Adjoint and Hamiltonian Input–Output Differential Equations

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Abstract— Based on recent developments in the theory of variational and Hamiltonian control systems by Crouch and van der Schaft, this paper answers two questions: given an input–output differential equation description of a nonlinear system, what is the adjoint variational system in input-output differential form and what are the conditions for the system to be Hamiltonian, i.e., such that the variational and the adjoint variational systems coincide? This resulting set of conditions is then used to generalize classical conditions such as the well-known Helmholtz conditions for the inverse problem in classical mechanics.

I. INTRODUCTION

THE work we are describing in this paper has its roots in a very old problem in classical mechanics, where one asks which Newtonian systems correspond to Lagrangian, or variational systems; the so-called inverse problem. There are many variants of this problem, see Santilli [14], but the simplest one can be stated as follows.

If $q \in \mathbf{R}^n$ is a configuration variable, which satisfies the Newtonian system

$$F(q, \dot{q}, \ddot{q}) = 0 \tag{1}$$

for a smooth $\mathbf{R}^n$ valued mapping F, such that $\partial F / \partial \ddot{q}$ is a nonsingular matrix in a suitable open domain, then under what conditions does there exist a function L of q, $\dot{q}$, so that for some ordering of the variables of q

$$F(q, \dot{q}, \ddot{q}) = \frac{d}{dt}\frac{\partial L^T}{\partial \dot{q}}(q, \dot{q}) - \frac{\partial L^T}{\partial q}(q, \dot{q}). \tag{2}$$

The conditions under which this property holds are known as the classical Helmholtz conditions, see Santilli [14] where generalizations to functions F depending on arbitrary finite jets of q are also considered.

The condition on the rank of $\partial F / \partial \ddot{q}$ in the system (1) above enables those systems satisfying (2) to be written also as a

Hamiltonian set of equations

$$\dot{q} = \frac{\partial H^T}{\partial p}(q, p)$$
$$\dot{p} = -\frac{\partial H^T}{\partial q}(q, p) \tag{3}$$

where $H(p, q)$ is the Hamiltonian function of the system, and (p, q) are coordinates on the symplectic phase space $\mathbf{R}^{2n}$. A result of Brockett and Rahimi [1] is also of interest in this context and concerns the linear system Σ

$$\dot{x} = Ax + Bu; \quad x(0) = 0$$
$$y = Cx \tag{4}$$

in which $x \in \mathbf{R}^n$, $u, y \in \mathbf{R}^m$, and the so-called adjoint system Σ^a (see also [12])

$$\dot{p} = -A^T p - C^T u_a; \quad p(0) = 0$$
$$y_a = B^T p. \tag{5}$$

It was shown with the minimality of both systems, together with the "self-adjointness" condition that the input–output maps of Σ and Σ^a coincide, that this is equivalent to the fact that the system Σ has another internal representation as a linear "Hamiltonian Control" system

$$\dot{p} = -\frac{\partial H^T}{\partial q}(p, q, u)$$
$$\dot{q} = \frac{\partial H^T}{\partial p}(p, q, u)$$
$$y = \frac{\partial H^T}{\partial u}(p, q, u) \tag{6}$$

where for a linear Hamiltonian system

$$H(p, q, u) = \frac{1}{2}(p^T, q^T)F\begin{bmatrix} p \\ q \end{bmatrix} + (p^T, q^T)Gu$$

for some matrices F and G.

The term "self-adjointness" is also used to describe the conditions which ensure that a Newtonian system does correspond to a Lagrangian, or Hamiltonian, system. This is explained by performing integration by parts to give an expression

$$\int_0^T \xi^T \left[\frac{\partial F}{\partial q} q_v + \frac{\partial F}{\partial \dot{q}} \dot{q}_v + \frac{\partial F}{\partial \ddot{q}} \ddot{q}_v\right] dt$$
$$= \int_0^T q_v^T F^*(\xi, \dot{\xi}, \ddot{\xi}) dt$$
$$+ Q(q, \dot{q}, \ddot{q}, q_v, \dot{q}_v, \ddot{q}_v, \xi, \dot{\xi}, \ddot{\xi})|_0^T \tag{7}$$

for some functions F^* and Q. The adjoint variational system to the variational system $\frac{\partial F}{\partial q}q_v + \frac{\partial F}{\partial \dot{q}}\dot{q}_v + \frac{\partial F}{\partial \ddot{q}}\ddot{q}_v = 0$ corresponding to (1) is then

$$F^*(\xi, \dot{\xi}, \ddot{\xi}) = 0.$$

Self-adjointness of F is simply the statement that

$$F^*(\xi, \dot{\xi}, \ddot{\xi}) = \frac{\partial F}{\partial q}\xi + \frac{\partial F}{\partial \dot{q}}\dot{\xi} + \frac{\partial F}{\partial \ddot{q}}\ddot{\xi}.$$

Work by Crouch and van der Schaft [5], [6], made an extensive investigation and generalization of the result by Brockett and Rahimi [1] to nonlinear control systems, based on earlier work by van der Schaft [15]. In particular, a state space form of the variational and adjoint variational systems was introduced, generalizing the relationship between Σ and Σ^a to nonlinear systems. The concept of self-adjointness was correspondingly generalized, and under suitable hypotheses it was shown that self-adjointness is necessary and sufficient for Hamiltonian realizations of input–output maps. Moreover, the self-adjointness condition was successfully interpreted in terms of the Volterra series and Fliess series. See also Jakubczyk [12] for a generalization to control systems with control entering in a nonaffine manner.

This work, although implicitly generalizing the classical Helmholtz condition, fails to give conditions in terms of the differential equation representation of the input–output map, generalizing the system representation given by (1), and here represented by an equation of the form

$$F(y, y^{(1)}, \cdots, y^{(N)}, u, u^{(1)}, \cdots, u^{(N-1)}) \equiv 0 \qquad (8)$$

where F is a smooth vector valued function of its arguments. Furthermore, the self-adjointness conditions of [5] and [6] are difficult to check in practice, since in principle one needs to compute the state trajectories of the nonlinear system under consideration. The present paper gives the full generalization of the classical Helmholtz conditions to control systems described by (8). The resulting conditions are completely in terms of the mapping F and its partial derivatives. They are worked out in detail for control systems

$$F(y, y^{(1)}, y^{(2)}, u, u^{(1)}) = 0 \qquad (9)$$

so that they have a Hamiltonian representation by a system of the form (6). To the knowledge of the authors, the only previous results in this direction are those given by two of the current authors [4], when dealing with the scalar input-scalar output version of the system (9). It is interesting to note that many problems associated with the preceding analysis coincide with those met in the study of time-varying linear systems; see, e.g., [2], [9], [10], [11], and [13].

II. THE ADJOINT VARIATIONAL SYSTEM

We consider analytic (i.e., C^ω), complete, state-space systems which may be written in the form

$$\Sigma_s \quad \dot{x} = f(x, u), \quad y = h(x), \quad u \in \mathbf{R}^m, \quad y \in \mathbf{R}^p \qquad (10)$$

where $x = (x_1, \cdots, x_n)$ denote local coordinates for some state space manifold M and corresponding analytic input–output differential representations

$$\Sigma_{i/o} \quad F(y, \dot{y}, \cdots, y^{(N)}, u, \dot{u}, \cdots, u^{(N-1)}) = 0 \in \mathbf{R}^p. \qquad (11)$$

If system (10) is minimal in the sense of Crouch and van der Schaft [5, ch. 3 for the input-affine case and ch. 6 for the general case], the corresponding representation (11) will also be called a minimal representation. Note that we do not insist here on the relationship between state-space representations (10) and input–output differential representations (11); we assume that all conditions are met for obtaining one representation from the other representation.

A variational system Σ_s^v about a given trajectory $(x(t), u(t), y(t))$ of Σ_s is defined in the usual way (see [5]) as

$$\dot{v}(t) = F(t)v(t) + G(t)u_v(t), \qquad u_v \in \mathbf{R}^m, \quad v \in \mathbf{R}^n$$
$$y_v(t) = H(t)v(t), \qquad y_v \in \mathbf{R}^p \qquad (12)$$

with $F(t) := \frac{\partial f}{\partial x}(x(t), u(t))$, $G(t) := \frac{\partial f}{\partial u}(x(t), u(t))$, $H(t) = \frac{\partial h}{\partial x}(x(t))$, and v, u_v, y_v denoting, respectively, the variational state, variational input, and variational output. Note that (12) results from differentiation of a one-parameter family of solutions to (10), cf. [5]. The adjoint variational system Σ_s^a, along the same trajectory $(x(t), u(t), y(t))$, is defined as (see [5])

$$\dot{p}(t) = -F^T(t)p(t) - H^T(t)u_a(t), \qquad u_a \in \mathbf{R}^p, \quad p \in \mathbf{R}^n$$
$$y_a(t) = G^T(t)p(t), \qquad y_a \in \mathbf{R}^m \qquad (13)$$

with p, u_a, y_a denoting the adjoint variational state, input, and output, respectively. The fundamental connection between variational and adjoint variational systems (along the same trajectory of Σ_s) is [5, Lemma 2.1]

$$y_a^T(t)u_v(t) - u_a^T(t)y_v(t) = \frac{d}{dt}p^T(t)v(t). \qquad (14)$$

Now let us translate this to input–output differential representations $\Sigma_{i/o}$ given by (11). Clearly, the variational systems $\Sigma_{i/o}^v$ (along solutions $u(t)$, $y(t)$ of $\Sigma_{i/o}$) are defined by the system of equations

$$\frac{\partial F}{\partial y}y_v(t) + \frac{\partial F}{\partial \dot{y}}\dot{y}_v(t) + \cdots + \frac{\partial F}{\partial y^{(N)}}y_v^N(t) + \frac{\partial F}{\partial u}u_v(t)$$
$$+ \frac{\partial F}{\partial \dot{u}}\dot{u}_v(t) + \cdots + \frac{\partial F}{\partial u^{(N-1)}}u_v^{(N-1)}(t) = 0 \qquad (15)$$

where the solution $u(t), y(t)$ is substituted in $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial \dot{y}}$, $\cdots$, $\frac{\partial F}{\partial u}, \frac{\partial F}{\partial \dot{u}} \cdots$. Again, (15) results from differentiation of a one-parameter family of solutions $(u(t, \epsilon), y(t, \epsilon))$ to (11). Comparing $\Sigma_{i/o}^v$ to Σ_s^v there is a potential problem since (see [20]) the set of solutions u_v, y_v to (15) may be strictly larger than the set of solutions u_v, y_v generated by (12). This has to do with the form of the input–output differential representation (11) (note that this representation is far from unique); we will later on make an assumption on (11) which will eliminate this potential problem.

