Hamiltonicity of Regular 2-Connected Graphs

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ABSTRACT

Let G be a k -regular 2-connected graph of order n . Jackson proved that G is hamiltonian if $n \leq 3k$. Zhu and Li showed that the upper bound $3k$ on n can be relaxed to $\frac{22}{7}k$ if G is 3-connected and $k \geq 63$. We improve both results by showing that G is hamiltonian if $n \leq \frac{7}{2}k - 7$ and G does not belong to a restricted class $\mathcal{F}$ of nonhamiltonian graphs of connectivity 2. To establish this result we obtain a variation of Woodall's Hopping Lemma and use it to prove that if $n \leq \frac{7}{2}k - 7$ and G has a dominating cycle (i.e., a cycle such that the vertices off the cycle constitute an independent set), then G is hamiltonian. We also prove that if $n \leq 4k - 3$ and $G \notin \mathcal{F}$, then G has a dominating cycle. For $k \geq 4$ it is conjectured that G is hamiltonian if $n \leq 4k$ and $G \notin \mathcal{F}$. © 1996 John Wiley & Sons, Inc.

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1. INTRODUCTION AND MAIN RESULT

We use [3] for terminology and notation not defined here and consider finite simple graphs only.

In [8], the following result was obtained.

Theorem 1 (Jackson [8]). Every 2-connected k -regular graph on at most $3k$ vertices is hamiltonian.

As noted in [8], Theorem 1 is best possible for $k = 3$ in view of the Petersen graph, and essentially best possible for $k \geq 4$. For future reference also, we define three classes $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{J}$ of graphs illustrating the latter assertion. For a positive integer p , let $\mathcal{K}_p$ denote the set of all graphs consisting of three disjoint complete graphs, where each of the components has order at least p . Now $\mathcal{G}$ is the class of all spanning subgraphs of graphs that can be obtained as the join of K_2 and a graph in $\mathcal{K}_1$. The class $\mathcal{H}$ is the set of all spanning subgraphs of graphs that can be obtained from the join of K_1 and a graph G in $\mathcal{K}_2$ by adding the edges of a triangle between three vertices from distinct components of G . The class $\mathcal{J}$ is the set of all spanning subgraphs of graphs that can be obtained from a graph G in $\mathcal{K}_3$ by adding the edges of two triangles between two disjoint triples of vertices, each containing one vertex of each component of G . It is easy to check that all graphs in $\mathcal{G} \cup \mathcal{H} \cup \mathcal{J}$ are nonhamiltonian. (Indeed, $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{J}$ were first obtained by Watkins and Mesner [12] in a characterization of the 2-connected graphs which have three vertices which are not contained in a common cycle.) Furthermore, each of the classes $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{J}$ contains 2-connected k -regular graphs on $3k + 4$ vertices for even $k \geq 4$, and $3k + 5$ vertices for all $k \geq 3$. (Note that $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{J}$ are not pairwise disjoint.) We set $\mathcal{F} = \mathcal{G} \cup \mathcal{H} \cup \mathcal{J}$.

Theorem 1 has been extended in several papers, e.g., in [2], [7], and [15]. The strongest among these extensions is due to Hilbig [7]. Let Π denote the Petersen graph and Π^Δ the 3-regular graph obtained from Π by replacing one vertex by a triangle.

Theorem 2 (Hilbig [7]). Let G be a 2-connected k -regular graph on at most $3k + 3$ vertices. Then G is hamiltonian if and only if $G \notin \{\Pi, \Pi^\Delta\}$.

In [9], the following improvement of Theorem 1 for 3-connected graphs is conjectured. (Note that no graph in $\mathcal{F}$ is 3-connected.)

Conjecture 1. (JACKSON, LI, and ZHU [9]). For $k \geq 4$, every 3-connected k -regular graph on at most $4k$ vertices is hamiltonian. ■

Conjecture 1 is a special case of Häggkvist's conjecture, appearing in [8], that every m -connected k -regular graph ($k \geq 4$) on at most $(m + 1)k$ vertices is hamiltonian. However, for $k \equiv 0 \pmod{4}$, the graph $\overline{K_k} \vee (\overline{K_{k-1}} + 2K_{k+1})$ contains a nonhamiltonian $\frac{1}{2}k$ -connected k -regular spanning subgraph G_k , showing that Häggkvist's conjecture is not true in general. The graphs G_k were independently found by H. A. Jung and the third author. For a more detailed description we refer to [9] or [10]. The graphs G_k also show that Conjecture 1 would be best possible.

A first step toward proving Conjecture 1 was made in [9]. A cycle C of a graph G is called a *dominating cycle* if $V(G) \setminus V(C)$ is an independent set of G .

Theorem 3 (Jackson, Li, and Zhu [9]). Let G be a 3-connected k -regular graph on at most $4k$ vertices. Then for $k \geq 63$, every longest cycle of G is a dominating cycle.

In the graph G_k , every longest cycle is dominating. Still Theorem 4 is essentially best possible: for even $k \geq 8$, the graph $K_3 \vee (2K_k + 2K_{k+1})$ of order $4k + 5$ has a 3-connected k -regular spanning subgraph containing no dominating cycle.

In [14], Theorem 3 was used to obtain another result in the direction of Conjecture 1.

Theorem 4 (Zhu and Li [14]). For $k \geq 63$, every 3-connected k -regular graph on at most $\frac{22}{7}k$ vertices is hamiltonian.

Our main result is the following.

Theorem 5. Let G be a 2-connected k -regular graph on at most $\frac{7}{2}k - 7$ vertices. Then G is hamiltonian if and only if $G \notin \mathcal{F}$.

Since no graph in $\mathcal{F}$ is 3-connected, the following improvement of Theorem 4 is an immediate consequence of Theorem 5.

Corollary. Every 3-connected k -regular graph on at most $\frac{7}{2}k - 7$ vertices is hamiltonian.

The necessity of the condition for hamiltonicity in Theorem 5 is obvious. The sufficiency is an immediate consequence of the following two results.

Theorem 6. Let G be a k -regular graph on at most $\frac{7}{2}k - 7$ vertices. If G contains a dominating cycle, then G is hamiltonian.

Theorem 7. Let G be a 2-connected k -regular graph on at most $4k - 3$ vertices. Then G contains a dominating cycle or $G \in \mathcal{F}$.

Theorem 6 is proved in Section 3. The proof uses a variation of Woodall's Hopping Lemma [13], which is the subject of Section 2. The proof of Theorem 7, which is based on ideas from [5] and [11], is given in Section 4.

Our proof of Theorem 5 (via Theorems 6 and 7) uses several ideas from [4], where a relatively short proof of (an extension of) Theorem 1 occurs. In particular the idea of breaking the proof into two parts in the way reflected by Theorems 6 and 7, stems from [4].

In view of the above results we propose a strengthening of Conjecture 1.

