almost halved in general. For a graph G and an integer $k \geq 1$, define $\sigma_k(G)$ by

$$\sigma_k(G) = \min\{\sum_{v\in S} d_G(v) \mid S \subseteq V(G) \text{ is an independent set of size } k\}.$$

Now we can state our main result, the proof of which will be given in Section 2.

Theorem 4. *Let G be a 2-connected graph on n vertices that satisfies $\sigma_3(G) \geq n+2$ and assume G contains a 2-factor with at least k odd components. Then a longest cycle in G has length at least* $\min\{n, \frac{1}{3}\sigma_3(G) + \frac{1}{2}n + \frac{1}{2}k\}$.

The conclusion in Theorem 4 is best possible. This is shown by the graphs $G_{k,l,t} = K_t \vee (lK_3 + (k-l)K_2 + (t+l-k)K_1)$ ($\vee$ denotes the join of two graphs). For any k, l, t with $k \geq l \geq 1$ and $t \geq 2l+k+2$, $G_{k,l,t}$ is a 2-connected graph on $n = 2t+2l+k$ vertices that contains a 2-factor with k odd components, but no 2-factor with more than k odd components, and satisfies $\sigma_3(G_{k,l,t}) = 3t \geq n+2$. The length of a longest cycle in $G_{k,l,t}$ is $3l + 2(k-l) + (t-k) + t = 2t + l + k = \frac{1}{3}\sigma_3(G_{k,l,t}) + \frac{1}{2}n + \frac{1}{2}k$.

Also the bound $\sigma_3(G) \geq n+2$ in Theorem 4 cannot be relaxed as is shown by the graphs $H_l = K_2 \vee (K_{3l} + K_{3l} + K_{3l-2})$. For any $l \geq 1$, H_l is a 2-connected graph on $n = 9l$ vertices that has a 2-factor with $3l$ odd components and satisfies $\sigma_3(H_l) = 9l + 1 = n+1$. Furthermore, $\frac{1}{3}\sigma_3(H_l) + \frac{1}{2}n + \frac{1}{2}(3l) = 9l + \frac{1}{3}$, whereas a longest cycle in H_l has length $6l+2$ only.

From Theorem 4 we can derive the following corollaries concerning Hamilton cycles in graphs. Corollary 7 shows that (a sharper version of) Conjecture 3 is true.

Corollary 5. *Let G be a 2-connected graph on n vertices that satisfies $\sigma_3(G) \geq \max\{\frac{3}{2}(n-k) - 1, n+2\}$ and contains a 2-factor with at least $k-1$ odd components. Then G is hamiltonian.*

Proof. Let G be a graph that satisfies the conditions in the corollary. Then we have

$$\tfrac{1}{3}\sigma_3(G) + \tfrac{1}{2}n + \tfrac{1}{2}(k-1) \geq \tfrac{1}{2}(n-k) - \tfrac{1}{3} + \tfrac{1}{2}n + \tfrac{1}{2}(k-1) = n - \tfrac{5}{6} > n-1,$$

so, by Theorem 4 we can conclude that G contains a Hamilton cycle. □

Corollary 6. *Let G be a 2-connected graph on n vertices that satisfies $\sigma_2(G) \geq \max\{n-k, \frac{2}{3}n + 1\}$ and contains a 2-factor with at least $k-1$ odd components. Then G is hamiltonian.*

Proof. For any graph G and any three independent vertices u, v, w in $V(G)$ we have

$$d(u)+d(v)+d(w)=\tfrac{1}{2}[(d(u)+d(v))+(d(u)+d(w))+(d(v)+d(w))]$$
$$\geqslant \tfrac{1}{2}(3\sigma_2(G)).$$

So, for any graph G, $\sigma_3(G) \geqslant \frac{3}{2}\sigma_2(G)$ and Corollary 6 follows immediately from Corollary 5. □

Corollary 7. *Let G be a 2-connected graph on n vertices that satisfies $\sigma_2(G) \geqslant n-k$ and contains a 2-factor with at least $k+1$ odd components. The G is hamiltonian.*

Proof. Let G be graph that satisfies the conditions in the corollary. Since G contains a 2-factor with at least $k+1$ odd components, we have $n \geqslant 3(k+1)$, or $k \leqslant \frac{1}{3}n-1$. This means $\sigma_2(G) \geqslant n-k \geqslant \frac{2}{3}n+1$, and the result follows from Corollary 6. □