The next logical question is how to define the adjoint variational system $\Sigma_{i/o}^a$. This is not immediate from the

definition of Σ_s^a. Relation (14) should provide the clue to a proper definition, although the right-hand side of (14) is expressed in the variational and adjoint variational state. Direct calculation based on Σ_s^v and Σ_s^a provides the following alternative to (14)

$$y_a^T(t)u_v(t) - u_a^T(t)y_v(t)$$
$$= \frac{d}{dt}\left[-\int_{-\infty}^t \int_{-\infty}^t u_a^T(\tau)H(\tau)\Phi(\tau, s)G(s)u_v(s)\, ds\, dt\right]$$
$$(16)$$

with $\Phi(\tau, s)$ being the transition matrix of $F(t)$ (and the variational and adjoint variational states initialized at 0 at time $-\infty$). Although the right-hand side of (16) is an integral expression in u_a, u_v, this partly motivates the following definition (another motivation is provided by the classical definition of adjoint variational systems for sets of differential equations, cf. [14]).

Definition 1 (see also [5]): Consider the variational system (15) along a solution $(u(t), y(t))$ of $\Sigma_{i/o}$. The adjoint variational system $\Sigma_{i/o}^a$ consists of the following set of input–output pairs $(u_a(\cdot), y_a(\cdot))$: there exists a function $Q(y, \dot{y}, \cdots, u, \dot{u}, \cdots, y_v, \dot{y}_v, \cdots, u_v, \dot{u}_v, \cdots, y_a, \dot{y}_a, \cdots, u_a, \dot{u}_a, \cdots)$ such that for all $t \in R$

$$y_a^T(t)u_v(t) - u_a^T(t)y_v(t) = \frac{d}{dt}Q \qquad (17)$$

for all input–output pairs $(u_v(\cdot), y_v(\cdot))$ which are solutions of $\Sigma_{i/o}^v$.

To justify this definition we first have to prove that Definition 1 uniquely characterizes the adjoint variational system.

Proposition 1: Suppose that there exists a function

$$\tilde{Q}(y, \dot{y}, \cdots, u, \dot{u}, \cdots, y_v, \dot{y}_v, \cdots, u_v, \dot{u}_v, \cdots,$$
$$\tilde{y}_a, \dot{\tilde{y}}_a, \cdots, u_a, \dot{u}_a \cdots)$$

such that for all solutions $(u_v(t), y_v(t))$ to $\Sigma_{i/o}^v$ and all $u_a(t)$

$$\tilde{y}_a^T(t)u_v(t) - u_a^T(t)y_v(t) = \frac{d}{dt}\tilde{Q} \qquad (18)$$

for all $t \in R$, then $\tilde{y}_a(t) = y_a(t)$, $t \in R$, and $\tilde{Q} = Q$ modulo a constant.

Proof: Subtracting (17) from (18) yields

$$(\tilde{y}_a - y_a)^T u_v = \frac{d}{dt}(\tilde{Q} - Q)$$

for all u_v. Hence

$$\int_{t_1}^{t_2} [\tilde{y}_a(t) - y_a(t)]^T u_v(t)\, dt = |\tilde{Q} - Q|_{t_1}^{t_2}. \qquad (19)$$

Now take a fixed function u_a on $[t_1, t_2]$ and corresponding fixed functions y_a, $\tilde{y}_a$ on $[t_1, t_2]$ and arbitrary u_v. It follows that the left-hand side of (19) only depends on $u_v(t_1)$ and $u_v(t_2)$ (since the right-hand side does), thus implying that both sides of (19) are zero, and $\tilde{y}_a = y_a$ and $\tilde{Q} = Q$ modulo a constant. ∎

The next thing we have to do is to show that Definition 1 is consistent with the definition of the adjoint variational system Σ_s^a. Comparing (17) to (14) we see that this

means that $p^T v$ has to be expressible as a function Q of $y_v, \dot{y}_v, \cdots, u_v, \dot{u}_v, \cdots, y_a, \dot{y}_a, \cdots, u_a, \dot{u}_a, \cdots$ (and of course $y, \dot{y}, \cdots, u, \dot{u}, \cdots$). To do so we make fundamental use of some results obtained by Ilchmann et al. [10] on time-varying linear systems and Coron [3] and Sontag [17] on the relation between nonlinear state space systems Σ_s and the variational systems Σ_s^v; see also [3], [18], and [7]. Indeed, in [3], [17] the following is shown. Consider the minimal state space system Σ_s. Let I be an open interval of R, and denote by $C^\infty(I; R^m)$ the set of smooth input functions $u: I \to R^m$, equipped with the Whitney topology. Then the set of all u in $C^\infty(I; R^m)$ such that all corresponding solutions $(x(t), u(t))$ of Σ_s defined on I have the property that the variational systems (12) along $(x(t), u(t))$ satisfy

$$\dim \text{span}\left\{\left(\frac{d}{dt} - F(t)\right)^i G(t)w; \quad w \in R^m, i \geq 0\right\} = n$$
$$(20)$$

for all $t \in I$, is a dense subset of $C^\infty(I; R^m)$; see [3, Corollary 1.8]. (Note that the definition of the strong accessibility algebra used in [3] is the usual definition in the case of input-affine systems Σ_s, while for general systems $\dot{x} = f(x, u)$ it corresponds exactly to the definition given in [5, ch. 6].)

Furthermore, see [3, Corollary 1.15], the set of all u in $C^\infty(I; R^m)$ such that all corresponding solutions $(x(t), u(t))$ of Σ_s defined on I have the property that the variational systems (12) along $(x(t), u(t))$ satisfy

$$\dim \text{span}\left\{\left(\frac{d}{dt} + F^T(t)\right)^i H^T(t)w; \quad w \in R^p, i \geq 0\right\} = n$$
$$(21)$$

for all $t \in I$, is also a dense subset of $C^\infty(I; R^m)$. Properties (20) and (21) express well-known controllability, respectively, observability, properties of the time-varying linear systems given by the variational systems Σ_s^v, and thus the above statements imply, loosely speaking, that in case Σ_s is minimal then its variational systems are controllable and observable for a dense subset of input functions.

To use now the fundamental results obtained in Ilchmann et al. [10] and Ilchmann [11] we will now restrict to analytic (C^ω) input functions on the time-interval I. Since (10) is assumed to be analytic this will mean that the variational and adjoint variational systems (12), respectively (13), are analytic, i.e., the entries of $F(t)$, $G(t)$, and $H(t)$ are analytic functions on I. Let $\mathcal{M}$ denote the meromorphic functions on I, and denote by $\mathcal{M}[D]$ the set of polynomials

$$\sum_{i=0}^k f_i D^i$$

in D with coefficients from $\mathcal{M}$ (D will represent the differentiation operator $\frac{d}{dt} \in \text{End}(\mathcal{M})$, the algebra of R-linear maps from $\mathcal{M}$ to $\mathcal{M}$). Considering also the multiplication in $\text{End}(\mathcal{M})$, we arrive at the skew-polynomial ring $\mathcal{M}[D]$ (see [10], [11]) with multiplication rule

$$D(fg) = fD(g) + D(f)g = (fD + D(f))g,$$
$$f, g \in \text{End}(\mathcal{M}). \qquad (22)$$

The set of $m \times n$ matrices over $\mathcal{M}[D]$ will be denoted by $\mathcal{M}[D]^{m \times n}$. A useful property (see [10]) is that a left inverse of a square matrix in $\mathcal{M}[D]^{n \times n}$ is also a right inverse, and vice versa.

Let us now denote the variational and adjoint variational system in shorthand notation by

$$\left(I\frac{d}{dt} - F\right)v - Gu_v = 0$$

$$y_v - Hv = 0 \qquad (23)$$

$$\left(I\frac{d}{dt} + F^T\right)p + H^T u_a = 0$$

$$y_a - G^T p = 0. \qquad (24)$$

Throughout we will restrict to the dense subset of analytic input functions for which (23) satisfies the controllability and observability properties (20) and (21). Then by [10, Lemma 3.3 and Theorem 6.4] there exists an invertible matrix

$$\begin{bmatrix} \tilde{U}\left(\frac{d}{dt}\right) & \tilde{W}\left(\frac{d}{dt}\right) \\ \tilde{V}\left(\frac{d}{dt}\right) & \tilde{Z}\left(\frac{d}{dt}\right) \end{bmatrix} \in \mathcal{M}[D]^{(n+m)\times(n+m)}$$

such that

$$\left[I\frac{d}{dt} - F, -G\right]\begin{bmatrix} \tilde{U}\left(\frac{d}{dt}\right) & \tilde{W}\left(\frac{d}{dt}\right) \\ \tilde{V}\left(\frac{d}{dt}\right) & \tilde{Z}\left(\frac{d}{dt}\right) \end{bmatrix} = [I_n, 0] \qquad (25)$$

and an invertible matrix

$$\begin{bmatrix} \tilde{P}\left(\frac{d}{dt}\right) & \tilde{Q}\left(\frac{d}{dt}\right) \\ \tilde{S}\left(\frac{d}{dt}\right) & \tilde{R}\left(\frac{d}{dt}\right) \end{bmatrix} \in \mathcal{M}[D]^{(n+p)\times(n+p)}$$

such that

$$\left[I\frac{d}{dt} + F^T, H^T\right]\begin{bmatrix} \tilde{P}\left(\frac{d}{dt}\right) & \tilde{Q}\left(\frac{d}{dt}\right) \\ \tilde{S}\left(\frac{d}{dt}\right) & \tilde{R}\left(\frac{d}{dt}\right) \end{bmatrix} = [I_n, 0]. \qquad (26)$$

We will now apply the following "integration by parts" procedure to (25), (26). This will be a key tool in all of our subsequent developments.