Conjecture 2. Let G be a 2-connected k -regular graph on at most $4k$ vertices. Then for $k \geq 4$, G is hamiltonian if and only if $G \notin \mathcal{F}$. ■

2. A HOPPING LEMMA

We first develop some additional terminology and notation.

Let C be a cycle of a graph G . We call C *extendable* if there exists an *extension* of C , i.e., a cycle C' with $V(C) \subseteq V(C')$ and $V(C) \neq V(C')$. For $v \in V(G) \setminus V(C)$, the cycle C is *v-extendable* if there exists a *v-extension* of C , i.e., an extension with vertex set $V(C) \cup \{v\}$.

We denote by $\vec{C}$ the cycle C with a given orientation, and by $\bar{\vec{C}}$ the cycle C with the reverse orientation. If $u, v \in V(C)$, then $u\vec{C}v$ denotes the consecutive vertices of C from u to v in the direction specified by $\vec{C}$. The same vertices, in reverse order, are given by

$v \vec{C} u$. We will consider $u\vec{C}v$ and $v\vec{C}u$ both as paths and as vertex sets. We use u^+ to denote the successor of u on $\vec{C}$ and u^- to denote its predecessor. If $Z \subseteq V(C)$, then $Z^+ = \{z^+ | z \in Z\}$ and $Z^- = \{z^- | z \in Z\}$. Similar notation is used for paths. When more than one cycle or path is under consideration, we write u^{+C}, u^{-C} instead of just u^+, u^- in order to avoid ambiguity.

If A and B are subsets of $V(G)$, then $\varepsilon(A, B)$ denotes the number of edges with one end in A and the other in B ; the edges with both ends in $A \cap B$ are counted twice. For $a \in V(G)$, we write $\varepsilon(a, B)$ instead of $\varepsilon(\{a\}, B)$.

In our variation of Woodall's Hopping Lemma [13], Lemma 8 below, we use the following hypotheses and definitions.

Let G be a graph, $\vec{C}$ a cycle of G with $V(C) \neq V(G)$, and a a vertex in $V(G) \setminus V(C)$. Assume C is not a -extendable. Set

$$X_1 = N(a) \cap V(C),$$

and for $i \geq 1$,

$$\begin{aligned} Y_i &= X_i^+ \cap X_i^-, \\ U_i &= X_i^+ \setminus Y_i, \\ W_i &= X_i^- \setminus Y_i, \end{aligned}$$

$$\begin{aligned} X'_{i+1} &= \{v \in V(C) | \text{there exist six neighbors } w_1, u_1, w_2, u_2, w_3, u_3 \text{ of } v \\ &\quad \text{such that } w_j \in W_i, u_j \in U_i \text{ and } w_j^+ \vec{C} u_j^- \subseteq X_i \cup Y_i \ (j = 1, 2, 3)\}, \end{aligned}$$

$$X''_{i+1} = N(Y_i) \cap V(C),$$

$$X_{i+1} = X_i \cup X'_{i+1} \cup X''_{i+1}.$$

Then $X_1 \subseteq X_2 \subseteq \dots$ and $Y_1 \subseteq Y_2 \subseteq \dots$. Set

$$X = \bigcup_{i=1}^{\infty} X_i,$$

$$Y = \bigcup_{i=1}^{\infty} Y_i.$$

Then

$$N(Y) \cap V(C) \subseteq X. \tag{1}$$

The *height* $h(x)$ of $x \in X$ is defined by

$$h(x) = \min\{i | x \in X_i\}.$$

A path $P = x_1 \vec{P} x_2$ is called a *hopping path* if each of the following conditions is satisfied:

$$x_1, x_2 \in X; \tag{2}$$

$$V(P) = V(C); \quad (3)$$

$$\begin{aligned} \text{if } 1 \leq i < \max\{h(x_1), h(x_2)\} \text{ and } y \in Y_i \setminus \{x_1, x_2\}, \\ \text{then } \{y^{-P}, y^{+P}\} = \{y^{-C}, y^{+C}\}; \end{aligned} \quad (4)$$

$$\begin{aligned} \text{if } 1 \leq i < \max\{h(x_1), h(x_2)\}, \text{ then } X_i \setminus \{x_1, x_2\} \\ \text{contains at most one vertex } x \text{ for which } \{x^{-P}, x^{+P}\} \neq \{x^{-C}, x^{+C}\}. \end{aligned} \quad (5)$$

The height $h(P)$ of a hopping path $P = x_1 \vec{P} x_2$ is defined by

$$h(P) = \max\{h(x_1), h(x_2)\}.$$

Note that our definitions differ from those given in [13] in that:

- we do not require $N(a) \subseteq V(C)$;
- we add the sets X'_i to X ;
- our conditions (4) and (5) for a hopping path are more restrictive.

Lemma 1. There exists no hopping path.

Proof. Assuming there exists a hopping path, let $P = x_1 \vec{P} x_2$ be one such that

$$h(P) \text{ is minimum, and} \quad (6)$$

$$\text{subject to (6), } h(x_1) + h(x_2) \text{ is minimum.} \quad (7)$$

Assume without loss of generality that $h(x_1) \geq h(x_2)$. We have $h(P) \neq 1$, otherwise $x_1, x_2 \in X_1 = N(a) \cap V(C)$ and $x_1 \vec{P} x_2 a x_1$ is an a -extension of C , contradicting our assumptions. Hence $h(x_1) = h(P) \geq 2$. Set $i = h(x_1) - 1$, so that $x_1 \in X_{i+1} \setminus X_i$. We derive contradictions in two cases.

Case 1. $h(x_2) < h(x_1)$.

We distinguish two subcases (not necessarily mutually exclusive).

Case 1.1. $x_1 \in X'_{i+1} \setminus X_i$.

By (5) and the definition of X'_{i+1} there exists at least one pair w, u with $w \in W_i, u \in U_i, w^+ \vec{C} u^- \subseteq X_i \cup Y_i$ and $w x_1, u x_1 \in E(G)$ such that $w \vec{C} u$ contains neither x_2 nor a vertex $x \in X_i \setminus \{x_2\}$ with $\{x^{-P}, x^{+P}\} \neq \{x^{-C}, x^{+C}\}$. It follows that $w \vec{C} u$ is a subpath of P . We have

$$u, w \notin X_i, \quad (8)$$

otherwise $u \vec{C} u^{-C}$ or $w^{+C} \vec{C} w$ is a hopping path of height at most i , contradicting (6). In particular, $u, w \neq x_2$. Suppose $u \in x_1 \vec{P} w$. (The case $w \in x_1 \vec{P} u$ is completely analogous.) Set $Q = w^{-P} \vec{P} x_1 w \vec{P} x_2$. We claim that Q is a hopping path of height at most i , contradicting (6). Since $w \vec{C} u$ is a subpath of P , we have $w^{-P} = w^{+C} \in X_i$, and so Q satisfies (2). Since P satisfies (3) and $V(Q) = V(P)$, Q satisfies (3). We now establish (4) and (5) for Q . Since (4) and (5) hold for P and $\{v^{-Q}, v^{+Q}\} = \{v^{-P}, v^{+P}\}$ for each internal vertex v of Q with $v \neq x_1, w$, it suffices to show that $x_1, w \notin X_{i-1} \cup Y_{i-1}$ if $i \geq 2$. Since $x_1 \in X_{i+1} \setminus X_i, x_1 \notin X_{i-1}$. Also, $x_1 \notin Y_{i-1}$, otherwise $w \in X_i$, contradicting (8).