Notice that for $k \leqslant \frac{1}{3}(n-5)$ we have $\lceil \frac{3}{2}(n-k)-1 \rceil \geqslant n+2$. So the bound $\sigma_3(G) \geqslant \max\{\frac{3}{2}(n-k)-1, n+2\}$ in Corollary 5 can be replaced by $\sigma_3(G) \geqslant \frac{3}{2}(n-k)-1$ if we add the extra condition $k \leqslant \frac{1}{3}(n-5)$. Analogously, the bound $\sigma_2(G) \geqslant \max\{n-k, \frac{2}{3}n+1\}$ in Corollary 6 can be replaced by $\sigma_2(G) \geqslant n-k$ if we add the condition $k \leqslant \frac{1}{3}(n-3)$. This is not a strong limitation, because we already have the condition that G contains a 2-factor with at least $k-1$ odd components, which means $n \geqslant 3(k-1)$, or $k \leqslant \frac{1}{3}(n+3)$.

The graphs $G_{k-1,1,t} = K_t \vee (K_3 + (k-2)K_2 + (t-k+2)K_1)$ show that the bounds in Corollaries 5 and 6 are sharp. For any k, t with $k \geqslant 2$ and $t \geqslant k+3$, $G_{k-1,1,t}$ is a 2-connected graph on $n = 2t+k+1$ vertices that contains a 2-factor with $k-1$ odd components, satisfies $\sigma_3(G_{k-1,1,t}) = 3t = \frac{3}{2}(n-k)-1-\frac{1}{2}$ and $\sigma_2(G_{k-1,1,t}) = 2t = n-k-1$, but contains no Hamilton cycle.

2. Proof of Theorem 4

Let C be a cycle of a graph G. If $V(G)-V(C)$ is a independent set, then C is called a *dominating cycle* of G. By $\vec{C}$ we denote the cycle C with a given orientation. If $u \in V(C)$, then u^+ denotes the successor of u on $\vec{C}$. If $A \subseteq V(C)$, then $A^+ = \{v^+ \mid v \in A\}$.

The following two lemmas are essential in our proof of Theorem 4. The first part of Lemma 8 is a result from [3]; the second part is implicit in the proof of [2, Theorem 10]. Lemma 9 is [2, Lemma 8]. Both lemmas also appear in [1].

Lemma 8 (Bauer et al. [1, 2], Bondy [3]). *Let G be a 2-connected graph on n vertices that satisfies $\sigma_3(G) \geqslant n+2$. Then every longest cycle of G is a dominating cycle. Moreover, if G is nonhamiltonian, then G contains a longest cycle C such that $\max\{d_G(v) \mid v \in V(G)-V(C)\} \geqslant \frac{1}{3}\sigma_3(G)$.*

Lemma 9 (Bauer et al. [1,2]). *Let G be a graph on n vertices with $\delta(G) \geqslant 2$ and $\sigma_3(G) \geqslant n$. Assume that G contains a longest cycle $\vec{C}$ which is a dominating cycle. If $v \in V(G) - V(C)$, then $(V(G) - V(C)) \cup (N(v))^+$ is an independent set of vertices.*

For the remainder of this section we assume that G is a nonhamiltonian, 2-connected graph on n vertices that satisfies $\sigma_3(G) \geqslant n+2$ and contains a 2-factor F with at least k odd components. By Lemma 8, we can choose a longest cycle C in G and a vertex $a \in V(G) - V(C)$ such that $N(a) \subseteq V(C)$ and $d_G(a) \geqslant \frac{1}{3}\sigma_3(G)$. Let c be the length of C and choose an orientation $\vec{C}$ of C. Define $A = (V(G) - V(C)) \cup (N(a))^+$ and $B = V(G) - A$. By Lemmas 8 and 9, A is an independent set of vertices of size $|A| \geqslant n - c + \frac{1}{3}\sigma_3(G)$.

Since A is an independent set, all edges in G either have one end vertex in A and the other end vertex in B, or both end vertices in B. Of course the same holds for the edges of the 2-factor F. Let $e_F(B)$ be the number of edges of F with both end vertices in B. Every odd component of F contains at least one edge with both end vertices in B, hence $e_F(B) \geqslant k$. Counting the edges of F between A and B in two ways we see

$$2|A| = 2|B| - 2e_F(B).$$

Since $|B| = n - |A|$, this is equivalent to

$$4|A| = 2n - 2e_F(B).$$

Using that $|A| \geqslant n - c + \frac{1}{3}\sigma_3(G)$ and $e_F(B) \geqslant k$ we obtain

$$\tfrac{4}{3}\sigma_3(G) + 4n - 4c \leqslant 2n - 2k,$$

which gives

$$c \geqslant \tfrac{1}{3}\sigma_3(G) + \tfrac{1}{2}n + \tfrac{1}{2}k.$$

This completes the proof of Theorem 4.

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