Consider two matrices $M\left(\frac{d}{dt}\right)$, $N\left(\frac{d}{dt}\right)$ over $\mathcal{M}[D]$, of appropriate dimensions such that

$$M\left(\frac{d}{dt}\right)N\left(\frac{d}{dt}\right) = C \qquad (27)$$

with C some constant matrix, say of dimension $k \times l$. (Note that $M\left(\frac{d}{dt}\right)$ acts also on the entries of $N\left(\frac{d}{dt}\right)$, cf. (22); the same of course applies to the expressions (25) and (26).)

Let now $\xi(t)$ be an l-vector of analytic functions, and consider the k-vector of analytic functions

$$M\left(\frac{d}{dt}\right)N\left(\frac{d}{dt}\right)\xi(t). \qquad (28)$$

Now premultiply (28) by a k-row vector of analytic functions $\eta^T(t)$, i.e.,

$$\eta^T(t)M\left(\frac{d}{dt}\right)N\left(\frac{d}{dt}\right)\xi(t) \qquad (29)$$

and apply integration by parts to the integral

$$\int_{t_1}^{t_2} \eta^T(t)M\left(\frac{d}{dt}\right)N\left(\frac{d}{dt}\right)\xi(t)\,dt$$

(with $[t_1, t_2] \subset I$) in order to shift the differentiations on $\xi(t)$ to differentiations on $\eta(t)$. It follows that

$$\int_{t_1}^{t_2} \eta^T(t)M\left(\frac{d}{dt}\right)N\left(\frac{d}{dt}\right)\xi(t)\,dt$$

$$= \int_{t_1}^{t_2} \xi^T(t)\tilde{N}\left(\frac{d}{dt}\right)\tilde{M}\left(\frac{d}{dt}\right)\eta(t)\,dt + \text{remainders} \quad (30)$$

for certain matrices $\tilde{N}\left(\frac{d}{dt}\right)$, $\tilde{M}\left(\frac{d}{dt}\right)$ over $\mathcal{M}[D]$ (in fact, $\tilde{M}$ and $\tilde{N}$ are dimensioned as M^T, respectively N^T.) Futhermore, from (27) and (30) it follows that the differential operator $\tilde{N}\left(\frac{d}{dt}\right)\tilde{M}\left(\frac{d}{dt}\right)$ equals the constant linear mapping C^T on all functions $\eta(t)$ of support within (t_1, t_2). However, this means that $\tilde{N}\left(\frac{d}{dt}\right)\tilde{M}\left(\frac{d}{dt}\right)$ equals C^T, and thus the remainders in (30) are necessarily zero.

Applying this procedure to (25) and (26) yields

$$\begin{bmatrix} U\left(\frac{d}{dt}\right) & V\left(\frac{d}{dt}\right) \\ W\left(\frac{d}{dt}\right) & Z\left(\frac{d}{dt}\right) \end{bmatrix}\begin{bmatrix} -I\frac{d}{dt} - F^T \\ -G^T \end{bmatrix} = \begin{bmatrix} I_n \\ 0 \end{bmatrix} \qquad (31)$$

$$\begin{bmatrix} P\left(\frac{d}{dt}\right) & S\left(\frac{d}{dt}\right) \\ Q\left(\frac{d}{dt}\right) & R\left(\frac{d}{dt}\right) \end{bmatrix}\begin{bmatrix} -I\frac{d}{dt} + F \\ H \end{bmatrix} = \begin{bmatrix} I_n \\ 0 \end{bmatrix} \qquad (32)$$

respectively, for invertible matrices $\begin{bmatrix} U & V \\ W & Z \end{bmatrix}$, $\begin{bmatrix} P & S \\ Q & R \end{bmatrix}$ obtained from $\begin{bmatrix} \tilde{U} & \tilde{W} \\ \tilde{V} & \tilde{Z} \end{bmatrix}$, $\begin{bmatrix} \tilde{P} & \tilde{Q} \\ \tilde{S} & \tilde{R} \end{bmatrix}$ by partial integration.

The action of the differential operator $\begin{bmatrix} P & S \\ Q & R \end{bmatrix}$ on (23) thus results in the equivalent system of equations

$$-PGu_v + Sy_v = v$$

$$-QGu_v + Ry_v = 0 \qquad (33)$$

and, similarly, a combination of (31) and (24) yields

$$UH^T u_a - Vy_a = p$$

$$WH^T u_a - Zy_a = 0. \qquad (34)$$

It thus follows that $v(t) = -P\left(\frac{d}{dt}\right)G(t)u_v(t) + S\left(\frac{d}{dt}\right)y_v(t)$ and $p(t) = U\left(\frac{d}{dt}\right)H^T(t)u_a(t) - V\left(\frac{d}{dt}\right)y_a(t)$, implying that $\frac{d}{dt}p^T(t)v(t)$ appearing in the right-hand side of (14) can be expressed as a function Q in y_v, u_v, u_a, y_a and their time-derivatives (and implicitly of $y, \dot{y}, \cdots, u, \dot{u}, \cdots$). This shows that Definition 1 is consistent with the definition of the adjoint variational system Σ_s^a. Furthermore, as additional information we obtain from (33), (34) that input–output differential representations of Σ_s^v and Σ_s^a are given by

$$-Q\left(\frac{d}{dt}\right)G(t)u_v(t) + R\left(\frac{d}{dt}\right)y_v(t) = 0 \qquad (35)$$

respectively

$$W\left(\frac{d}{dt}\right)H^T(t)u_a(t) - Z\left(\frac{d}{dt}\right)y_a(t) = 0. \qquad (36)$$

Now compare (35) to (15), and write (15) in a more convenient notation as

$$\Sigma_{i/o}^v: \qquad D\left(\frac{d}{dt}\right)y_v(t) - N\left(\frac{d}{dt}\right)u_v(t) = 0. \qquad (37)$$

The requirement, as alluded to before, that the set of solutions u_v, y_v generated by Σ_s^v equals the solution set of $\Sigma_{i/o}^v$ can thus be rephrased as the following.

Assumption 1: There exists an invertible matrix $E\left(\frac{d}{dt}\right) \in \mathcal{M}[D]^{p \times p}$ such that

$$E\left(\frac{d}{dt}\right)N\left(\frac{d}{dt}\right) = Q\left(\frac{d}{dt}\right)G(t),$$

$$E\left(\frac{d}{dt}\right)D\left(\frac{d}{dt}\right) = R\left(\frac{d}{dt}\right). \qquad (38)$$

From now on we will suppose throughout that Assumption 1 holds.

Our next objective is to give a procedure to compute an input–output differential representation of the adjoint variational system directly in terms of the input–output differential representation $\Sigma_{i/o}^v$ of the variational system (without going through the state space representation). Thus, let us consider $\Sigma_{i/o}^v$ as given by (15), denoted in shorthand notation by (37). Premultiply (37) by a p-dimensional row-vector $\xi^T(t)$ and again apply partial integration to

$$0 = \int_{t_1}^{t_2} \xi^T(t) \left[D\left(\frac{d}{dt}\right)y_v(t) - N\left(\frac{d}{dt}\right)u_v(t) \right] dt \qquad (39)$$

to shift all differentiations in $y_v(t)$, $u_v(t)$ to differentiation in $\xi(t)$. This results in

$$0 = \int_{t_1}^{t_2} y_v^T(t) \left[\tilde{D}\left(\frac{d}{dt}\right)\xi(t) \right] - u_v^T(t) \left[\tilde{N}\left(\frac{d}{dt}\right)\xi(t) \right] dt$$
$$+ R(y, \dot{y}, \cdots, u, \dot{u}, \cdots, y_v, \dot{y}_v, \cdots, u_v, \dot{u}_v, \cdots, \xi, \dot{\xi}, \cdots)|_{t_1}^{t_2} \qquad (40)$$

for certain matrices $\tilde{D}$, $\tilde{N}$ in $\mathcal{M}[D]$, and remainders R.

Comparing this to (17) (with Q playing the role of the remainders R) motivates the following definition of the adjoint variational system by $(\Sigma_{i/o}^a)$

$$u_a(t) = \tilde{D}\left(\frac{d}{dt}\right)\xi(t)$$

$$y_a(t) = \tilde{N}\left(\frac{d}{dt}\right)\xi(t). \qquad (41)$$

Note that (41) is an image representation, in contrast with the kernel representation (37). (For a linear time-invariant system (41) corresponds to a right factorization, while (37) corresponds to a left factorization of the transfer matrix.) Indeed, the (analytic) input–output behavior of (41) consists of all analytic time-functions $y_a(t)$, $u_a(t)$ satisfying (41) for some analytic function $\xi(t)$.

Theorem 1: The equations (41) are an image representation of the adjoint variational system defined in Definition 1.

Proof: We only have to show that R in (40) can also be expressed as a function of $y, \dot{y}, \cdots, u, \dot{u}, \cdots, y_v, \dot{y}_v, \cdots, u_v,$ $\dot{u}_v, \cdots, y_a, \dot{y}_a, \cdots, u_a, \dot{u}_a, \cdots$.