Since $w \notin X_i$, certainly $w \notin X_{i-1}$. Finally, $w \notin Y_{i-1}$, otherwise $x_1 \in X_i$. Furthermore, $h(Q) = \max\{h(w^{-P}), h(x_2)\} \leq i$.

Case 1.2. $x_1 \in X_{i+1} \setminus X_i$.

Then there exists a vertex $y \in Y_i \setminus Y_{i-1}$ with $x_1y \in E(G)$. We have

$$y \notin X_i, \quad (9)$$

otherwise $y^{+C}\vec{C}y$ would be a hopping path of height at most i . In particular, $y \neq x_2$. By (4), $y^{-P} \in X_i$. Set $Q = y^{-P} \vec{P} x_1 y \vec{P} x_2$. We obtain a contradiction to (6) by showing that Q is a hopping path of height at most i . It suffices to show that $x_1, y \notin X_{i-1} \cup Y_{i-1}$ if $i \geq 2$ (as in Case 1.1). Clearly, $x_1 \notin X_{i-1}$. Also, $x_1 \notin Y_{i-1}$, otherwise $y \in X_i$, contradicting (9). Since $y \notin X_i$, certainly $y \notin X_{i-1}$. Finally, $y \in Y_i \setminus Y_{i-1}$, so $y \notin Y_{i-1}$.

Case 2. $h(x_2) = h(x_1) (= i + 1)$.

Again we distinguish two subcases.

Case 2.1. $x_1 \in X'_{i+1} \setminus X_i$ (or $x_2 \in X'_{i+1} \setminus X_i$).

Define w and u as in Case 1.1. We have

$$u, w \notin X_{i+1}, \quad (10)$$

otherwise $u\vec{C}u^{-C}$ or $w^{+C}\vec{C}w$ is a hopping path that contradicts (6) or (7). In particular, $u, w \neq x_2$. Assume for example that $w \in x_1\vec{P}u$ and set $Q = u^{-P} \vec{P} x_1 u \vec{P} x_2$. Now Q is a hopping path of height $i + 1$ that contradicts (7). To see this, it suffices to show that $x_1, u \notin X_i \cup Y_i$. Clearly, $x_1 \notin X_i$. Also, $x_1 \notin Y_i$, otherwise the hopping path $x_1^{+C}\vec{C}x_1$ contradicts (7). By (10), $u \notin X_i$. Finally, $u \notin Y_i$ by definition of U_i .

Case 2.2. $x_1, x_2 \in X''_{i+1} \setminus X_i$.

Then there exist vertices $y_1, y_2 \in Y_i \setminus Y_{i-1}$ with $x_1y_1, x_2y_2 \in E(G)$. We have

$$y_j \notin X_{i+1} \quad (j = 1, 2), \quad (11)$$

otherwise $y_j^{+C}\vec{C}y_j$ is a hopping path that contradicts (6) or (7). Now $y_1^{-P} \vec{P} x_1 y_1 \vec{P} y_2 x_2 \vec{P} y_2^{+P}$ (if $y_1 \in x_1\vec{P}y_2$) or $y_2^{-P} \vec{P} x_1 y_1 \vec{P} y_2 x_2 \vec{P} y_1^{+P}$ (if $y_2 \in x_1\vec{P}y_1$) is a hopping path of height at most i , contradicting (6). To see this, it suffices to show that $x_1, x_2, y_1, y_2 \notin X_{i-1} \cup Y_{i-1}$ if $i \geq 2$. Clearly, $x_1, x_2 \notin X_{i-1}$ and $y_1, y_2 \notin Y_{i-1}$. Also, $x_1, x_2 \notin Y_{i-1}$, otherwise $y_1 \in X_i$ or $y_2 \in X_i$, contradicting (11). Finally, by (11), $y_1, y_2 \notin X_{i-1}$. ■

We shall now briefly indicate how Lemma 1 will be used to prove Theorem 6. In the remainder of this section we first apply Lemma 1 to obtain several other lemmas concerning the existence of edges with one end in $X^+ \cup X^-$.

In Section 3 we suppose that G is a nonhamiltonian graph satisfying the hypotheses of Theorem 6. We apply the lemmas of this section to a nonextendable dominating cycle in G , in order to restrict the number of edges between $X^+ \cup X^-$ and $V(G) \setminus X$. The regularity condition then implies that there must be “many” edges between $X^+ \cup X^-$ and X . We then obtain a contradiction to k -regularity by showing that this number of edges is greater than $k|X|$.

Lemma 2. $X \cap X^+ = \emptyset$.

Proof. Suppose $X \cap X^+$ contains a vertex x . Then $x\vec{C}x^-$ is a hopping path, contradicting Lemma 1. ■

Set

$$U = X^+ \setminus Y, \quad W = X^- \setminus Y.$$

Lemma 3.

- (a) U and W are independent sets.
- (b) If $z_1, z_2 \in U$ or $z_1, z_2 \in W$, then there exists no pair of vertices $v, v^+ \in z_1\vec{C}z_2$ such that $z_1v^+, z_2v \in E(G)$.
- (c) If $u \in U$ and $w \in W$, then there exists no pair of vertices $v, v^+ \in w^+\vec{C}u^-$ such that $uv, wv^+ \in E(G)$ or $uv^+, wv \in E(G)$.
- (d) If $x \in X, v \in (U \cup W) \setminus \{x^+, x^-\}$ and $xv \in E(G)$, then $N(x^+) \cap v^+\vec{C}x^- \cap (U \cup W) = N(x^-) \cap x^+\vec{C}v^- \cap (U \cup W) = \emptyset$.

Proof. (a) Suppose $u_1, u_2 \in U$ and $u_1u_2 \in E(G)$. By definition of U and by Lemma 2, $u_1, u_2 \notin X \cup Y$. But then $u_1^- \vec{C} u_2u_1\vec{C}u_2^-$ is a hopping path, contradicting Lemma 1. Hence U , and similarly W , is an independent set.

(b) Suppose $z_1, z_2 \in U, v, v^+ \in z_1\vec{C}z_2$ and $z_1v^+, z_2v \in E(G)$. By (a), $v \neq z_1$ and $v^+ \neq z_2$. As in the proof of (a), $z_1, z_2 \notin X \cup Y$. Hence by (1), $v, v^+ \notin Y$. By Lemma 2, at most one of the vertices v and v^+ is in X . We conclude that $z_1^- \vec{C} z_2v \vec{C} z_1v^+\vec{C}z_2^-$ is a hopping path. This contradiction proves one half of (b). The other half is proved similarly.

