Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Systems & Control Letters 52 (2004) 247–255
SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

Transfer functions for infinite-dimensional systems

Hans Zwart*

Department of Applied Mathematics, University of Twente, P.O. Box 217, NL-7500 AE Enschede, Netherlands

Received 21 October 2003; received in revised form 21 December 2003; accepted 9 February 2004

Abstract

In this paper, we study three definitions of the transfer function for an infinite-dimensional system. The first one defines the transfer function as the expression $C(sI - A)^{-1}B + D$. In the second definition, the transfer function is defined as the quotient of the Laplace transform of the output and input, with initial condition zero. In the third definition, we introduce the transfer function as the quotient of the input and output, when the input and output are exponentials. We show that these definitions always agree on the right-half plane bounded to the left by the growth bound of the underlying semigroup, but that they may differ elsewhere.

Keywords: Transfer function; Infinite-dimensional system

1. Introduction

The notion of transfer function is classical in systems theory. One could even state that without transfer functions there is no systems theory. There are, however, different definitions of a transfer function, such as the Laplace transform of the impulse response, or the quotient of the Laplace transform of the output and the Laplace transform of the input, or just $C(sI - A)^{-1}B + D$ for a state linear system. For finite-dimensional systems all these definitions lead to same rational function, see Polderman and Willems [4, Chapter 8], provided one makes the necessary analytic continuation. However, as is shown in Curtain and Zwart [2, Example 4.3.8], these notions may differ for infinite-dimensional systems. Hence there is a need for clarification on this point. In this paper, we start by studying transfer functions for the state linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,$$
$$y(t) = Cx(t) + Du(t), \tag{1}$$

where A is the infinitesimal generator of a C_0-semigroup on the state space X, B is a bounded linear operator from input space U to X, C is a bounded linear operator from X to the output space Y, and D is a bounded operator from U to Y. The spaces X, U and Y are assumed to be Banach spaces. To simplify notation, we denote by $\mathscr{L}(U,Y)$ the space of bounded linear operators from U to Y. For system (1), we introduce the following notions of a transfer function. To avoid confusion, we give different names to the different definitions of transfer functions.

* Corresponding author. Tel.: +31-534893464; fax: +31-534893800.
E-mail address: h.j.zwart@math.utwente.nl (H. Zwart).

0167-6911/$ - see front matter
doi:10.1016/j.sysconle.2004.02.002

Definition 1.1. For system (1) we introduce

1. The characteristic function of system (1) is defined as
$$\mathfrak{G}(s) = C(sI - A)^{-1}B + D, \quad s \in \rho(A), \tag{2}$$
where $\rho(A)$ denotes the resolvent set of A.
2. Assume that $x_0 = 0$, and let $\hat{u}(s)$ and $\hat{y}(s)$ denote the (one-sided) Laplace transforms of u and y, respectively. If there exists a real α such that for all input–output pairs whose Laplace transform exists on $\{s \in \mathbb{C} | \mathrm{Re}(s) > \alpha\}$ we can write $\hat{y}(s) = H(s)\hat{u}(s)$ on $\mathrm{Re}(s) > \alpha$, and $H(s)$ is a $\mathscr{L}(U, Y)$-valued function of a complex variable defined for $\mathrm{Re}(s) > \alpha$, then we call $H(s)$ the input–output transfer function of (1).

For these definitions we summarize the following results, see [2, Lemma 4.3.6]:

- The function H as defined above exists, and equals the Laplace transform of the impulse response $h(t) := CT(t)B + D\delta(t)$, where $T(t)$ is the C_0-semigroup generated by A. The region of convergence of this Laplace transform is the right-half plane $\{s \in \mathbb{C} | \mathrm{Re}(s) > \beta\}$. Furthermore, $\beta \leqslant \omega$, where ω is the growth bound of the semigroup.
- On the right-half plane $\mathbb{C}_\omega^+ := \{s \in \mathbb{C} | \mathrm{Re}(s) > \omega\}$, we have that $\mathfrak{G}(s) = H(s)$.
- On the right-half plane $\mathbb{C}_\beta^+ := \{s \in \mathbb{C} | \mathrm{Re}(s) > \beta\}$, one may have that $\mathfrak{G}(s) \neq H(s)$, see Example 4.3.8 of [2].

Hence in the right-half plane bounded by the growth bound of the semigroup there is no confusion about the notion of the transfer function. However, one would like to know how one may extend this. Recall that for finite-dimensional systems one normally defines the transfer function as the characteristic function, and this equals the analytic continuation of $H(s)$. Since for finite-dimensional systems the transfer function is rational, it is clear what is meant by the analytic continuation. However, for an infinite-dimensional system this is not clear at all. For example, consider a system with impulse response

$$h(t) = \frac{1}{\sqrt{t}}.$$

Its Laplace transform equals

$$H(s) = \sqrt{\frac{\pi}{s}} \quad \text{for } s \in \mathbb{C} \text{ with } \mathrm{Re}(s) > 0.$$

By standard Laplace theory, we have that this function is analytic on its domain. Unlike the rational case, there does not exist “the” analytic continuation for this function. First one has to specify the branch cut of $\sqrt{s}$. Normally, one chooses the negative real line, but any (straight) line starting at zero and contained in the open left half plane will do.