From (25) we obtain

$$\begin{bmatrix} I\frac{d}{dt} - F & -G & 0 \\ -H & 0 & I_p \end{bmatrix} \begin{bmatrix} \tilde{U} & 0 \\ \tilde{V} & 0 \\ 0 & I_p \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ K & I_p \end{bmatrix} \qquad (42)$$

for some K in $\mathcal{M}[D]$. Hence, from (32) and (42) we obtain

$$\begin{bmatrix} P & S \\ Q & R \end{bmatrix} \begin{bmatrix} I\frac{d}{dt} - F & -G & 0 \\ -H & 0 & I_p \end{bmatrix} \begin{bmatrix} \tilde{U} & 0 \\ \tilde{V} & 0 \\ 0 & I_p \end{bmatrix}$$

$$= \begin{bmatrix} -I_n & -PG & S \\ 0 & -QG & R \end{bmatrix} \begin{bmatrix} \tilde{U} & 0 \\ \tilde{V} & 0 \\ 0 & I_p \end{bmatrix}$$

$$= \begin{bmatrix} P & S \\ Q & R \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -K & I_p \end{bmatrix}. \qquad (43)$$

Clearly the right-hand side of (43) is an invertible matrix, and thus "postmultiplication" of (43) by this inverse yields

$$-Q\left(\frac{d}{dt}\right)G(t)\overline{B}\left(\frac{d}{dt}\right) + R\left(\frac{d}{dt}\right)\overline{A}\left(\frac{d}{dt}\right) = I_p \qquad (44)$$

for some matrices $\overline{A}\left(\frac{d}{dt}\right)$, $\overline{B}\left(\frac{d}{dt}\right)$ in $\mathcal{M}[D]$. Now recall that an input–output differential representation of Σ_s^v is given by (35), while also (38) holds. Thus there also exist matrices $A\left(\frac{d}{dt}\right)$, $B\left(\frac{d}{dt}\right)$ in $\mathcal{M}[D]$ such that

$$D\left(\frac{d}{dt}\right)A\left(\frac{d}{dt}\right) - N\left(\frac{d}{dt}\right)B\left(\frac{d}{dt}\right) = I_p. \qquad (45)$$

Applying the partial integration procedure [see (27)–(30)] to (45) yields

$$\tilde{A}\left(\frac{d}{dt}\right)\tilde{D}\left(\frac{d}{dt}\right) - \tilde{B}\left(\frac{d}{dt}\right)\tilde{N}\left(\frac{d}{dt}\right) = I_p \qquad (46)$$

for certain matrices $\tilde{A}$, $\tilde{B}$ in $\mathcal{M}[D]$, and with $\tilde{D}$, $\tilde{N}$ as given by (41). Thus, by (41)

$$\xi(t) = \tilde{A}\left(\frac{d}{dt}\right)\tilde{D}\left(\frac{d}{dt}\right)\xi(t) - \tilde{B}\left(\frac{d}{dt}\right)\tilde{N}\left(\frac{d}{dt}\right)\xi(t)$$

$$= \tilde{A}\left(\frac{d}{dt}\right)u_a(t) - \tilde{B}\left(\frac{d}{dt}\right)y_a(t)$$

showing that ξ can be expressed into y_a, u_a and their time-derivatives. ∎

Remark 1: The above notions seem to be also useful for analyzing the controllability properties of an input–output differential system (11). Consider the variational and adjoint variational systems of (11) given by (37), respectively (41), where the entries of the matrix differential operators D, N, and $\tilde{D}$, $\tilde{N}$ are seen as functions of $y, \dot{y}, \cdots, u, \dot{u}, \cdots$. Then one may construct matrix differential operators D_a, N_a (with entries depending on $y, \dot{y}, \cdots, u, \dot{u}, \cdots$) of maximal rank such that $D_a\tilde{N} - N_a\tilde{D} = 0$, implying that

$$D_a\left(\frac{d}{dt}\right)y_a(t) - N_a\left(\frac{d}{dt}\right)u_a(t) = 0 \qquad (47)$$

for all trajectories u_a, y_a generated by (41). Integration by parts applied to the kernel representation (47) yields the "adjoint of the adjoint system," given in image representation

$$u_v(t) = \tilde{D}_a\left(\frac{d}{dt}\right)\eta(t)$$

$$y_v(t) = \tilde{N}_a\left(\frac{d}{dt}\right)\eta(t) \qquad (48)$$

where we have suggestively denoted the inputs and outputs by u_v and y_v, since we want to compare (48) to the variational system (37). Indeed, it is easily checked that all trajectories u_v, y_v generated by (48) satisfy (37), while equality of the behavior of (48) and (37) seems to correspond to some form of controllability of the original nonlinear system (11) [(48) defines the "controllable" part of the system]. This is an area for future research.

In the next section we will use the representation of the adjoint variational system $\Sigma_{i/o}^a$ as given by (41) in order to give a convenient characterization of Hamiltonian systems.

III. CHARACTERIZATION OF HAMILTONIAN SYSTEMS FROM THE INPUT–OUTPUT DIFFERENTIAL REPRESENTATION

We will derive in this section a complete characterization of Hamiltonian systems using the representation of the adjoint variational systems as in (41). In Section IV we will use an alternative approach based on the adjoint variational system Σ_s^a found in [5]. Note that for systems described as

$$\Sigma_{i/o} \qquad F(y, \dot{y}, \ddot{y}, u, \dot{u}) = 0 \in R, \qquad u \in R, y \in R \qquad (49)$$

the conditions under which (49) represents a Hamiltonian system have been found already in an earlier paper by Crouch and Lamnabhi [4], i.e.,

$$
\begin{array}{ll}
\text{i)} & \frac{\partial F}{\partial \dot{u}} = 0 \\
\text{ii)} & \frac{\partial F}{\partial u}\left(\frac{d}{dt}\frac{\partial F}{\partial \dot{y}} - \frac{\partial F}{\partial y}\right) = \frac{\partial F}{\partial y}\frac{d}{dt}\frac{\partial F}{\partial u}
\end{array} \qquad (50)
$$

for every solution $(y, \dot{y}, \ddot{y}, u, \dot{u})$ satisfying (49).

Using the characterization of the adjoint variational system given in Section II we now obtain similar conditions for a general input–output differential representation (11).

Theorem 2: Consider a minimal input–output differential representation $\Sigma_{i/o}$ given by (11) with $p = m$, and its variational systems $\Sigma_{i/o}^v$ given by (37) satisfying Assumption 1. Compute the adjoint variational system $\Sigma_{i/o}^a$ given by (41). Then $\Sigma_{i/o}$ is an input–output representation of a Hamiltonian system if and only if

$$D\left(\frac{d}{dt}\right)\tilde{N}\left(\frac{d}{dt}\right) - N\left(\frac{d}{dt}\right)\tilde{D}\left(\frac{d}{dt}\right) = 0 \qquad (51)$$

along every analytic solution $(y(t), u(t))$ of $\Sigma_{i/o}$.

Proof: Observe that (51) is equivalent to $\Sigma_{i/o}^v = \Sigma_{i/o}^a$, i.e., the input–output behavior defined by (37) is the same as the input–output behavior defined by (41). In the terminology of [5], [6] this means that every variational system Σ_s^v along an analytic solution of $\Sigma_{i/o}$ is self-adjoint. In [5, ch. 4] it is shown that Σ_s is Hamiltonian if and only if every variational system along trajectories resulting from piecewise constant inputs are self-adjoint. We finally note that by the Approximation Lemma [19, Lemma 1] the approximation of piecewise constant input functions by analytic input functions will result in state trajectories converging to the state trajectories corresponding to the piecewise constant input functions. ∎

We will now work out in detail the self-adjointness condition (51) in case $N = 2$, i.e., we consider

$$\Sigma_{i/o}: \qquad F(y, \dot{y}, \ddot{y}, u, \dot{u}) = 0 \in R^m, \qquad u, y \in R^m \qquad (52)$$

and

$$\Sigma_{i/o}^v: \qquad Ay_v + B\dot{y}_v + C\ddot{y}_v + Du_v + E\dot{u}_v = 0,$$
$$u_v, y_v \in R^m \qquad (53)$$

where the (i, k)th elements of the $m \times m$ matrices A, B, C, D, E are given by

$$A_{ik} = \frac{\partial F_i}{\partial y^k}, \quad B_{ik} = \frac{\partial F_i}{\partial \dot{y}^k}, \quad C_{ik} = \frac{\partial F_i}{\partial \ddot{y}_k},$$
$$D_{ik} = \frac{\partial F_i}{\partial u^k}, \quad E_{ik} = \frac{\partial F_i}{\partial \dot{u}^k}. \qquad (54)$$

To compute the adjoint variational system $\Sigma_{i/o}^a$ two consecutive partial integrations yield (we are using summation convention)

$$
\begin{aligned}
0 &= \int_{t_1}^{t_2} \xi^i [A_{ik}y_v^k + B_{ik}\dot{y}_v^k + C_{ik}\ddot{y}_v^k + D_{ik}u_v^k + E_{ik}\dot{u}_v^k]\, dt \\
&= \int_{t_1}^{t_2} [\xi^i A_{ik}y_v^k - (\xi^i B_{ik})^{(1)}y_v^k + (\xi^i C_{ik})^{(2)}y_v^k \\
&\quad + \xi^i D_{ik}u_v^k - (\xi^i E_{ik})^{(1)}u_v^k]\, dt \\
&\quad + |\xi^i B_{ik}y_v^k + \xi^i C_{ik}\dot{y}_v^k + \xi^i E_{ik}u_v^k - (\xi^i C_{ik})^{(1)}y_v^k|_{t_1}^{t_2}.
\end{aligned}
$$
$$(55)$$

Thus the adjoint variational system is

$$\Sigma_{i/o}^a: \quad \begin{cases} u_a^k = \xi^i A_{ik} - (\xi^i B_{ik})^{(1)} + (\xi^i C_{ik})^{(2)} \\ y_a^k = -\xi^i D_{ik} + (\xi^i E_{ik})^{(1)} \end{cases}$$
$$k = 1, \cdots, m \qquad (56)$$

while Q is given by the terms between the brackets $|\cdot|$. The self-adjointness condition (51) now becomes

$$
\begin{aligned}
\sum_{k=1}^{m} &- A_{jk}D_{ik}\xi^i + A_{jk}(E_{ik}\xi^i)^{(1)} - B_{jk}(D_{ik}\xi^i)^{(1)} \\
&+ B_{jk}(E_{ik}\xi^i)^{(2)} - C_{jk}(D_{ik}\xi^i)^{(2)} + C_{jk}(E_{ik}\xi^i)^{(3)} \\
&+ D_{jk}A_{ik}\xi^i - D_{jk}(B_{ik}\xi^i)^{(1)} + D_{jk}(C_{ik}\xi^i)^{(2)} \\
&+ E_{jk}(A_{ik}\xi^i)^{(1)} - E_{jk}(B_{ik}\xi^i)^{(2)} + E_{jk}(C_{ik}\xi^i)^{(3)} = 0
\end{aligned}
$$
$$(57)$$

along every solution $(u(t), y(t))$ of $\Sigma_{i/o}$. This gives the following conditions.