(c) Suppose $u \in U, w \in W, v, v^+ \in w^+\vec{C}u^-$ and either $uv, wv^+ \in E(G)$ or $uv^+, wv \in E(G)$. Arguing as in the proof of (b) we have $u, w \notin X \cup Y, v, v^+ \notin Y$ and $|\{v, v^+\} \cap X| \leq 1$. It follows that $u^- \vec{C} v^+w \vec{C} uv \vec{C} w^+$ (if $uv, wv^+ \in E(G)$) or $u^- \vec{C} v^+u\vec{C}wv \vec{C} w^+$ (if $uv^+, wv \in E(G)$) is a hopping path, a contradiction.

(d) This is an immediate consequence of (a), (b), and (c). ■

Set

$$t = |U| (= |W|)$$

and

$$S = V(C) \setminus (X \cup Y).$$

For $Z \subseteq V(G)$ define

$$e(Z) = \varepsilon(U \cup W, Z).$$

For $z, z' \in V(G)$ we write $e(z)$ instead of $e(\{z\})$ and $e(z, z')$ instead of $e(\{z, z'\})$.

Lemma 4.

- (a) If $x \in X$ and $x^+ \in U$, then $e(x^+) \leq t - \frac{1}{2}e(x) + 1$. If $x \in X$ and $x^- \in W$, then $e(x^-) \leq t - \frac{1}{2}e(x) + 1$.
- (b) If $x \in X, x^+ \in U, x^- \in W$ and $N(x) \cap ((U \cup W) \setminus \{x^+, x^-\}) \neq \emptyset$, then $e(x^+, x^-) \leq t - \frac{1}{2}e(x) + 2$.
- (c) If $v, v^+ \in S$, then $e(v, v^+) \leq t + 3$.

Proof. (a) It suffices to prove the first half, the second is symmetric. Assume $x \in X$ and $x^+ \in U$. If $N(x) \cap (U \cup W) \subseteq \{x^+, x^-\}$, then we are done, since $e(x^+) \leq t$ by Lemma 6(a). In the remaining case, let p be the first vertex of $N(x) \cap (U \cup W)$ on $x^{++}\vec{C}x^{--}$ and set

$$a_1 = |U \cap x^+\vec{C}p|, \quad a_2 = |U \cap p\vec{C}x^-|,$$

$$b_1 = |W \cap x^+\vec{C}p|, \quad b_2 = |W \cap p\vec{C}x^-|.$$

Then $b_1 + b_2 = t + |W \cap \{p\}|$, $e(x) \leq 1 + a_2 + b_2$ and by Lemma 3(a) and (d), $e(x^+) \leq b_1$. Hence

$$e(x^+) + \frac{1}{2}e(x) \leq b_1 + \frac{1}{2}(1 + a_2 + b_2). \quad (12)$$

We complete the proof of (a) by showing that the right-hand side of (12) is at most $t + 1$. If $p \in U$, then $a_2 = b_2$ and hence

$$b_1 + \frac{1}{2}(1 + a_2 + b_2) = b_1 + b_2 + \frac{1}{2} = t + \frac{1}{2}.$$

If $p \in W$, then $a_2 = b_2 - 1$ and hence

$$b_1 + \frac{1}{2}(1 + a_2 + b_2) = b_1 + b_2 = t + 1.$$

(b) Let x satisfy the stated conditions. First assume $e(x) = 3$. Let p be the unique vertex in $N(x) \cap ((U \cup W) \setminus \{x^+, x^-\})$ and assume without loss of generality that $p \in U$. Using Lemma 3(a) and (d) we obtain

$$\begin{aligned} e(x^+, x^-) + \frac{1}{2}e(x) &\leq |W \cap x^+\vec{C}p^-| + |U \cap p\vec{C}x^-| + \frac{3}{2} \\ &= |U \cap x^+\vec{C}p^-| + |U \cap p\vec{C}x^-| + \frac{3}{2} = t + \frac{3}{2} < t + 2. \end{aligned}$$

Now assuming $e(x) \geq 4$, let p be the first and q the last vertex of $N(x) \cap (U \cup W)$ on $x^{++}\vec{C}x^{--}$. Set

$$a_1 = |U \cap x^+\vec{C}p|, \quad a_2 = |U \cap p\vec{C}q|, \quad a_3 = |U \cap q\vec{C}x^-|,$$

$$b_1 = |W \cap x^+\vec{C}p|, \quad b_2 = |W \cap p\vec{C}q|, \quad b_3 = |W \cap q\vec{C}x^-|.$$

Then $b_1 + b_2 + b_3 = t + |W \cap \{p, q\}|$, $e(x) \leq 2 + a_2 + b_2$ and by Lemma 3(a) and (d), $e(x^+, x^-) \leq b_1 + a_3$. Hence

$$e(x^+, x^-) + \frac{1}{2}e(x) \leq b_1 + a_3 + 1 + \frac{1}{2}(a_2 + b_2). \quad (13)$$

We complete the proof of (b) by showing that the right-hand side of (13) is at most $t + 2$. If $p, q \in U$, then $a_2 = b_2 + 1$, $a_3 = b_3$ and hence

$$b_1 + a_3 + 1 + \frac{1}{2}(a_2 + b_2) = b_1 + b_2 + b_3 + \frac{3}{2} = t + \frac{3}{2}.$$

The case $p, q \in W$ is symmetric. If $p \in U, q \in W$, then $a_2 = b_2$, $a_3 = b_3 - 1$ and hence

$$b_1 + a_3 + 1 + \frac{1}{2}(a_2 + b_2) = b_1 + b_2 + b_3 = t + 1.$$

Finally, if $p \in W, q \in U$, then $a_2 = b_2, a_3 = b_3$ and hence

$$b_1 + a_3 + 1 + \frac{1}{2}(a_2 + b_2) = b_1 + b_2 + b_3 + 1 = t + 2.$$

(c) Assume $v, v^+ \in S$. We distinguish two cases.

Case 1. $N(v) \cap W = \emptyset$ or $N(v^+) \cap U = \emptyset$.