As can be seen from the fact that $H(s) = C(sI - A)^{-1}B + D$ on $\mathbb{C}_\omega^+ = \{s \in \mathbb{C} | \mathrm{Re}(s) > \omega\}$, it is natural to relate an analytic continuation of H to that of $C(sI - A)^{-1}B + D$, i.e., to that of $(sI - A)^{-1}$. Starting from the resolvent operator $(sI - A)^{-1}$ defined on $\mathbb{C}_\omega^+$, there is a natural domain for its analytic continuation. This domain is the largest component of the resolvent set containing $\mathbb{C}_\omega^+$, and is denoted by $\rho_\infty(A)$, see [2, Section 2.5]. On $\rho_\infty(A)$ the resolvent operator has a unique analytic continuation which equals $(sI - A)^{-1}$, as is easy to see. The analytic continuation of $H(s)$ from $\mathbb{C}_\omega^+$ to $\rho_\infty(A)$ equals $\mathfrak{G}(s)$.

In Lemma 4.3.6 of [2] it is claimed that

$$H(s) = \mathfrak{G}(s) \quad \text{on } \rho_\infty(A).$$

In Example 2.2 we show that this is wrong. The reason for this lies in the construction of the analytic continuation. We have that $H(s) = \mathfrak{G}(s)$ on $\mathbb{C}_\omega^+$. As mentioned above, we can see $\mathfrak{G}(s)$ as the analytic continuation of $H(s)$ from $\mathbb{C}_\omega^+$ to $\rho_\infty(A)$. On the other hand, the analytic continuation of $H(s)$ from $\mathbb{C}_\omega^+$ to $\mathbb{C}_\beta^+$ equals $H(s)$. Suppose now that $\beta = 0$, and that the spectrum of A is the positive half circle, see Fig. 1. So the point $\frac{1}{2}$ is an element of $\mathbb{C}_\beta^+$ and an element of $\rho_\infty(A)$, but any path contained in $\rho_\infty(A)$ connecting $\frac{1}{2}$ with 2 must leave $\mathbb{C}_\beta^+$. Since an analytic continuation is completely dependent on the allowed paths, it is likely that the value obtained in $\frac{1}{2}$ using these paths will differ from the value obtained by going from 2 to $\frac{1}{2}$ over the real axis. This is what happens in Example 3.2.

In Definition 1.1 we gave two definitions for transfer function of system (1). For finite-dimensional systems there is another characterization of a transfer function, namely the exponential input, $e^{\lambda t}, t \in \mathbb{R}$, gives as output the same exponential multiplied by a complex number. This number equals the transfer function $G(\lambda)$. When working with (finite-dimensional)

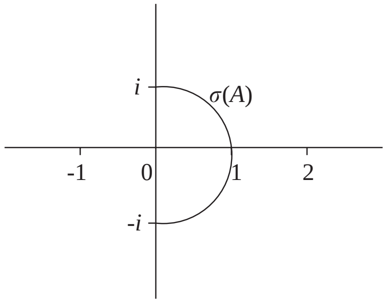

Fig. 1. A situation in which $\rho_\infty(A) \cap \mathbb{C}_\beta^+$ consists of two components.

systems on the time axis $[0, \infty)$, one has to make the proper choice for the initial condition in other to obtain an exponential output. For infinite-dimensional systems this notion of a transfer function has hardly been investigated. In the next section we study this notion and relate it to the other notions of transfer functions. Maybe the most surprising result is that $G(\lambda)$ need not have a unique value at λ. Note that working on the time axis $[0, \infty)$ is the natural choice for infinite-dimensional state linear systems, since the abstract differential equation $\dot{x}(t) = Ax(t)$ may have only the trivial solution on the time axis $\mathbb{R}$, even when A is the infinitesimal generator of C_0-semigroup.

In Section 3, we give some examples showing the difference between the different notions. Throughout most of the paper we assume that B and C are bounded operators. In Section 4, we shall summarize the results if this assumption does no longer hold. Please note that the difficulties arising for transfer functions for infinite-dimensional systems are not caused by the unboundedness of A. It is purely a consequence of the fact that the state space is infinite-dimensional.

2. Transmission functions

We begin by giving a signal characterization of the characteristic function for state linear system (1). We leave the proof up to the reader.

Lemma 2.1. *Consider system (1). For every $\lambda \in \rho(A)$ there exists an input-state-output triple $(u(t), x(t), y(t))$ of the form $(u_0 e^{\lambda t}, x_0 e^{\lambda t}, y_0 e^{\lambda t})$, $t \geqslant 0$, satisfying (1). Furthermore, for a given u_0, the initial condition x_0 and y_0 are unique and are given by $(\lambda I - A)^{-1} B u_0$ and $\mathfrak{G}(\lambda) u_0$, respectively.*

In the next definition, we remove the assumption that the state must be of the form $x_0 e^{\lambda t}$.

Definition 2.2. Consider system (1). For $\lambda \in \mathbb{C}$ we define an element $y_\lambda \in Y$ to be a transmission value for $u_0 \exp(\lambda t)$ if for the input $u_0 \exp(\lambda t)$, $t \geqslant 0$, there exists an initial condition $x_\lambda(0) \in X$ such that the output of (1) equals $y_\lambda \exp(\lambda t)$ for $t \geqslant 0$. We say that $G(\lambda)$ is a transmission function (at λ) if for every $u_0 \in U$ y_λ can be written as $y_\lambda = G(\lambda) u_0$.