Terms with $(\xi^i)^{(3)}$:

$$\sum_{k=1}^{m} [C_{jk}E_{ik} + E_{jk}C_{ik}] = 0.$$

Terms with $(\xi^i)^{(2)}$:

$$\sum_{k=1}^{m} [B_{jk}E_{ik} - C_{jk}D_{ik} + 3C_{jk}\dot{E}_{ik} + D_{jk}C_{ik}$$
$$- E_{jk}B_{ik} + 3E_{jk}\dot{C}_{ik}] = 0.$$

Terms with $(\xi^i)^{(1)}$:

$$\sum_{k=1}^{m} [A_{jk}E_{ik} - B_{jk}D_{ik} + 2B_{jk}\dot{E}_{ik} - 2C_{jk}\dot{D}_{ik}$$
$$+ 3C_{jk}\ddot{E}_{ik} - D_{jk}B_{ik} + 2D_{jk}\dot{C}_{ik} + E_{jk}A_{ik}$$
$$- 2E_{jk}\dot{B}_{ik} + 3E_{jk}\ddot{C}_{ik}] = 0.$$

Terms with ξ^i:

$$\sum_{k=1}^{m} [-A_{jk}D_{ik} + A_{jk}\dot{E}_{ik} -_{jk} \dot{D}_{ik} + B_{jk}\ddot{E}_{ik} - C_{jk}\ddot{D}_{ik}$$
$$+ C_{jk}E_{ik}^{(3)} + D_{jk}A_{ik} - D_{jk}\dot{B}_{ik} + D_{jk}\ddot{C}_{ik}$$
$$+ E_{jk}\dot{A}_{ik} - E_{jk}\ddot{B}_{ik} + E_{jk}C_{ik}^{(3)}] = 0.$$

$i, j = 1, \cdots, m.$

Hence, we have the following theorem.

Theorem 3: Consider a minimal system (52). Then it is Hamiltonian if and only if the following conditions

$\mathcal{L}_1)$ $CE^T + EC^T = 0,$

$\mathcal{L}_2)$ $BE^T - CD^T + 3C\dot{E}^T + DC^T - EB^T + 3E\dot{C}^T = 0,$

$\mathcal{L}_3)$ $AE^T - BD^T + 2B\dot{E}^T - 2C\dot{D}^T + 3C\ddot{E}^T$
$\qquad - DB^T + 2D\dot{S}^T + EA^T - 2E\dot{B}^T + 3E\ddot{C}^T = 0,$

$\mathcal{L}_4)$ $-AD^T + A\dot{E}^T - B\dot{D}^T + B\ddot{E}^T - C\ddot{D}^T$
$\qquad + CE^{(3)T} + DA^T - D\dot{B}^T + D\ddot{C}^T$
$\qquad + E\dot{A}^T - E\ddot{B}^T + EC^{(3)T} = 0$

$$(58)$$

hold along every solution $(u(t), y(t))$ of (52).

*Elaboration of the Conditions $\mathcal{L}_1$, $\mathcal{L}_2$,
$\mathcal{L}_3$, and $\mathcal{L}_4$ in Special Cases*

Case 1: Let us first consider the case $m = 1$, i.e., (47). If we assume $C = \frac{\partial F}{\partial \ddot{y}} \neq 0$, then $\mathcal{L}_1$ yields

$$E = \frac{\partial F}{\partial \dot{u}} = 0$$

which is the condition (50)-i). Then $\mathcal{L}_2$ reduces to

$$-\frac{\partial F}{\partial \ddot{y}}\frac{\partial F}{\partial u} + \frac{\partial F}{\partial u}\frac{\partial F}{\partial \ddot{y}} = 0$$

which is automatically satisfied. Furthermore $\mathcal{L}_3$ amounts to

$$-\frac{\partial F}{\partial \ddot{y}}\frac{\partial F}{\partial u} - \frac{\partial F}{\partial u}\frac{\partial F}{\partial \ddot{y}} - 2\frac{\partial F}{\partial \ddot{y}}\left(\frac{\partial F}{\partial u}\right)^{(1)} + 2\frac{\partial F}{\partial u}\left(\frac{\partial F}{\partial \ddot{y}}\right)^{(1)} = 0$$

or

$$-\frac{\partial F}{\partial \ddot{y}}\frac{\partial F}{\partial u} - \frac{\partial F}{\partial \ddot{y}}\left(\frac{\partial F}{\partial u}\right)^{(1)} + \frac{\partial F}{\partial u}\left(\frac{\partial F}{\partial \ddot{y}}\right)^{(1)} = 0$$

which is the condition (50)-ii), and it is easily checked that $\mathcal{L}_4$ is precisely the time-derivative of (50)-ii).

Let us now derive a more explicit expression for the remainder Q in this particular case. Since $E = 0$, from (55)

$$Q = \xi B y_v + \xi C \dot{y}_v - (\xi C)^{(1)} y_v$$

and from (56)

$$\xi = -D^{-1} y_a.$$

If the conditions (50) hold we readily obtain

$$Q = [y_a, \dot{y}_a]^T \begin{bmatrix} 0 & C/D \\ -C/D & 0 \end{bmatrix} \begin{bmatrix} y_v \\ \dot{y}_v \end{bmatrix}. \qquad (59)$$

Now let us take a closer look at (50). Writing out

$$\left(\frac{\partial F}{\partial \ddot{y}}\right)^{(1)} = \frac{\partial^2 F}{\partial \ddot{y}^2}y^{(3)} + \frac{\partial^2 F}{\partial \dot{y}\partial \ddot{y}}\ddot{y} + \frac{\partial^2 F}{\partial y\partial \ddot{y}}\dot{y} + \frac{\partial^2 F}{\partial u\partial \ddot{y}}\dot{u}$$

and

$$\left(\frac{\partial F}{\partial u}\right)^{(1)} = \frac{\partial^2 F}{\partial u\partial \ddot{y}}y^{(3)} + \frac{\partial^2 F}{\partial \ddot{y}\partial u}\ddot{y} + \frac{\partial^2 F}{\partial y\partial u}\dot{y} + \frac{\partial^2 F}{\partial u^2}\dot{u}$$

and collecting terms first with $y^{(3)}$ and then $\dot{u}$ we obtain

$$\begin{cases} \frac{\partial F}{\partial u}\frac{\partial^2 F}{\partial \ddot{y}^2} - \frac{\partial^2 F}{\partial u\partial \ddot{y}}\frac{\partial F}{\partial \ddot{y}} = 0 \\ \frac{\partial F}{\partial u}\frac{\partial^2 F}{\partial u\partial \ddot{y}} - \frac{\partial^2 F}{\partial u^2}\frac{\partial F}{\partial \ddot{y}} = 0 \end{cases} \qquad (60)$$

which means that the elements of the matrix given in (59) only depend on y, $\dot{y}$ (the state of the system), and thus (59) defines a symplectic form ω on the state space $\{(y, \dot{y}) | y \in R, \dot{y} \in R\}$ with coordinate expression

$$\omega = \begin{pmatrix} 0 & \frac{\partial F}{\partial \ddot{y}} / \frac{\partial F}{\partial u} \\ -\frac{\partial F}{\partial \ddot{y}} / \frac{\partial F}{\partial u} & 0 \end{pmatrix}.$$

Case 2: Let us assume that the input–output representation F_i, $i = 1, \cdots, m$, have the following particular form

$$F_i(y, \dot{y}, \ddot{y}, u, \dot{u}) = S_i(y, \dot{y}, \ddot{y}) - u_i. \qquad (61)$$

This is the classical case (see [14]). Conditions $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, and $\mathcal{L}_4$ reduce to

$\mathcal{L}_1)$ void

$\mathcal{L}_2)$ $\frac{\partial S}{\partial \dot{y}} - \left(\frac{\partial S}{\partial \dot{y}}\right)^T = 0$

$\mathcal{L}_3)$ $\frac{\partial S}{\partial y} + \left(\frac{\partial S}{\partial y}\right)^T - 2\frac{d}{dt}\left(\frac{\partial S}{\partial \dot{y}}\right)^T = 0$

$\mathcal{L}_4)$ $\frac{\partial S}{\partial y} - \left(\frac{\partial S}{\partial y}\right)^T + \frac{d}{dt}\left(\frac{\partial S}{\partial \dot{y}}\right)^T - \frac{d^2}{dt^2}\left(\frac{\partial S}{\partial \ddot{y}}\right)^T = 0$

at every point $(y, \dot{y}, \ddot{y})$.

These conditions are precisely the conditions (2.1.17) in [14]. It follows that (61) represents the input–output behavior of a Hamiltonian system if and only if

$$S_i = R_{ik}(y, \dot{y})\ddot{y}^k + T_i(y, \dot{y}), \qquad i = 1, \cdots, m$$

where R_{ik} and T_i satisfy (2.2.9) in [14].