Assume without loss of generality that $N(v) \cap W = \emptyset$. If $N(v) \cap U = \emptyset$, then $e(v, v^+) = e(v^+) \leq t + 2$, otherwise v^+ would belong to X'_i for some $i \geq 2$. Thus we may assume $N(v) \cap U \neq \emptyset$. Let p be the first vertex of $N(v) \cap U$ on $v^+ \vec{C}v$. Since $N(v) \cap W = \emptyset$, we have $N(v) \cap v^+ \vec{C}p^- \cap (U \cup W) = \emptyset$. By Lemma 3(b), (c) we have $N(v^+) \cap p^+ \vec{C}v \cap (U \cup W) = \emptyset$. Also, $|N(v^+) \cap (U \cup W) \cap v^+ \vec{C}p| \leq \frac{1}{2}|(U \cup W) \cap v^+ \vec{C}p| + 2$, otherwise v^+ would belong to X . We conclude that

$$\begin{aligned} e(v, v^+) &\leq |U \cap p \vec{C}v| + \frac{1}{2}|(U \cup W) \cap v^+ \vec{C}p| + 2 \\ &= |U \cap p \vec{C}v| + |U \cap v^+ \vec{C}p| + 2 = t + 3. \end{aligned}$$

Case 2. $N(v) \cap W \neq \emptyset$ and $N(v^+) \cap U \neq \emptyset$.

Let p be the last vertex of $N(v) \cap W$ on $v^+ \vec{C}v$ and q the first vertex of $N(v^+) \cap U$ on $v^+ \vec{C}v$. (Possibly, $p = v^+$ and/or $q = v$.) Using Lemma 3(b), (c) we obtain

$$q \in p \vec{C}v,$$

$$(N(v) \cup N(v^+)) \cap U \subseteq q \vec{C}v, \quad (14)$$

$$(N(v) \cup N(v^+)) \cap W \subseteq v^+ \vec{C}p.$$

Furthermore, by Lemma 3(b),

$$\begin{aligned} \varepsilon(\{v, v^+\}, U \cap q \vec{C}v) &\leq |U \cap q \vec{C}v| + 1, \\ \varepsilon(\{v, v^+\}, W \cap v^+ \vec{C}p) &\leq |W \cap v^+ \vec{C}p| + 1. \end{aligned} \quad (15)$$

By (14), addition of the inequalities in (15) yields

$$\begin{aligned} e(v, v^+) &\leq |U \cap q \vec{C}v| + |W \cap v^+ \vec{C}p| + 2 \\ &= |W \cap q \vec{C}v| + |W \cap v^+ \vec{C}p| + 3 \leq |W| + 3 = t + 3. \quad \blacksquare \end{aligned}$$

Let Z_U be the set of all vertices $v \in S$ such that $v^+ \in S \cap N(u_1)$ and $v^- \in S \cap N(u_2)$ for some $u_1, u_2 \in U$ with $v \in u_1 \vec{C}u_2$. Define Z_W analogously.

Lemma 5.

- (a) If $u_1, u_2 \in U$, then $\varepsilon(\{u_1, u_2\}, S) \leq |S|$. If $w_1, w_2 \in W$, then $\varepsilon(\{w_1, w_2\}, S) \leq |S|$.
- (b) If $u_1, u_2, u_3 \in U$, then $\varepsilon(\{u_1, u_2, u_3\}, S) \leq |S| + t + |Z_U|$. If $w_1, w_2, w_3 \in W$, then $\varepsilon(\{w_1, w_2, w_3\}, S) \leq |S| + t + |Z_W|$.
- (c) If $z \in Z_U$ or $z \in Z_W$, then $\varepsilon(z, S) \leq \frac{1}{2}|S|$.

Proof. We only prove the assertions concerning U ; those concerning W follow similarly.

(a) Let $u_1, u_2 \in U$ and let S' be the vertex set of one of the components of $C - (X \cup Y)$. We can assume, without loss of generality, that $S' \subseteq u_2 \vec{C} u_1$. Let u be the unique vertex in $U \cap S'$ and w the unique vertex in $W \cap S'$. Now define

$$\begin{aligned} S_1 &= \{v \in S' \mid u_1 v \in E(G)\}, \\ S_2 &= \{v \in S' \setminus \{w\} \mid u_2 v^+ \in E(G)\}. \end{aligned}$$

By Lemma 3(b) we have $S_1 \cap S_2 = \emptyset$. Using Lemma 3(a) this gives

$$\varepsilon(\{u_1, u_2\}, S') = |S_1| + |S_2| \leq |S'|.$$

Summing over all components of $C - (X \cup Y)$ we obtain $\varepsilon(\{u_1, u_2\}, S) \leq |S|$.

(b) Let $u_1, u_2, u_3 \in U$ and let S' be the vertex set of one of the components of $C - (X \cup Y)$. We can assume, without loss of generality, that $u_2 \in u_1 \vec{C} u_3$ and $S' \subseteq u_3 \vec{C} u_1$. Let u be the unique vertex in $U \cap S'$ and w the unique vertex in $W \cap S'$. Now define

$$\begin{aligned} S_1 &= \{v \in (S' \cup \{w^+\}) \setminus \{u\} \mid u_1 v^- \in E(G)\}, \\ S_2 &= \{v \in S' \mid u_2 v \in E(G)\}, \\ S_3 &= \{v \in S' \setminus \{w\} \mid u_3 v^+ \in E(G)\}. \end{aligned}$$

By Lemma 3(b) we have $S_1 \cap S_2 = S_2 \cap S_3 = \emptyset$. Also, by definition, $S_1 \cap S_3 \subseteq Z_U$. Using Lemma 3(a) this gives

$$\varepsilon(\{u_1, u_2, u_3\}, S') = |S_1| + |S_2| + |S_3| \leq |S'| + 1 + |Z_U \cap S'|.$$

Summing over all components of $C - (X \cup Y)$ we obtain $\varepsilon(\{u_1, u_2, u_3\}, S) \leq |S| + t + |Z_U|$.

(c) Let $z \in Z_U$ and suppose $u_1, u_2 \in U$ such that $z \in u_1 \vec{C} u_2$, $z^+ \in S \cap N(u_1)$ and $z^- \in S \cap N(u_2)$. By Lemma 3(b), $\varepsilon(z, U) = 0$. Furthermore, there exists no pair of vertices $v, v^+ \in S$ such that $zv, zv^+ \in E(G)$, otherwise there would exist a hopping path. For example, if $v, v^+ \in S \cap u_2 \vec{C} u_1$ and $zv, zv^+ \in E(G)$, then $u_1^- \vec{C} v^+ zv \vec{C} u_2 z^- \vec{C} u_1 z^+ \vec{C} u_2^-$ is a hopping path. We conclude that $\varepsilon(z, S) \leq \frac{1}{2}|S|$. ■

Recall that so far C was only assumed not to be a -extendable. In our next lemma we make a stronger assumption about C . Set

$$R = V(G) \setminus (V(C) \cup \{a\}).$$

Lemma 6. Assume C is not extendable. If $z_1, z_2 \in X^+$ or $z_1, z_2 \in X^-$, then there exists no (z_1, z_2) -path with all internal vertices in R .