We can solve Eq. (1) for $u_0 \exp(\lambda t)$, by simply taking the Laplace transform of this equation. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \max\{\mathrm{Re}(\lambda), \omega\}$, where ω is the growth bound of semigroup generated by A, we have

$$sX_\lambda(s) - x_\lambda(0) = AX_\lambda(s) + Bu_0 \frac{1}{s-\lambda}, \tag{3}$$

$$Y_\lambda(s) = CX_\lambda(s) + Du_0 \frac{1}{s-\lambda}. \tag{4}$$

Or equivalently

$$X_\lambda(s) = (sI - A)^{-1} x_\lambda(0) + (sI - A)^{-1} Bu_0 \frac{1}{s-\lambda}, \tag{5}$$

$$Y_\lambda(s) = C(sI - A)^{-1} x_\lambda(0) + C(sI - A)^{-1} Bu_0 \frac{1}{s-\lambda} + Du_0 \frac{1}{s-\lambda}. \tag{6}$$

With the concept of characteristic function, i.e., $\mathfrak{G}(s) = C(sI - A)^{-1} B + D$, we can write the last equation as

$$Y_\lambda(s) = C(sI - A)^{-1} x_\lambda(0) + \mathfrak{G}(s) u_0 \frac{1}{s-\lambda}. \tag{7}$$

So we have that y_λ is a transmission value for $u_0 \exp(\lambda t)$ if and only if

$$y_\lambda = C(sI - A)^{-1}(s-\lambda) x_\lambda(0) + C(sI - A)^{-1} Bu_0 + Du_0 \tag{8}$$

on $\mathbb{C}^+_{\max\{\omega, Re(\lambda)\}}$.

From Lemma 2.1 the following result is immediately.

Lemma 2.3. *For every λ in the resolvent set of A, we have that $\mathfrak{G}(\lambda) := C(\lambda I - A)^{-1}B + D$ is a transmission function at λ.*

Since we have that $\mathfrak{G}(\lambda)$ is always a transmission function, one might expect that this is the only one. However, there can be more as the following example shows.

Example 2.4. Let A be the left shift on $\ell_2(\mathbb{Z})$. So

$$(Az)_k = z_{k+1}, \quad k \in \mathbb{Z}.$$

We have that

$$\sigma(A) = \sigma_c(A) = \{s \in \mathbb{C} : |s| = 1\}.$$

We define B as the vector in $\ell_2(\mathbb{Z})$ having all coefficients zero except the coefficient at the position -1 which is taken to be 1, i.e., $B_k = \delta_{-1,k}$. The output operator C is defined as $Cz = z_0$ and $D = 0$. The growth bound of the semigroup generated by A is one, and a simple calculation shows that if $|\lambda| > 1$, then

$$((\lambda I - A)^{-1}B)_k = \lambda^k \quad \text{for } k < 0; \text{ otherwise } 0.$$

On the other hand, if $|\lambda| < 1$, then

$$((\lambda I - A)^{-1}B)_k = -\lambda^k \quad \text{for } k \geqslant 0; \text{ otherwise } 0.$$

So the characteristic function $\mathfrak{G}(\lambda) = C(\lambda I - A)^{-1}B = -1$ if $|\lambda| < 1$, but otherwise it is zero.

Let $\lambda \in \rho(A)$ and choose $x_\lambda(0)$ in (6) as $p(\lambda)(\lambda I - A)^{-1}B$, where $p(\lambda)$ is an arbitrary constant, then we find that

$$Y_\lambda(s) = C(sI - A)(\lambda I - A)^{-1}B\,p(\lambda) + 0$$
$$= p(\lambda)\frac{1}{s - \lambda}\,C(\lambda I - A)^{-1}B.$$

Thus for $|\lambda| > 1$ this is zero, but for $|\lambda| < 1$, we find that any multiple, $p(\lambda)$, can be a transmission value for $u_0 \exp(\lambda t)$. So we find that the transmission function and value are only unique on some region of the complex plane.

As we have seen in the previous example, we can only expect that the transmission value and function are unique in some region of the complex plane. The following theorem shows that this indeed holds. In order to prove this theorem, we need some results which are listed next. First, we show that any analytic continuation of $C(sI - A)^{-1}$, $s \in \mathbb{C}_\omega^+$ to a point in $\lambda \in \rho(A)$ equals $C(\lambda I - A)^{-1}$. This lemma is an extension of Lemma 2.3 of Curtain [1].

Lemma 2.5. *Let $\lambda \in \rho(A)$, and assume further that the operator $C(sI - A)^{-1}$ has the analytic continuation $\Psi(s)$ from $\mathbb{C}_\omega^+ = \{s \in \mathbb{C} | \mathrm{Re}(s) > \omega\}$, to the (connected) region Ω containing λ and this right-half plane $\mathbb{C}_\omega^+$. Then $\Psi(\lambda) = C(\lambda I - A)^{-1}$. Thus any analytic continuation of $C(sI - A)^{-1}$ from $\mathbb{C}_\omega^+$ to λ has the same value.*

Proof. Let Ψ denote the analytic continuation of $C(sI - A)^{-1}$ from $\mathbb{C}_\omega^+$ to Ω. On $\mathbb{C}_\omega^+$ we have that for all $x_0 \in D(A)$

$$\Psi(s)(sI - A)x_0 = C(sI - A)^{-1}(sI - A)x_0 = Cx_0.$$

Since Ψ has an analytic continuation to Ω and since $(sI - A)x_0$, and Cx_0 are entire functions, we have that the above relation also holds on Ω. In particular, there holds

$$\Psi(\lambda)(\lambda I - A)x_0 = Cx_0.$$

This holds for every $x_0 \in D(A)$, and so we have that

$$\Psi(\lambda) = C(\lambda I - A)^{-1}.$$

So the analytic continuation of $C(sI - A)^{-1}$ to λ is unique. □

In order to see the particular nature of this theorem consider the function $\sqrt{s}$. This function is analytic in $\mathbb{C}_0^+$, but its value at -1 depends on the way we have reached this point. Note that the result of the above lemma does not hold for $C(sI - A)^{-1}x_0$, as can be seen from Example 2.4. For this function we can prove the following lemma. In this result we use $\rho_\infty(A)$, the largest component of the resolvent set that contains an interval $[r, \infty)$.