Case 3:

$$F_i(y, \dot{y}, \ddot{y}, u, \dot{u}) = S_i(y, \dot{y}) - u_i, \qquad i = 1, \cdots, m.$$

From the result of Santilli, it follows that

$$S_i = X_{ik}(y)\dot{y}^k + Y_i(y), \qquad i = 1, \cdots, m$$

where X_{ik} and Y_i satisfy

$$\begin{cases} X_{ik} + X_{ki} = 0 \\ \frac{\partial X_{ij}}{\partial y^k} + \frac{\partial X_{jk}}{\partial y^i} + \frac{\partial X_{ki}}{\partial y^j} = 0 \\ \frac{\partial Y_j}{\partial y^i} - \frac{\partial Y_i}{\partial y^j} = 0. \end{cases}$$

Thus $X = (X_{ij})$ defines the symplectic form ω. This also follows from (18), since $y_a^k = \xi^k$ and $Q = y_a^i X_{ik} y_v^k$.

Example 1: As an example we consider the system given by the equations

$$\dot{q} = u$$
$$\dot{p} = \sin q + \alpha p$$
$$y = p.$$

The corresponding input–output differential equation is given by

$$F(y, \dot{y}, \ddot{y}, u, \dot{u}) = (\ddot{y} - \alpha\dot{y})^2 + (\dot{y} - \alpha y)^2 u^2 - u^2 \equiv 0. \quad (62)$$

Clearly, by inspecting the state representation of the system, we see that the system is Hamiltonian with the Hamiltonian function $H = \cos q + up$, if and only if $\alpha = 0$. This is not so evident, however, from the form of the input–output differential equation above. We shall employ condition (50) of Crouch and Lamnabhi [4] to show this. Condition i), $\partial F/\partial \dot{u} \equiv 0$, is trivially satisfied. We calculate the quantity

$$Z = \frac{\partial F}{\partial u}\left[\frac{d}{dt}\frac{\partial F}{\partial \ddot{y}} - \frac{\partial F}{\partial \dot{y}}\right] - \frac{\partial F}{\partial \ddot{y}}\frac{d}{dt}\frac{\partial F}{\partial u}$$

directly from F to obtain

$$Z = (-4u)(1 - (\dot{y} - \alpha y)^2)[(y^{(3)} - \alpha\ddot{y})$$
$$+ \alpha(\ddot{y} - \alpha\dot{y}) - u^2(\dot{y} - \alpha y)]$$
$$- 4(\ddot{y} - \alpha\dot{y})[2u(\dot{y} - \alpha y)(\ddot{y} - \alpha\dot{y})$$
$$- \dot{u}(1 - (\dot{y} - \alpha y)^2)].$$

From $F(y, \dot{y}, \ddot{y}, u, \dot{u}) \equiv 0$ we obtain

$$(y^{(3)} - \alpha\ddot{y}) = \frac{u\dot{u}}{(\ddot{y} - \alpha\dot{y})} - (\dot{y} - \alpha y)u^2 - \frac{u\dot{u}(\dot{y} - \alpha y)^2}{(\ddot{y} - \alpha\dot{y})}.$$

Substituting this expression into the expression for Z and again using the definition of F, we see that $Z \equiv 0$ if and only if

$$\alpha u(1 - (\dot{y} - \alpha y)^2)(\ddot{y} - \alpha\dot{y}) \equiv 0.$$

This equation can be satisfied for all $y(0)$, $\dot{y}(0)$, $u(0)$ only if $\alpha = 0$, so we conclude that the system (62) is Hamiltonian if and only if $\alpha = 0$, as we previously concluded.

It is also interesting to compare the adjoint system with equation (62). The variational system is given by

$$u_v((\dot{y} - \alpha y)^2 u - u) + y_v(-\alpha u^2(\dot{y} - \alpha y))$$
$$+ \dot{y}_v(u^2(\dot{y} - \alpha y) - \alpha(\ddot{y} - \alpha\dot{y})) + \ddot{y}_v(\ddot{y} - \alpha(\dot{y})) = 0. \quad (63)$$

Using the method of integration by parts introduced in (56), we see that the adjoint system is given by the equation

$$y_a = \xi((\dot{y} - \alpha y)^2 u - u)$$

$$u_a = \xi\alpha u^2(\dot{y} - \alpha y) + \overline{\dot{\xi}(u^2(\dot{y} - \alpha y) - \alpha(\ddot{y} - \alpha\dot{y}))} - \overline{\ddot{\xi}(\ddot{y} - \alpha\dot{y})}.$$

Thus $\xi = y_a/((\dot{y} - \alpha y)^2 u - u)$. An explicit expression for the adjoint variational system in input–output differential representation (34) is obtained by eliminating ξ from the equation above. Our theory guarantees that the resulting equation coincides with equation (63) if and only if $\alpha = 0$.

Example 2: In the case of a linear time-invariant system

$$Ay + B\dot{y} + C\ddot{y} + Du + E\dot{u} = 0, \qquad u, y \in \boldsymbol{R}^m$$

with A, B, C, D, E constant $m \times m$ matrices, the conditions $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$ reduce to

$$CE^T + EC^T = 0$$

$$BE^T - CD^T + DC^T - EB^T = 0$$

$$AE^T - BD^T - DB^T + EA^T = 0$$

$$-AD^T + DA^T = 0,$$

which is equivalently to the equality

$$(D + Es)(A^T - B^Ts + C^Ts^2)$$
$$= (A + Bs + Cs^2)(D^T - E^Ts)$$

for all $s \in C$. This last equality in turn is equivalent to the transfer matrix $G(s) = (A + Bs + Cs^2)^{-1}(D + Es)$ of the system to satisfy the condition (cf. [1], [15]) $G(s) = G^T(-s)$.

Example 3: It can be straightforwardly checked that the Euler–Lagrange equations with external forces u

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = u, \qquad u, y \in \boldsymbol{R}^m$$

for any Lagrangian function $L(y, \dot{y})$ satisfy (56) (see also Santilli [14]).

Consider now an Euler–Lagrange system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) - \frac{\partial L}{\partial y_1} = v_1, \qquad v_1, y_1 \in \boldsymbol{R}$$

in interconnection with a static nonlinearity N (see van der Schaft [15] for details of interconnections), and assume for simplicity that $L(y, \dot{y}_1) = \frac{1}{2}m\dot{y}_1^2 - V(y_1)$, while the nonlinearity N is described by a differentiable function $y_2 = h(v_2)$. The interconnected system with outputs y_1, y_2 and inputs u_1, u_2 is given as (after elimination of v_1 and v_2)

$$m\ddot{y}_1 + \frac{\partial V}{\partial y_1}(y_1) - y_2 - u_1 = 0$$
$$h(y_1 + u_2) - y_2 = 0.$$

Computing the matrices A, B, C, D, E as in (51) yields

$$A = \begin{bmatrix} \frac{\partial^2 V}{\partial y_1^2}(y_1) & -1 \\ \frac{\partial h}{\partial v_2}(y_1 + u_2) & -1 \end{bmatrix},$$

$$C = \begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & \frac{\partial h}{\partial v_2}(y_1 + u_2) \end{bmatrix}$$

while both B and E are zero. It is readily checked that conditions $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$ are satisfied, and thus the interconnected system is a Hamiltonian system for every scalar differentiable nonlinearity $y_2 = h(v_2)$.

In the multivariable case with $y_1, y_2, u_1, u_2, v_1, v_2 \in \boldsymbol{R}^m$ and $L(y_1, \dot{y}_1) = \frac{1}{2}\dot{y}_1^T M\dot{y}_1 - V(y_1)$, with $M = M^T > 0$,

it is straightforwardly checked that the interconnected system satisfies $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$ if and only if the nonlinearity $y_2 = h(v_2)$ satisfies the integrability condition $\frac{\partial h}{\partial v_2} = \left(\frac{\partial h}{\partial v_2}\right)^T$, and thus there exists (locally) a potential function $P(v_2)$ such that $y_2 = \frac{\partial P}{\partial v_2}(v_2)$.

IV. CHARACTERIZATION OF HAMILTONIAN SYSTEMS FROM THE STATE SPACE REPRESENTATION

In this section we derive another formulation of the criteria for self adjointness of the variational system (12) corresponding to an input–output differential representation (11). In this derivation, however, we work directly with the state space representations of the variational system (12) rather than the input–output differential representation (15) (which we did in the previous section). To do this we obtain a direct correspondence between the representations (12) and (15).