Proof. Assume C is not extendable and, without loss of generality, $z_1, z_2 \in X^+$. Suppose z_1 and z_2 are joined by a path with all internal vertices in R . Set $G' = G + z_1 z_2$. Since C is not extendable in G , C is not a -extendable in G' . Define X', Y' , and U' for G' in the same way as X, Y , and U were defined for G . Clearly, $X \subseteq X'$. Hence by Lemma 2 (applied to G'), $z_1, z_2 \notin X'$. Since $z_1 z_2 \in E(G')$ and $N_{G'}(Y') \cap V(C) \subseteq X'$, it also follows from Lemma 2 that $z_1, z_2 \notin Y'$. But then $z_1, z_2 \in U'$, contradicting Lemma 3(a). ■

We finally note that, *mutatis mutandis*, the above results obviously remain valid if, instead of just a vertex a outside C , a subset A of $V(G)$ is considered such that $A \cap V(C) = \emptyset$ and A induces a connected subgraph of G . The condition that C is not a -extendable is then replaced by the requirement that there exists no extension of C whose vertex set is contained in $V(C) \cup A$, while X_1 is defined as $N(A) \cap V(C)$ and R as $V(G) \setminus (V(C) \cup A)$.

3. PROOF OF THEOREM 6

Using the results of Section 2 we now prove Theorem 6.

Proof of Theorem 6. Let G be a k -regular graph of order n such that G contains a dominating cycle and

$$n \leq \frac{7}{2}k - 7. \quad (16)$$

By Dirac's Theorem [6], G is hamiltonian if $n \leq 2k$. Thus we may assume

$$k \geq 6. \quad (17)$$

Assuming G is nonhamiltonian, let $\vec{C}$ be a nonextendable dominating cycle of G and a be a vertex in $V(G) \setminus V(C)$. Define $X, Y, U, W, t, S, Z_U, Z_W, R$ and the function $e: 2^{V(G)} \rightarrow \mathbb{N}$ as in Section 2. Additionally define

$$\begin{aligned} X_{YY} &= \{v \in X | v^-, v^+ \in Y\}, \\ X_{WY} &= \{v \in X | v^- \in W, v^+ \in Y\}, \\ X_{YU} &= \{v \in X | v^- \in Y, v^+ \in U\}, \\ X_{WU} &= \{v \in X | v^- \in W, v^+ \in U\}, \\ X_0 &= \{v \in X_{UU} | e(v) = 2\}. \end{aligned}$$

Set $x = |X|, y = |Y|, s = |S|, r = |R|, x_{YY} = |X_{YY}|, x_{WY} = |X_{WY}|, x_{YU} = |X_{YU}|, x_{WU} = |X_{WU}|$ and $x_0 = |X_0|$.

We make a few observations that will be used repeatedly. Clearly,

$$x \geq |N(a) \cap V(C)| = |N(a)| = k, \quad (18)$$

$$t = x - y, \quad (19)$$

and by (16),

$$s = n - r - x - y - 1 \leq \frac{7}{2}k - r - x - y - 8. \quad (20)$$

By Lemma 6 and the fact that C is a dominating cycle,

$$\varepsilon(Y \cup \{a\}, R) \leq r.$$

Hence by (1) and the fact that G is k -regular,

$$\varepsilon(Y \cup \{a\}, X) \geq k(y + 1) - r. \quad (21)$$

Let X' be a subset of X , of cardinality x' . Clearly,

$$\varepsilon(Y \cup \{a\}, X \setminus X') \leq (x - x')(y + 1).$$

Hence by (21),

$$\varepsilon(Y \cup \{a\}, X') \geq (k - x + x')(y + 1) - r,$$

implying that

$$e(X') \leq kx' - (k - x + x')(y + 1) + r. \quad (22)$$

Since $x_{YU} = x_{WY} = y - x_{YY}$, we have

$$x_{WU} = x - x_{YY} - x_{YU} - x_{WY} = x - 2y + x_{YY}, \quad (23)$$

implying that

$$x_{YY} = 2y - x + x_{WU} \geq 2y - x. \quad (24)$$

We now distinguish several cases and subcases and derive contradictions in each of them.

Case 1. $t \geq 5$.

Using Lemma 4(a), (b), (c) and observing that $x_{WY} = x_{YU} = t - x_{WU}$ we obtain

$$\begin{aligned} 2e(S) &= 2 \sum_{v \in S} e(v) = \sum_{v \in X_{WU} \setminus X_0} e(v^-, v^+) + \sum_{v \in X_0} e(v^-) + \sum_{v \in X_0} e(v^+) \\ &\quad + \sum_{v \in X_{WY}} e(v^-) + \sum_{v \in X_{YU}} e(v^+) + \sum_{v, v^+ \in S} e(v, v^+) \\ &\leq \sum_{v \in X_{WU} \setminus X_0} (t - \tfrac{1}{2}e(v) + 2) + 2x_0t + \sum_{v \in X_{WY}} (t - \tfrac{1}{2}e(v) + 1) \\ &\quad + \sum_{v \in X_{YU}} (t - \tfrac{1}{2}e(v) + 1) + (s - t)(t + 3) = (x_{WU} - x_0)(t + 2) \\ &\quad + 2x_0t + 2(t - x_{WU})(t + 1) + (s - t)(t + 3) - \tfrac{1}{2}e(X \setminus (X_0 \cup X_{YY})) \\ &= t(t - x_{WU} + x_0 + s - 1) - 2x_0 + 3s - \tfrac{1}{2}e(X \setminus (X_0 \cup X_{YY})). \end{aligned} \quad (25)$$

By (1), $e(Y) = 0$ and by Lemma 6, $e(R) \leq 2r$. Hence by (25),

$$\begin{aligned} e(X) &= 2tk - e(S) - e(R) \geq \tfrac{1}{2}t(4k - t + x_{WU} - x_0 - s + 1) \\ &\quad + x_0 - \tfrac{3}{2}s - 2r + \tfrac{1}{4}e(X \setminus (X_0 \cup X_{YY})). \end{aligned} \quad (26)$$

Since

$$\begin{aligned} e(X) &= e(X_0) + e(X_{YY}) + e(X \setminus (X_0 \cup X_{YY})) \\ &= 2x_0 + e(X_{YY}) + e(X \setminus (X_0 \cup X_{YY})), \end{aligned}$$

(26) implies

$$e(X_{YY}) + \tfrac{3}{4}e(X \setminus (X_0 \cup X_{YY})) \geq \tfrac{1}{2}t(4k - t + x_{WU} - x_0 - s + 1) - x_0 - \tfrac{3}{2}s - 2r. \quad (27)$$

On the other hand, by (22),

$$\begin{aligned}
& e(X_{YY}) + \frac{3}{4}e(X \setminus (X_0 \cup X_{YY})) \\
&= \frac{1}{4}e(X_{YY}) + \frac{3}{4}e(X \setminus X_0) \leq \frac{1}{4}(kx_{YY} - (k - x + x_{YY})(y + 1) + r) \quad (28) \\
&+ \frac{3}{4}(k(x - x_0) - (k - x_0)(y + 1) + r) = \frac{1}{4}(k(x_{YY} + 3x - 3x_0) \\
&- (4k - x + x_{YY} - 3x_0)(y + 1) + 4r).
\end{aligned}$$