Lemma 2.6. *Let x_0 be an element of X. Then the following assertions are equivalent:*

1. *$C(sI - A)^{-1}x_0 = 0$ for all s in Ω, where $\Omega \subset \rho_\infty(A)$ contains an accumulation point;*
2. *$C(sI - A)^{-1}x_0 = 0$ for all $s \in \rho_\infty(A)$;*
3. *x_0 is non-observable, i.e., $CT(t)x_0 =$ for all $t \geqslant 0$.*

Proof. The implications (3) $\Leftrightarrow$ (1) follow easily by taking the Laplace transform of $CT(t)x_0$.

It is clear that (2) implies (1). Let us assume that (1) holds, and let s be an arbitrary element in $\rho_\infty(A)$. We can see $(sI-A)^{-1}$ as the inverse of $(sI-A)$, but we can also regard it as the evaluation at s of the analytic continuation of the resolvent from Ω to $\rho_\infty(A)$. Since both interpretations give the same value, we have that $C(sI-A)^{-1}x_0$ equals the analytic continuation of the zero function on Ω. Thus $C(sI-A)^{-1}x_0=0$. □

Theorem 2.7. *Let $\lambda\in\rho(A)$. Then the transmission value for $u_0\exp(\lambda t)$ is non-unique if and only if there exists a x_0, such that $C(sI-A)^{-1}x_0=0$ on $\rho_\infty(A)$ and $C(\lambda I-A)^{-1}x_0\neq 0$.*

Furthermore, all transmission values for $\exp(\lambda t)u_0$ *are characterized as elements of the set*

$$C(\lambda I-A)^{-1}Bu_0+Du_0+C(\lambda I-A)^{-1}\mathcal{N}, \tag{9}$$

where $\mathcal{N}=\{x_0\in X\,|\,C(sI-A)^{-1}x_0=0 \text{ on } \rho_\infty(A)\}$. Thus they form a (affine) linear subspace of Y. Note that $\mathcal{N}$ is the non-observable subspace, see Lemma 2.6.

Proof. Suppose that there exists a x_0 with the properties as stated above, then choose $x_\lambda(0)=(\lambda I-A)^{-1}Bu_0+p_\lambda(\lambda I-A)^{-1}x_0$, with $p_\lambda\in\mathbb{C}$.

$$\begin{aligned}
Y_\lambda(s) &= C(sI-A)^{-1}x_\lambda(0)\\
&\quad +C(sI-A)^{-1}Bu_0\frac{1}{s-\lambda}+Du_0\frac{1}{s-\lambda}\\
&= C(sI-A)^{-1}(\lambda I-A)^{-1}Bu_0\\
&\quad +p_\lambda C(sI-A)^{-1}(\lambda I-A)^{-1}x_0\\
&\quad +C(sI-A)^{-1}Bu_0\frac{1}{s-\lambda}+Du_0\frac{1}{s-\lambda}\\
&= C(\lambda I-A)^{-1}Bu_0\frac{1}{s-\lambda}\\
&\quad +p_\lambda C(\lambda I-A)^{-1}x_0\frac{1}{s-\lambda}\\
&\quad -p_\lambda C(sI-A)^{-1}x_0\frac{1}{s-\lambda}+Du_0\frac{1}{s-\lambda}\\
&= C(\lambda I-A)^{-1}Bu_0\frac{1}{s-\lambda}+Du_0\frac{1}{s-\lambda}\\
&\quad +p_\lambda C(\lambda I-A)^{-1}x_0\frac{1}{s-\lambda}+0,
\end{aligned}$$

where we have used the resolvent identity. So we have that $C(\lambda I-A)^{-1}Bu_0+Du_0+p_\lambda C(\lambda I-A)^{-1}x_0$ is a transmission value for $\exp(\lambda t)u_0$. Since $C(\lambda I-A)^{-1}x_0\neq 0$, this expression has infinitely many values, depending on the choice of p_λ. Furthermore, the transmission value is in set (9).

Now we shall prove the converse. Suppose that $y_{1,\lambda}$ and $y_{2,\lambda}$ are both transmission values for $\exp(\lambda t)u_0$ and let $x_{1,\lambda}(0)$ and $x_{2,\lambda}(0)$ be the corresponding initial conditions. From Eq. (8) we see that

$$y_{1,\lambda}-y_{2,\lambda}=C(sI-A)^{-1}(s-\lambda)[x_{1,\lambda}(0)-x_{2,\lambda}(0)],\quad s\in\mathbb{C}^+_{\max\{\omega,Re(\lambda)\}}. \tag{10}$$

When we take the limit for $s\to\infty$, we obtain that

$$y_{1,\lambda}-y_{2,\lambda}=C[x_{1,\lambda}(0)-x_{2,\lambda}(0)].$$

Writing in (10) $s-\lambda$ as $sI-A+A-\lambda I$, and using the above relation we see that

$$\begin{aligned}
y_{1,\lambda}-y_{2,\lambda} &= C[x_{1,\lambda}(0)-x_{2,\lambda}(0)]+C(A-\lambda I)\\
&\quad\times(sI-A)^{-1}[x_{1,\lambda}(0)-x_{2,\lambda}(0)]\\
&= y_{1,\lambda}-y_{2,\lambda}+C(A-\lambda I)(sI-A)^{-1}\\
&\quad\times[x_{1,\lambda}(0)-x_{2,\lambda}(0)].
\end{aligned}$$