We first observe that the input–output map of the variational system Σ_s^v in equation (12) may be expressed in the form

$$y_v(t) = \int_0^t W_v(t, \sigma, u, x_0) u_v(\sigma)\, d\sigma \qquad (64)$$

where we assume that $v(0) = 0$, and x_0 is the initial condition of the corresponding state space system (10). We also note that the input–output map of the adjoint variational system Σ_s^a in (13) is expressed in the form

$$y_a(t) = -\int_0^t W_v^T(\sigma, t, u, x_0) u_a(\sigma)\, d\sigma \qquad (65)$$

where we assume that $p(0) = 0$. Thus (as formulated in [5], [6]) self-adjointness of the variational systems may be simply expressed as the statement

$$W_v(t, \sigma, u, x_0) = -W_v^T(\sigma, t, u, x_0) \qquad (66)$$

for all $t \geq \sigma \geq 0$, all piecewise constant controls u, and one initial state x_0. As we argued above, it is sufficient to check this identity for analytic controls u. Using the notation of (12), we may express the kernel W_v in the form

$$W_v(t, \sigma, x_0) = H(t)\Phi(t, \sigma)G(\sigma)(= W_v(t, \sigma))$$

where $\Phi(t, \sigma)$ is the transition matrix of $F(t)$. For analytic controls u, H, Φ, and G are all analytic in their arguments. Thus the self-adjointness criteria (66) may be expressed in the form

$$H(t)\Phi(t, \sigma)G(\sigma) = -G(t)^T\Phi(\sigma, t)^T H(t)^T$$
$$t \geq \sigma \geq 0. \qquad (67)$$

Our first task is to give an equivalent formulation of the conditions (67) in terms of standard (time varying) linear system objects. We define the sequence of time varying $p \times n$ matrices $\Gamma_k(t)$ by setting

$$\Gamma_k^T(t) = \left(I\frac{d}{dt} + F^T(t)\right)^k H^T(t);$$
$$\Gamma_0(t) = H(t), \, k \geq 0 \qquad (68)$$

and we define the sequence of time varying $n \times m$ matrices $\Sigma_k(t)$ by setting

$$\Sigma_k(t) = (-1)^k\left(I\frac{d}{dt} - F(t)\right)^k G(t);$$
$$\Sigma_0(t) = G(t), \, k \geq 0. \qquad (69)$$

Noting that

$$\frac{\partial}{\partial t}\Phi(t, \sigma) = F(t)\Phi(t, \sigma);$$

$$\frac{\partial}{\partial \sigma}\Phi(t, \sigma) = -\Phi(t, \sigma)F(\sigma)$$

we easily obtain

$$\frac{\partial^k}{\partial t^k}H(t)\Phi(t, \sigma) = \Gamma_k(t)\Phi(t, \sigma); \qquad k \geq 0 \qquad (70)$$

$$\frac{\partial^k}{\partial \sigma^k}\Phi(t, \sigma)G(\sigma) = (-1)^k\Phi(t, \sigma)\Sigma_k(\sigma); \qquad k \geq 0. \qquad (71)$$

Lemma 1: In the case of analytic data the identity (67) is equivalent to the sequence of identities

$$(-1)^{k+l+1}\Gamma_k(t)\Sigma_l(t) = \Sigma_k(t)^T\Gamma_l(t)^T; \qquad k, l \geq 0. \qquad (72)$$

Proof: By applying (70) and (71) to (67) we obtain

$$(-1)^l\Gamma_k(t)\Phi(t, \sigma)\Sigma_l(\sigma) = -(-1)^k\Sigma_k(t)^T\Phi(\sigma, t)^T\Gamma_l(\sigma)^T.$$

By setting $t = \sigma$ we obtain (72). Conversely, from (72) and analyticity we recover the identity

$$(-1)^{k+1}\Gamma_k(t)\Phi(t, \sigma)G(\sigma) = \Sigma_k(t)^T\Phi(\sigma, t)^T H(\sigma)^T.$$

For $k = 0$, this is the desired identity (67). ∎

Out main interest is to show that we may replace the infinite set of conditions represented by (72), with a suitable finite subset. We require the following results.

Lemma 2: Consider a time varying state space system

$$\dot{x} = F(t)x + G(t)u, \qquad x(0) = 0, \quad x \in R^l$$
$$y = H(t)x. \qquad (73)$$

Then

$$\frac{d^p}{dt^p}y(t) = \sum_{k=0}^{p-1}\frac{d^k}{dt^k}(\Gamma_{p-1-k}\Sigma_0 u)(t)$$
$$+ \int_0^t \Gamma_p(t)\Phi(t, \sigma)G(\sigma)u(\sigma)\, d\sigma. \qquad (74)$$

Proof: Since $\Gamma_0 = H(t)$, the statement (74) is true for $p = 0$. Assume by induction that (74) is true. Then

$$\frac{d^{p+1}}{dt^{p+1}}y(t)$$
$$= \sum_{k=0}^{p-1}\frac{d^{k+1}}{dt^{k+1}}(\Gamma_{p-1-k}\Sigma_0 u)(t) + \Gamma_p(t)G(t)u(t)$$
$$+ \int_0^t \Gamma_{p+1}(t)\Phi(t, \sigma)G(\sigma)u(\sigma)\, d\sigma$$
$$= \sum_{k=1}^{p}\frac{d^k}{dt^k}(\Gamma_{p-k}\Sigma_0 u)(t) + \Gamma_p(t)\Sigma_0(t)u(t)$$
$$+ \int_0^t \Gamma_{p+1}(t)\Phi(t, \sigma)G(\sigma)u(\sigma)\, d\sigma.$$

This is, then, the identity (74) with p replaced by $p + 1$. ∎

Lemma 3: The input–output map of the controllable time varying linear system (73) satisfies the system

$$\sum_{k=0}^{N} A_k(t) y^{(k)}(t) = \sum_{k=0}^{N-1} B_k(t) u^{(k)}(t) \qquad (75)$$

if and only if

$$\sum_{k=0}^{N} A_k(t) \Gamma_k(t) = 0 \qquad (76)$$

$$B_k(t) = + \sum_{q=k}^{N-1} A_{q+1}(t) \sum_{p=k}^{q} (\Gamma_{p-k}(t) \Sigma_0(t))^{(p-k)} \binom{p}{k}. \quad (77)$$

Proof: The input–output map of system (73) is given by

$$y(t) = \int_0^t H(t) \Phi(t, \sigma) G(\sigma) u(\sigma) \, d\sigma.$$

If this satisfies (75), we obtain the following expression using lemma (2)

$$\sum_{p=1}^{N} A_p(t) \sum_{k=0}^{p-1} (\Gamma_{p-1-k} \Sigma_0 u)^{(k)}(t)$$

$$+ \sum_{p=0}^{N} A_p(t) \Gamma_p(t) \int_0^t \Phi(t, \sigma) G(\sigma) u(\sigma) \, d\sigma$$

$$- \sum_{k=0}^{N-1} B_k(t) u^{(k)}(t) = 0. \qquad (78)$$

Thus

$$\sum_{p=0}^{N} A_p(t) \Gamma_p(t) \int_0^t \Phi(t, \sigma) G(\sigma) u(\sigma) \, d\sigma = 0$$

$$\sum_{p=1}^{N} A_p(t) \sum_{k=0}^{p-1} \sum_{r=0}^{k} (\Gamma_{p-1-k} \Sigma_0)^{(k-r)}(t) u^{(r)}(t) \binom{k}{r}$$

$$- \sum_{k=0}^{N-1} B_k(t) u^{(k)}(t) = 0.$$

Controllability of system (73) and reordering summations now yields the desired identities (76) and (77). Conversely, the identities (76) and (77) yield (78) which by Lemma 2 ensures that the input–output map of (73) satisfies (75) as desired. ∎

We wish to employ Lemma 3 in the context of the variational system Σ_s^v in (12) and the corresponding input–output representation $\Sigma_{i/o}^v$ in (15). However, (76) is not written in terms of purely input–output quantities which we require for our purposes. We therefore make the following observation.

If the system (73) is controllable, in the sense that

$$\text{rank} \, [\Sigma_0(t)|\Sigma_1(t)|, \cdots |\Sigma_{N-1}(t)] = l \qquad (79)$$

for all times t and all controls u, then the condition (76) may be replaced by the equivalent condition

$$\sum_{k=0}^{N} A_k(t) \Gamma_k(t) [\Sigma_0(t)|\Sigma_1(t)| \cdots |\Sigma_{N-1}(t)] = 0. \qquad (80)$$

We may now state and prove our main result in this section.

Theorem 4: Consider a minimal input–output differential representation $\Sigma_{i/o}$, given by (11) with $p = m$, associated minimal state space representation Σ_s given by (11), and the associated variational system Σ_s^v. Then under the assumption (1) and the assumption that $\frac{\partial F}{\partial y^{(N)}}$ is invertible for all analytic solutions (u, y) of (11), $\Sigma_{i/o}$ is an input–output representation of a Hamiltonian system Σ_s if and only if

$$(-1)^{k+l+1} \Gamma_k(t) \Sigma_l(t) = \Sigma_k(t)^T \Gamma_l(t)^T; \quad 0 \le k, l \le N \qquad (81)$$

for all t and all analytic controls u.

Proof: As in [5] and [6], we know that Σ_s is Hamiltonian if and only if (66) holds for all analytic controls u. We have shown that these conditions are equivalent to the conditions (72), (Lemma 1). These conditions imply those of (81), hence establishing the necessity of the conditions (81). To prove necessity we argue as follows. We first compute the quantities Σ_k and Γ_k for the adjoint variational system Σ_s^a in (13). We make the substitutions

$$F \to -F^T; \quad G \to -H^T; \quad H \to G^T$$

in definitions (68) and (69). We obtain

$$\Sigma_k^a(t) = (-1)^k \left(I \frac{d}{dt} + F(t)^T \right)^k (-H(t)^T)$$

$$= (-1)^{k+1} \Gamma_k^v(t)^T (= (-1)^{k+1} \Gamma_k(t)^T)$$

$$\Gamma_k^{aT}(t) = \left(I \frac{d}{dt} - F(t) \right)^k G(t)$$

$$= (-1)^k \Sigma_k^v(t) (= (-1)^k \Sigma_k(t)).$$

Thus

$$\Sigma_k^a(t) = (-1)^{k+1} \Gamma_k(t)^T; \quad \Gamma_k^a(t) = (-1)^k \Sigma_k(t)^T. \quad (82)$$

We wish to generate conditions under which the input–output map (65) of Σ_s^a coincides with that of the input–output map (64) of Σ_s^v. By Assumption 1, the set of solutions (y_v, u_v) generated by Σ_s^v equals the set of solutions of $\Sigma_{i/o}^v$ represented by (15) or

$$\sum_{k=0}^{N} A_k(t) y_v^{(k)}(t) = \sum_{k=0}^{N-1} B_k(t) u_v^{(k)}(t). \qquad (83)$$