Combining (27) and (28) we obtain

$$\begin{aligned}
& 2t(4k - t + x_{WU} - x_0 - s + 1) - 4x_0 - 6s - 8r \\
& \leq k(x_{YY} + 3x - 3x_0) - (4k - x + x_{YY} - 3x_0)(y + 1) + 4r. \quad (29)
\end{aligned}$$

By (20), the inequality (29) remains valid if we replace s by $\frac{7}{2}k - r - x - y - 8$. Doing so and eliminating t and x_{WU} via (19) and (23), we obtain

$$\begin{aligned}
& x_0(2x + y - 3k + 7) - x_{YY}(2x - y - k + 1) - 2x^2 + 3xy \\
& + 2kx - 3ky - 2rx + 2ry - 23x + 12y + 17k + 6r - 48 \geq 0. \quad (30)
\end{aligned}$$

Since $t \geq 5$ by hypothesis, we have

$$-2rx + 2ry + 6r = -2rt + 6r \leq -4r \leq 0.$$

Hence by (30),

$$\begin{aligned}
& x_0(2x + y - 3k + 7) - x_{YY}(2x - y - k + 1) - 2x^2 + 3xy + 2kx \\
& - 3ky - 23x + 12y + 17k - 48 \geq 0. \quad (31)
\end{aligned}$$

Case 1.1. $2x + y - 3k + 7 > 0$.

Clearly, $x_0 \leq x_{WU}$. Hence by (23), in (31) we may replace x_0 by $x - 2y + x_{YY}$ to obtain

$$-2x_{YY}(k - y - 3) - 2y^2 - kx + 3ky - 16x - 2y + 17k - 48 \geq 0. \quad (32)$$

Case 1.1.1. $y \leq \frac{1}{2}k$.

Then by (17), $k - y - 3 \geq 0$. Hence by (32),

$$-2y^2 - kx + 3ky - 16x - 2y + 17k - 48 \geq 0. \quad (33)$$

Since $y \leq \frac{1}{2}k$, the left-hand side of (33) is an increasing function of y . Hence in (33) we may replace y by $\frac{1}{2}k$ to obtain

$$-k(x - k) - 16(x - k) - 48 \geq 0,$$

contradicting (18).

Case 1.1.2. $\frac{1}{2}k < y < k - 3$.

Then by (24) and (32),

$$-2(2y - x)(k - y - 3) - 2y^2 - kx + 3ky - 16x - 2y + 17k - 48 \geq 0, \quad (34)$$

or equivalently,

$$x - y \leq \frac{17k - 12y - 48}{2y - k + 22}. \quad (35)$$

Since $y > \frac{1}{2}k$, (35) implies

$$x - y < \frac{11k - 48}{22} < \frac{1}{2}k.$$

Hence by (19) and the hypothesis of Case 1,

$$5 \leq t < \frac{1}{2}k. \quad (36)$$

By (18) and (19), $y = x - t \geq k - t$, so (35) also implies

$$t \leq \frac{17k - 12(k - t) - 48}{2(k - t) - k + 22},$$

or equivalently,

$$(t - 5)(2t - k) - 48 \geq 0,$$

contradicting (36).

Case 1.1.3. $y \geq k - 3$.

If $x_{YY} = y$, then $x = y$ and $t = 0$. This contradicts the hypothesis of Case 1, so we must have $x_{YY} \leq y - 1$. Hence by (32),

$$-2(y - 1)(k - y - 3) - 2y^2 - kx + 3ky - 16x - 2y + 17k - 48 \geq 0,$$

or equivalently,

$$x - y \leq \frac{19k - 14y - 54}{k + 16}. \quad (37)$$

Since $y \geq k - 3$, (37) implies

$$t = x - y \leq \frac{5k - 12}{k + 16} < 5,$$

contradicting the hypothesis of Case 1.

Case 1.2. $2x + y - 3k + 7 \leq 0$.

Then by (31),

$$-x_{YY}(2x - y - k + 1) - 2x^2 + 3xy + 2kx - 3ky - 23x + 12y + 17k - 48 \geq 0, \quad (38)$$

or equivalently,

$$-x_{YY}(2x - y - k + 1) + (x - k)(3y - 2x) - 23x + 12y + 17k - 48 \geq 0. \quad (39)$$

Case 1.2.1. $y \leq \frac{1}{2}k$.

Then from (18) and (39) we deduce that

$$0 \leq -23x + 12y + 17k - 48 \leq -23x + 23k - 48 < 0,$$

a contradiction.

Case 1.2.2. $y > \frac{1}{2}k$.

Clearly, $y \leq x$, implying that

$$2x - y - k + 1 \geq x - k + 1 > 0.$$

Hence by (24) and (38),

$$\begin{aligned} & -(2y - x)(2x - y - k + 1) - 2x^2 + 3xy + 2kx - 3ky - 23x \\ & + 12y + 17k - 48 \geq 0, \end{aligned}$$

which is equivalent to (34). We obtain a contradiction as in Case 1.1.2. (Note that after (34) the assumption $y < k - 3$ in the hypothesis of Case 1.1.2 is no longer used.)

Case 2. $t \leq 4$.

Let $S_1, \dots, S_t$ be the vertex sets of the components of $C - (X \cup Y)$, so that $S = \cup_{i=1}^t S_i$. For $i = 1, \dots, t$, let u_i be the (unique) vertex of U in S_i and w_i the vertex of W in S_i .

Assume $v \in R$. Since C is a nonextendable dominating cycle, $N(v) \subseteq V(C)$ and $N(v) \cap N(v)^+ = \emptyset$. Furthermore, by Lemma 6, there is at most one value of i for which $N(v) \cap S_i$ contains both u_i and w_i . Hence by (18), (19), (20) and the hypothesis of Case 2,

$$\begin{aligned} \varepsilon(v, S) & \leq \frac{1}{2}(s + 1) \leq \frac{1}{2} \left(\frac{7}{2}k - r - x - y - 7 \right) \\ & = \frac{1}{2} \left(\frac{7}{2}k - r - 2x + t - 7 \right) \leq \frac{1}{2} \left(\frac{3}{2}k - r - 3 \right). \end{aligned} \quad (40)$$

By Lemma 6, $\varepsilon(v, Y) \leq 1$. Hence by (40),

$$\varepsilon(v, X) \geq k - \frac{1}{2} \left(\frac{3}{2}k - r - 3 \right) - 1 = \frac{1}{4}(k + 2r + 2),$$

implying that

$$\varepsilon(R, X) \geq \frac{1}{4}r(k + 2r + 2). \quad (41)$$

Case 2.1. $t \leq 1$.