Using Lemma 2.6 we obtain that for all $s\in\rho_\infty(A)$

$$C(A-\lambda I)(sI-A)^{-1}[x_{1,\lambda}(0)-x_{2,\lambda}(0)]=0. \tag{11}$$

Take s_0 a fixed point in $\rho_\infty(A)$, then it follows from the above equation that

$$\begin{aligned}
0 &= C(A-\lambda I)(s_0I-A)^{-1}[x_{1,\lambda}(0)x_{2,\lambda}(0)]\\
&\quad -C(A-\lambda I)(sI-A)^{-1}[x_{1,\lambda}(0)-x_{2,\lambda}(0)]\\
&= (s-s_0)C(A-\lambda I)(s_0I-A)^{-1}(sI-A)^{-1}\\
&\quad\times[x_{1,\lambda}(0)-x_{2,\lambda}(0)]\\
&= (s-s_0)C(sI-A)^{-1}(A-\lambda I)(s_0I-A)^{-1}\\
&\quad\times[x_{1,\lambda}(0)-x_{2,\lambda}(0)].
\end{aligned}$$

Defining x_0 as $x_0=(A-\lambda I)(s_0I-A)^{-1}[x_{1,\lambda}(0)-x_{2,\lambda}(0)]$, we see that $C(sI-A)^{-1}x_0=0$, and

$$\begin{aligned}
C(\lambda I-A)^{-1}x_0 &= C(\lambda I-A)^{-1}(A-\lambda I)(s_0I-A)^{-1}\\
&\quad\times[x_{1,\lambda}(0)-x_{2,\lambda}(0)]
\end{aligned}$$

$$= -C(s_0 I - A)^{-1}[x_{1,\lambda}(0) - x_{2,\lambda}(0)]$$

by (11)

$$= \frac{-1}{s_0 - \lambda}[y_{1,\lambda} - y_{2,\lambda}] \neq 0. \qquad \square$$

From the above theorem we can make some simple observations:

- If the transmission value is non-unique for a $u_0 \in U$, then it is non-unique for every $u \in U$. Even for $u_0 = 0$.
- The undetermined part of the transmission value is independent of u_0.
- If the transmission value for $u_0 \exp(\lambda t)$ is unique, then the transmission function at λ exists and equals the characteristic function at λ.

From Theorem 2.7 we have the following direct consequences.

Corollary 2.8. (1) *For $\lambda \in \rho_\infty(A)$ the transmission value is unique, and is given by*

$$y_\lambda = C(\lambda I - A)^{-1}Bu_0 + Du_0.$$

Furthermore, the transmission function equals the characteristic function.

(2) *If $\Sigma(A, -, C)$ is approximately observable, then the transmission value is unique on $\rho(A)$. Furthermore, the transmission function equals the characteristic function.*

So on $\rho_\infty(A)$ the transmission value and function are unique, but as we have seen from Example 2.4 it can be completely undetermined on another component of the spectrum. Since the three notions of the transfer functions agree on $\mathbb{C}_\omega^+$, we may define this to be the transfer function of the system.

Definition 2.9. Consider system (1), and let ω denote the growth bound of the semigroup generated by A. The function

$$\mathfrak{G}(s) = C(sI - A)^{-1}B + D, \quad s \in \mathbb{C}_\omega^+$$

is defined as the transfer function of (1).

Using the characterization of the transmission value, we obtain conditions when it is unique.

Lemma 2.10. *For the transmission value and function we have the following uniqueness results*:

1. *If $\Sigma(A, -, C)$ is output stable, then the transmission value and function are unique at every point in the intersection of the closed right-half plane and the resolvent set.*
2. *Let $\lambda \in \rho(A)$. Assume further that the operator $C(sI - A)^{-1}$ has the analytic continuation $\Psi(s)$ from $\mathbb{C}_\omega^+$ to the (connected) region Ω containing λ. Then the transmission value and function at λ are unique, and the transmission function at λ equals the characteristic function at λ.*

Proof. (1) Let $\lambda \in \rho(A) \cap \{s \in \mathbb{C} | \mathrm{Re}(s) \geqslant 0\}$ and let $y_{1,\lambda}$ and $y_{2,\lambda}$ be transmission values for $\exp(\lambda t)u_0$. Then, see (10),

$$y_{1,\lambda} - y_{2,\lambda} = C(sI - A)^{-1}(s - \lambda)[x_{1,\lambda}(0) - x_{2,\lambda}(0)].$$

Or, equivalently,

$$C(sI - A)^{-1}[x_{1,\lambda}(0) - x_{2,\lambda}(0)] = \frac{y_{1,\lambda} - y_{2,\lambda}}{s - \lambda}$$

for $s \in \mathbb{C}^+_{\max\{\omega, \mathrm{Re}(\lambda)\}}$. In time-domain the above equation reads as

$$CT(t)[x_{1,\lambda}(0) - x_{2,\lambda}(0)] = [y_{1,\lambda} - y_{2,\lambda}]\exp(\lambda t).$$

Since the system is output stable and since $\mathrm{Re}(\lambda) \geqslant 0$, this implies that $y_{1,\lambda} - y_{2,\lambda} = 0$. Thus the transmission function at λ is unique.