(We also have that the set of solutions (y_v, u_v) of Σ_s^v corresponding to zero initial conditions $v(0) = 0$, is equal to the subspace through the origin of the set of solutions of $\Sigma_{i/o}^v$.) Hence, we may check self-adjointness of Σ_s^v, simply by checking that the input–output map (65) of Σ_s^a satisfies (83). By Lemma 3, however, the input–output map (65) satisfies (83) if and only if the (76) and (77) hold with Γ_k and Σ_k replaced by Γ_k^a and Σ_k^a, given in (82). Note that the controllability assumption required in Lemma 3 is translated into controllability of Σ_s^a, which is simply observability of Σ_s^u. Moreover, we may substitute the condition (76) by (80) as long as the condition (79) holds for Σ_s^a. Thus we obtain the

following conditions by substituting (82) into (80) and (77)

$$\sum_{k=0}^{N} A_k(t)\Sigma_k(t)^T(-1)^k[-\Gamma_0(t)^T|\Gamma_1(t)^T|$$

$$-\Gamma_2(t)^T \cdots |(-1)^N\Gamma_{N-1}(t)^T] = 0$$

$$B_k(t) = \sum_{q=k}^{N-1} A_{q+1}(t)\sum_{p=k}^{q}(-1)^{p-k+1}$$

$$\cdot (\Sigma_{p-k}^T(t)\Gamma_0(t)^T)^{(p-k)}\binom{p}{k}. \quad (84)$$

We now note that conditions (81) are indeed sufficient to ensure that (84) may be simply rewritten as (77) and (80). These are satisfied by virtue of the fact that by assumption the solutions (y_v, u_v) of Σ_s^v are solutions of $\Sigma_{i/o}^v$. It follows that we have shown sufficiency of the conditions (81), under the apparent further assumption that (79) holds for Σ_s^a.

To evaluate the further assumption it is useful to make an explicit construction, which is required later on. It is easily seen that we may rewrite the system $\Sigma_{i/o}^v$ given in (83) in the form

$$\sum_{k=0}^{N}(\hat{A}_k(t)y_v(t))^{(k)} = \sum_{k=0}^{N-1}(\hat{B}_k(t)u_v(t))^k$$

where $\hat{A}_N(t) = A_N(t) = \frac{\partial F}{\partial y^{(N)}}$. (Clearly the matrices $\hat{A}_k$ and $\hat{B}_k$ are related to the operators $\tilde{D}$ and $\tilde{N}$ defining the adjoint variational system in (41).) Under the assumption that $\frac{\partial F}{\partial y^{(N)}}$ is invertible we may rewrite $\Sigma_{i/o}^v$ in the form

$$y_v^{(N)}(t) + \sum_{k=0}^{N-1}(\overline{A}_k(t)y_v(t))^k = \sum_{k=0}^{N-1}(\overline{B}_k(t)u_v(t))^k. \quad (85)$$

In this form we may write down the observable canonical form for the system (85), which will be a particular realization of Σ_s^v.

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} -\overline{A}_{N-1}(t) & I & 0 & & 0 \\ -\overline{A}_{N-2}(t) & 0 & I & & 0 \\ & \ddots & & \ddots & \\ -\overline{A}_0(t) & 0 & 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

$$+ \begin{pmatrix} \overline{B}_{N-1}(t) \\ \overline{B}_{N-2}(t) \\ \vdots \\ \overline{B}_0(t) \end{pmatrix} u_v(t)$$

$$y(t) = (I, 0, \cdots, 0)(x_1, x_2, \cdots, x_N)^T. \quad (86)$$

Clearly this time varying system satisfies the condition

$$\text{Rank}\,[\Gamma_0(t)^T|\Gamma_1(t)^T|\cdots|\Gamma_{N-1}(t)^T] = Np \quad (87)$$

for all t and all analytic controls u. Thus the corresponding adjoint system does satisfy the controllability condition (79). Note that the conditions (81) are independent of the particular realization of Σ_s^v which is chosen, and in particular there is no necessity for the chosen realization to be minimal (as a time varying system). ∎

Finally in this section we point out the relationship between the conditions (81) and the conditions (49) derived in the previous section. Although the conditions (81) are apparently expressed in terms of the system Σ_s^v, or any other realization, we may interpret them directly in terms of $\Sigma_{i/o}^v$. In particular we may apply the conditions (81) to the realization (85) constructed above. The presentation of the resulting conditions on the matrices $A_k(t)$, $B_k(t)$ defining $\Sigma_{i/o}^v$, as in (83) turns out to be different, but equivalent, to the conditions obtained by applying Theorem 2 in the previous section. We demonstrate the conditions obtained in this section on the system (52). We write the corresponding variational system in the form of (53) but assume $C = I_m$, the identity matrix for ease of explanation. (The general conditions may be obtained by replacing A, B, D, and E by $C^{-1}A$, $C^{-1}B$, $C^{-1}D$, and $C^{-1}E$, respectively.) Now if the variational system is written as

$$A(t)y_v + B(t)\dot{y}_v + \ddot{y}_v = -E(t)\dot{u}_v - D(t)u_v$$

this may be rewritten in the form of (85) as follows

$$\ddot{y}_v + \overline{\dot{B(t)y_v}} + (A(t) - \dot{B}(t))y_v = \overline{(-E(t)u_v)} + (-D(t) + \dot{E}(t))u_v.$$

Thus the corresponding observable canonical form (86) becomes

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -B(t) & I \\ \dot{B}(t) - A(t) & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -E(t) \\ \dot{E}(t) - D(t) \end{pmatrix}u_v$$

$$y(t) = (I, 0)(x, x_2)^T. \quad (88)$$

The first condition in (81) is simply

$$-\Gamma_0(t)\Sigma_0(t) = \Sigma_0(t)^T\Gamma_0(t)^T$$

or simply

$$-H(t)G(t) = G(t)^T H(t)^T.$$

Substituting from (88) we obtain

$$E(t) = -E(t)^T.$$

Replacing E by $C^{-1}E$ we obtain

$$E(t)C(t)^T + C(t)E(t)^T = 0. \quad (89)$$

But this is simply $\mathcal{L}_1$, in Theorem 3.

The next condition in (81) is simply

$$\Gamma_1(t)\Sigma_0(t) = \Sigma_1(t)^T\Gamma_0(t)^T$$

or simply $(\dot{H}(t) + H(t)F(t))G(t) = -(\dot{G}(t)^T - G(t)^T F(t)^T)$ $H^T(t)$ or since $\dot{H}(t) \equiv 0$

$$H(t)F(t)G(t) = G(t)^T F(t)^T H(t)^T - \dot{G}(t)^T H(t)^T.$$

Substituting from (88) we obtain

$$[I, 0]\begin{bmatrix} -B & I \\ \dot{B} - A & 0 \end{bmatrix}\begin{bmatrix} -E \\ \dot{E} - D \end{bmatrix}$$

$$= [-E^T, \dot{E}^T - D^T]\begin{bmatrix} -B^T & \dot{B}^T - A^T \\ I & 0 \end{bmatrix}\begin{bmatrix} I \\ 0 \end{bmatrix}$$

$$- [-\dot{E}^T, \ddot{E}^T - \dot{D}]\begin{bmatrix} I \\ 0 \end{bmatrix}$$

which simply reduces to the condition

$$B(t)E(t) - E(t)^T B(t)^T + D(t)^T - D(t) + \dot{E}(t) - 2\dot{E}(t)^T = 0.$$

Replacing E, B, and D by $C^{-1}E$, $C^{-1}B$, and $C^{-1}D$, we obtain

$$C^{-1}BC^{-1}E - E^T C^{-T} B^T C^{-T} + D^T C^{-T} - C^{-1}D$$
$$+ \overline{(C^{-1}E)} - 2\overline{(C^{-1}E)^T} = 0.$$

However, from (89) we have

$$C^{-1}E = -E^T C^{-T}.$$

By expanding the above expression and using this identity we obtain

$$- C^{-1}BE^T C^{-T} + C^{-1}EB^T C^{-T} + D^T C^{-T}$$
$$- C^{-1}D + C^{-1}\dot{C}E^T C^{-T} + C^{-1}\dot{E}$$
$$- 2C^{-1}E\dot{C}^T C^{-T} - 2\dot{E}^T C^{-T} = 0$$

which may be reexpressed, using the invertibility of C, as

$$-BE^T + EB^T + CD^T - DC^T + \dot{C}E^T + \dot{E}C^T$$
$$- 2E\dot{C}^T - 2C\dot{E}^T = 0.$$

By differentiating (89), however, we obtain

$$\dot{C}E^T + \dot{E}C^T = -E\dot{C}^T - C\dot{E}^T.$$

We therefore obtain the expression

$$-BE^T + EB^T + CD^T - DC^T - 3E\dot{C}^T - 3C\dot{E}^T = 0.$$

Now it is easily seen that this is just condition $\mathcal{L}_2$ in Theorem 3. Clearly, the remaining conditions $\mathcal{L}_3$ and $\mathcal{L}_4$ will be contained in the conditions (81) for $N = 2$.

We make two final comments on (81). Clearly, by inspecting (77) and (80), the number of conditions in (81) may be reduced to $N \geq k \geq 0$, $N - 1 \geq l \geq 0$, $k \geq l$. For $N = 2$ this results in five conditions, but we already know from Theorem 3 that in the case $N = 2$, there are only four independent conditions. Thus we expect even the reduced set of conditions (81) to include many redundancies. Furthermore, Theorem 4 requires the assumption that $\frac{\partial F}{\partial y^{(N)}}$ is invertible, which Theorem 2 does not.

Conditions (81) do provide a satisfying generalization of the Brockett and Rahami result, discussed in the introduction. In particular, the condition that system (4) is Hamiltonian is simply given by the self-adjointness condition

$$Ce^{A(t-\sigma)}B = -B^T e^{-A^T(t-\sigma)}C^T.$$

By setting $\Gamma_k = CA^k$, $\Sigma_k = A^k B$, it is clear that self-adjointness is indeed equivalent to condition (81) for $N = n - 1$.

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