(2) If at λ the transmission value and function would not be unique, then by Theorem 2.7 we know that there would exist a x_0 such that $C(sI - A)^{-1}x_0 = 0$ on $\rho_\infty(A)$ and $C(\lambda I - A)^{-1}x_0 \neq 0$. On $\rho_\infty(A)$ we have that $\Psi(s)x_0 = C(sI - A)^{-1}x_0 = 0$, and since Ψ has an analytic continuation to Ω, we find that $\Psi(s)x_0 = 0$ on Ω. Especially, $\Psi(\lambda)x_0 = 0$. However, from Lemma 2.5 we know that $\Psi(\lambda) = C(\lambda I - A)^{-1}$. Thus we conclude that $C(\lambda I - A)^{-1}x_0 = 0$, providing a contradiction. $\square$

Examples 2.4 and 3.2 of the next section, show that the input–output transfer function can be different from the characteristic and/or the transmission function. In the following lemma we show that if the system is output stable, then they are all equal on the open right-half plane. For the proof we refer to Lemma 2.3 in [1].

Lemma 2.11. *If $\Sigma(A,B,C,D)$ is output stable, then the transmission function and the input–output transfer function are equal on $\rho(A)\cap\mathbb{C}_0^+$.*

3. Analytic continuations of the transfer function

In this section, we show that the input–output transfer function need not be a transmission function, even when the transmission function is unique. But first we need to discuss the following simple example.

Example 3.1. Consider the differential equation on $[-1,1]$

$$\frac{\partial}{\partial t}x(\xi,t)=\mathrm{i}\xi x(\xi,t)+(1-\xi^2)^{-1/4}u(t). \tag{12}$$

As state space we choose $L^2(-1,1)$. The system operator A_0 is given by

$$A_0\phi(\xi)=(\mathrm{i}\xi)\phi(\xi)$$

and this is a bounded operator, with bound 1, on $L^2(-1,1)$. Since it is a multiplication operator, it is easy to see that its spectrum equals $\sigma(A_0)=\{s\in\mathbb{C}\,|\,\mathrm{Re}(s)=0 \text{ and } |\mathrm{Im}(s)|\leqslant 1\}$. The input operator B_0 is defined as $B_0u=(1-\xi^2)^{-1/4}u$, and since $(1-\xi^2)^{-1/4}\in L^2(-1,1)$, we know that B_0 is a bounded operator from $\mathbb{C}$ to $L^2(-1,1)$. As output operator, we take the dual of B_0. Thus,

$$y(t)=\int_{-1}^{1}x(\xi,t)(1-\xi^2)^{-1/4}\,\mathrm{d}\xi. \tag{13}$$

The characteristic function of this system is given by the expression

$$\mathfrak{G}_0(s)=\int_{-1}^{1}(1-\xi^2)^{-1/2}\frac{1}{(s-\mathrm{i}\xi)}\,\mathrm{d}\xi. \tag{14}$$

We rewrite this in a form which gives the closed-form expression. Let s be a positive real number, then we have that

$$\begin{aligned}\mathfrak{G}_0(s)&=\int_{-1}^{1}\frac{1}{s-\mathrm{i}\xi}(1-\xi^2)^{-1/2}\,\mathrm{d}\xi\\&=\int_{-1}^{0}\frac{1}{s-\mathrm{i}\xi}(1-\xi^2)^{-1/2}\,\mathrm{d}\xi\\&\quad+\int_{0}^{1}\frac{1}{s-\mathrm{i}\xi}(1-\xi^2)^{-1/2}\,\mathrm{d}\xi\\&=\int_{0}^{1}\frac{1}{s+\mathrm{i}\xi}(1-\xi^2)^{-1/2}\,\mathrm{d}\xi\\&\quad+\int_{0}^{1}\frac{1}{s-\mathrm{i}\xi}(1-\xi^2)^{-1/2}\,\mathrm{d}\xi\\&=2\int_{0}^{1}\frac{s}{(s^2+\xi^2)\sqrt{1-\xi^2}}\,\mathrm{d}\xi.\end{aligned} \tag{15}$$

Now using the substitution $x=\sqrt{\xi^{-2}-1}$ this last integral becomes

$$2\int_0^\infty\frac{s}{s^2+1+s^2x^2}\,\mathrm{d}x=\frac{\pi}{\sqrt{s^2+1}}.$$

Note that for $s>0$ we have to take the usual (positive) square root of s^2+1. Thus $\mathfrak{G}_0(\frac{1}{2})=2\pi/\sqrt{5}$. For $s<0$ we have that $\mathfrak{G}_0(s)=-\mathfrak{G}_0(-s)$, see (15), e.g. $\mathfrak{G}_0(-\frac{1}{2})=-2\pi/\sqrt{5}$.

Concluding we have that the characteristic function is the (unique) analytic continuation of $\pi/\sqrt{s^2+1}$ to the resolvent set of A, i.e., every complex s except for the interval $[-\mathrm{i},\mathrm{i}]$ on the imaginary axis. Since the resolvent set is connected, we have that the characteristic function and the transmission function are the same on the whole resolvent set.

The impulse response of the system is

$$h_0(t)=C_0\mathrm{e}^{A_0t}B_0=\int_{-1}^{1}\frac{\mathrm{e}^{\mathrm{i}\xi t}}{\sqrt{1-\xi^2}}\,\mathrm{d}\xi,$$

which is π times the Bessel function of the first kind and the zero order.

Example 3.2. Consider the differential equation on $[-1,1]$

$$\begin{aligned}\frac{\partial}{\partial t}x(\xi,t)&=\frac{1+\mathrm{i}\xi}{1-\mathrm{i}\xi}x(\xi,t)\\&\quad+(1-\mathrm{i}\xi)^{-1}(1-\xi^2)^{-1/4}u(t).\end{aligned} \tag{16}$$

As state space we choose again $L^2(-1,1)$. The system operator A_1 is given by

$$A_1\phi(\xi)=\frac{1+\mathrm{i}\xi}{1-\mathrm{i}\xi}\phi(\xi)$$

and this is a bounded operator, with bound one, on $L^2(-1,1)$. Since it is a multiplication operator, it is easy to see that its spectrum equals $\sigma(A_1)=\{s\in\mathbb{C}\,|\,\mathrm{Re}(s)\geqslant 0 \text{ and } |s|=1\}$. The input operator B_1 is defined as $B_1u=(1-\mathrm{i}\xi)^{-1}(1-\xi^2)^{-1/4}u$, and since $(1-\mathrm{i}\xi)^{-1}(1-\xi^2)^{-1/4}\in L^2(-1,1)$, we know that B_1 is a bounded operator from $\mathbb{C}$ to $L^2(-1,1)$. As output equation we take

$$y(t)=2\int_{-1}^{1} x(\xi,t)(1-\mathrm{i}\xi)^{-1}(1-\xi^2)^{-1/4}\,\mathrm{d}\xi + \frac{\pi}{\sqrt{2}}u(t). \tag{17}$$

Thus the system has a feed-through operator equal to $D_1 := \pi/\sqrt{2}$ and an output operator equal to

$$C_1\phi = 2\int_{-1}^{1}\phi(\xi)(1-\mathrm{i}\xi)^{-1}(1-\xi^2)^{-1/4}\,\mathrm{d}\xi.$$

For the calculation of the characteristic function we need we the following simple, but important relations between the system in Example 3.1 and the system defined above:

$$A_1 = (I+A_0)(I-A_0)^{-1},$$

$$B_1 = (I-A_0)^{-1}B_0,$$

$$C_1 = 2C_0(I-A_0)^{-1}.$$

Furthermore, $D_1 = \mathfrak{G}_0(1)$. So we have that

$$\begin{aligned}\mathfrak{G}_1(s) &:= C_1(sI-A_1)^{-1}B_1 + D_1\\ &= 2C_0(I-A_0)^{-1}(sI-(I+A_0)(I-A_0)^{-1})^{-1}\\ &\quad\times(I-A_0)^{-1}B_0 + D_1\\ &= C_0[2(I-A_0)^{-1}(sI-(I+A_0)(I-A_0)^{-1})^{-1}\\ &\quad\times(I-A_0)^{-1} + (I-A_0)^{-1}]B_0.\end{aligned}$$

Since for $s\in\rho(A_1)$ and $s\neq -1$, we have that

$$\begin{aligned}&\left(\left(\frac{s-1}{s+1}\right)I - A_0\right)^{-1}\\ &\quad=(I-A_0)^{-1} + 2(I-A_0)^{-1}\\ &\qquad\times(sI-(I+A_0)(I-A_0)^{-1})^{-1}(I-A_0)^{-1},\end{aligned} \tag{18}$$

we see that

$$\mathfrak{G}_1(s) = \mathfrak{G}_0\left(\frac{s-1}{s+1}\right) = \frac{\pi}{\sqrt{((s-1)/(s+1))^2+1}} = \frac{\pi}{\sqrt{(2s^2+2)/(s+1)^2}},\quad s\neq -1. \tag{19}$$

We may simplify the above expression, but extra care should be taken. Let us first define on the set $\rho(A_1)$ the function

$$f(s) := \frac{\pi}{\sqrt{s^2+1}}.$$

Thus, we have taken the branch cut of the square root of s^2+1 equal to the positive half circle. This is possible, see [3, Part II, Section 12]. Now define on $\rho(A_1)$ the function

$$\mathfrak{G}_2(s) = \frac{1}{\sqrt{2}}(s+1)f(s).$$

Then this function is analytic on $\rho(A_1)$ and it equals $\mathfrak{G}_1(s)$ for $s\in(1,\infty)$. This last follows since for $\tilde{s}>0$, one has to take the positive square root in $\mathfrak{G}_0(\tilde{s})$, see (19). Since two analytic functions on a given domain are the same if they are equal on an interval, we conclude that $\mathfrak{G}_2(s)=\mathfrak{G}_1(s)$ on $\rho(A_1)$. Or equivalently,

$$\mathfrak{G}_1(s) = \frac{\pi}{\sqrt{2}}\frac{s+1}{\sqrt{s^2+1}}\quad\text{on } \rho(A_1).$$

Note that

$$\mathfrak{G}_1(s) \neq \frac{1}{\sqrt{2}}(s+1)\mathfrak{G}_0(s),$$

since they have a different domain. Direct calculations gives that $\mathfrak{G}_1(\frac{1}{2})=\mathfrak{G}_0(-\frac{1}{3})=-3\pi/\sqrt{10}$, whereas $(1/\sqrt{2})(\frac{1}{2}+1)\mathfrak{G}_0(\frac{1}{2}) = 3\pi/\sqrt{10}$.

Next we want to derive the impulse response. One might try to calculate the impulse response via $h_1(t)= C_1\mathrm{e}^{A_1t}B_1 + D_1\delta(t)$. However, this leads to an integral, which could not be solved directly, and so we choose another route. We know that the impulse response is the inverse Laplace transform of the input–output transfer function. Unfortunately, we do not know this input–output transfer function, since we do not know its region of convergence. However, one does not need its precise region of convergence for the calculation of the inverse Laplace transform. Knowing the input–output transfer function on some right-half plane is sufficient for obtaining its inverse Laplace transform. In this example the growth bound of the semigroup is one, and thus on $\mathbb{C}_1^+$, we know that the input–output transfer function equals the characteristic function. Thus, we must calculate the inverse Laplace transform of $(\pi(s+1))/\sqrt{2(s^2+1)}$. We find that this equals

$$h_1(t) = \frac{\pi}{\sqrt{2}}[J_0(t) - J_1(t) + \delta(t)], \tag{20}$$

where J_0 and J_1 denote the Bessel functions of the first kind and of the zeroth and first order, respectively.

Since the absolute values of $J_0(t)$ and $J_1(t)$ decay like $1/\sqrt{t}$, [6, Section 7.1], we see that the Laplace transform of $h_1(t)$ is analytic in the open right-half plane $\mathbb{C}_0^+$. However, this Laplace transform only equals the transmission function on the smaller right-half plane $\mathbb{C}_1^+$. It is not hard to see that the Laplace transform of $h_1(t)$ equals $H_1(s) = (1/\sqrt{2})(s+1)\mathfrak{G}_0(s)$ on $\mathbb{C}_0^+$.

In Example 2.4, we already saw that the Laplace transform of the impulse response only equals the transmission function on some right-half plane. One might have had the impression that this was caused by the non-uniqueness of the transmission value. However, in the present example we have that $\rho(A_1) = \rho_\infty(A_1)$, and hence the transmission function is unique, but still we do not have that the input–output transfer function equals the transmission function. The equality only holds on $\mathbb{C}_\omega^+$, with ω the growth bound of the semigroup.

Note that the above example also shows that there are different meanings for the continuation of $C(sI - A)^{-1}B$. One could either mean to take an analytic continuation of $(sI - A)^{-1}$ or to take an analytic continuation of the expression $C(sI - A)^{-1}B$. Please note that if one first specifies the domain, then there can be no confusion.

4. Transmission functions for well-posed linear systems

Many of the results which are presented in Section 2 also hold for a more general class of systems. We assume that U, Y and X are Hilbert spaces, and furthermore we assume that we have a well-posed system, see e.g. [5] for the precise definition. Given this well-posed linear system, there exist Hilbert spaces, X_1 and X_{-1}, with $X_1 \subset X \subset X_{-1}$ and which satisfy $(sI - A)^{-1}X \subset X_1$, and $(sI - A)^{-1}X_{-1} \subset X$. Furthermore, there exists a B, C with $B \in \mathscr{L}(U, X_{-1})$ and $C \in \mathscr{L}(X_1, Y)$, and an analytic $\mathscr{L}(U, Y)$-valued function $\mathfrak{G}$ on $\mathbb{C}_\omega^+$ such that

$$Y(s) = C(sI - A)^{-1}x(0) + \mathfrak{G}(s)U(s), \quad s \in \mathbb{C}_\omega^+ \tag{21}$$

and

$$\mathfrak{G}(s) - \mathfrak{G}(s_0) = C(sI - A)^{-1}(s_0 I - A)^{-1} \times B(s_0 - s), \quad s, s_0 \in \mathbb{C}_\omega^+. \tag{22}$$

The above results can be found in [7–9], see also [5] for a short summary of the results. Instead of describing the system via (1) we describe the system via (21). Since this equation is the same as (7) and since we have used Eq. (7) and not (1) in the proofs of Section 2, we see that all results as derived in Section 2 also hold for well-posed linear systems. It remains to define the characteristic function. This is easy via (22). Fix an element $s_0 \in \mathbb{C}_\omega^+$ then for $s \in \rho(A)$ we define the characteristic function as

$$\mathfrak{G}(s) = \mathfrak{G}(s_0) + C(sI - A)^{-1}(s_0 I - A)^{-1} \times B(s_0 - s). \tag{23}$$

It is easy to see that this definition is independent of the particular choice of s_0. Furthermore, this function is analytic on $\rho(A)$ and, by the resolvent identity, this function satisfies (22) for all $s, s_0 \in \rho(A)$.

Acknowledgements

The author wants to thank Ruth Curtain for the stimulating discussions which highly contributed to the final form of this paper. Furthermore, Lemma 2.5 originates from her.

References

[1] R.F. Curtain, Riccati equations for stable well-posed linear systems: the generic case, SIAM J. Control Optim. 42 (5) (2003) 1681–1702.
[2] R.F. Curtain, H.J. Zwart, An Introduction to Infinite-Dimensional Linear Systems Theory, Springer, New York, 1995.
[3] K. Knopp, Theory of Functions, Part I and II, Two volumes bound as one, Dover Publications, Mineola, New York, 1996.
[4] J.W. Polderman, J.C. Willems, Introduction to Mathematical Systems Theory, A Behavioral Approach, Springer, New York, 1998.
[5] O.J. Staffans, G. Weiss, Transfer functions of regular linear systems. Part II: the system operator and the Lax–Phillips semigroup, Trans. Amer. Math. Soc. 354 (8) (2002) 3229–3262.
[6] G.N. Watson, A Treatise on the Theory of Bessel Functions, 2nd Edition, Cambridge University Press, Cambridge, 1962.
[7] G. Weiss, Admissibility of unbounded control operators, SIAM J. Control Optim. 27 (3) (1989) 527–545.
[8] G. Weiss, Admissible observation operators for linear semigroups, Israel J. Math. 65 (1989) 17–43.
[9] G. Weiss, Transfer functions of regular linear systems. Part I: characterization of regularity, Trans. Amer. Math. Soc. 342 (2) (1994) 827–854.