By (21),

$$\varepsilon(R, X) \leq kx - k(y + 1) + r = k(t - 1) + r. \quad (42)$$

Since $t \leq 1$, (41) and (42) together imply that $r = 0$. Hence if $t = 0$, then (42) gives a contradiction right away. If $t = 1$, then by (21),

$$kx \geq \varepsilon(Y \cup \{a\}, X) + \varepsilon(\{u_1, w_1\}, X) \geq k(y + 1) + 2 = kx + 2,$$

again a contradiction.

Case 2.2. $t \in \{2, 3, 4\}$.

Using Lemmas 5(a) and 6, (18), (19), and (20) we obtain

$$\varepsilon(\{u_1, u_2\}, S \cup R) \leq s + r \leq \frac{7}{2}k - 2x + t - 8 \leq \frac{3}{2}k + t - 8,$$

implying that

$$\varepsilon(\{u_1, u_2\}, X) \geq 2k - \left(\frac{3}{2}k + t - 8 \right) = \frac{1}{2}k + 8 - t. \quad (43)$$

Similarly,

$$\varepsilon(\{w_1, w_2\}, X) \geq \frac{1}{2}k + 8 - t. \quad (44)$$

Using (19), (21), (41), (43), and (44) we obtain

$$\begin{aligned} kx &\geq \varepsilon(Y \cup \{a\}, X) + \varepsilon(R, X) + \varepsilon(\{u_1, u_2, w_1, w_2\}, X) \\ &\geq k(y + 1) - r + \frac{1}{4}r(k + 2r + 2) + k + 16 - 2t \\ &= k(x - t + 2) + \frac{1}{4}r(k + 2r - 2) + 16 - 2t. \end{aligned} \quad (45)$$

If $t = 2$ this gives a contradiction and hence we have $t \in \{3, 4\}$.

For $t = 3$ we have by (45),

$$kx \geq k(x - 1) + \frac{1}{4}r(k + 2r - 2) + 10,$$

and for $t = 4$ we obtain, again using (19), (21), (41), (43), and (44),

$$\begin{aligned} kx &\geq \varepsilon(Y \cup \{a\}, X) + \varepsilon(R, X) + \varepsilon(\{u_1, u_2, u_3, u_4, w_1, w_2, w_3, w_4\}, X) \\ &\geq k(y + 1) - r + \frac{1}{4}r(k + 2r + 2) + 2k + 32 - 4t \\ &= k(x - 1) + \frac{1}{4}r(k + 2r - 2) + 16. \end{aligned}$$

In both cases we conclude that

$$r \leq 3. \quad (46)$$

Using Lemmas 5(b) and 6, (18), (19), and (20) we obtain

$$\begin{aligned} \varepsilon(\{u_1, u_2, u_3\}, S \cup R) &\leq s + t + |Z_U| + r \\ &\leq \frac{7}{2}k - 2x + 2t - 8 + |Z_U| \leq \frac{3}{2}k + |Z_U|, \end{aligned}$$

implying that

$$\varepsilon(\{u_1, u_2, u_3\}, X) \geq 3k - \left(\frac{3}{2}k + |Z_U|\right) = \frac{3}{2}k - |Z_U|. \quad (47)$$

Similarly,

$$\varepsilon(\{w_1, w_2, w_3\}, X) \geq \frac{3}{2}k - |Z_W|. \quad (48)$$

Assume $z \in Z_U \cup Z_W$. Using Lemma 5(c), (18), (19), (20), and (46) we obtain

$$\varepsilon(z, S \cup R) \leq \frac{1}{2}s + r \leq \frac{1}{2}\left(\frac{7}{2}k + r - 2x + t - 8\right) \leq \frac{1}{2}\left(\frac{3}{2}k + r - 4\right) \leq \frac{1}{4}(3k - 2),$$

implying that

$$\varepsilon(Z_U \cup Z_W, X) \geq |Z_U \cup Z_W| \left(k - \frac{1}{4}(3k - 2)\right) = \frac{1}{4}|Z_U \cup Z_W|(k + 2). \quad (49)$$

Set

$$T = V(G) \setminus (X \cup Y \cup Z_U \cup Z_W \cup R \cup \{a, u_1, u_2, u_3, w_1, w_2, w_3\}).$$

If $t = 4$, then $u_4, w_4 \in T$, implying that $\varepsilon(T, X) \geq 2$. Hence for both $t = 3$ and $t = 4$,

$$\varepsilon(T, X) \geq 2(t - 3). \quad (50)$$

From (17), (21), (41), (47), (48), (49), and (50) we obtain

$$\begin{aligned}
 kx &\geq \varepsilon(Y \cup \{a\}, X) + \varepsilon(R, X) + \varepsilon(\{u_1, u_2, u_3\}, X) + \varepsilon(\{w_1, w_2, w_3\}, X) \\
 &\quad + \varepsilon(Z_U \cup Z_W, X) + \varepsilon(T, X) \geq k(y+1) - r + \frac{1}{4}r(k+2r+2) + 3k \\
 &\quad - |Z_U| - |Z_W| + \frac{1}{4}|Z_U \cup Z_W|(k+2) + 2(t-3) \geq k(y+1) + 3k + 2(t-3) \\
 &= kx + k(4-t) + 2(t-3) > kx,
 \end{aligned}$$

our final contradiction. ■

4. PROOF OF THEOREM 7

For the proof of Theorem 7 we need some additional terminology and notation.

Let λ be a positive integer. A cycle C of a graph G is called a D_λ -cycle if every component of $G - V(C)$ has order smaller than λ . Hence a D_1 -cycle is a Hamilton cycle and a D_2 -cycle is a dominating cycle.

Let X, X_1, X_2 be subgraphs of G . By $N(X)$ we denote the set $\{v \in V(G) \setminus V(X) \mid \varepsilon(v, X) > 0\}$. The subgraphs X_1 and X_2 are called *remote* if $V(X_1) \cap V(X_2) = N(X_1) \cap V(X_2) = \emptyset$. The number of components of X with at least λ vertices is denoted by $\omega_\lambda(X)$.

If $\vec{C}$ is a cycle of G and $v \in V(C)$, then a subgraph X of G is called a $(\vec{C}, v, \lambda)$ -subgraph if each of the following conditions is satisfied:

- (i) X is connected and has order λ ;
- (ii) $V(X) \cap V(C) = v\vec{C}w$ for some vertex $w \in V(C)$;
- (iii) if X' satisfies (i) and (ii), then $V(X) \cap V(C) \subseteq V(X') \cap V(C)$.

Proof of Theorem 7. Let G be a 2-connected k -regular graph of order $n \leq 4k - 3$ and assume G contains no dominating cycle. Set

$$\lambda + 1 = \min\{i \mid G \text{ has a } D_i\text{-cycle}\},$$

so that $\lambda \geq 2$. Let $\vec{C}$ be a $D_{\lambda+1}$ -cycle of G for which $\omega_\lambda(G - V(C))$ is minimum. Since G has no D_λ -cycle, $G - V(C)$ has a component X_0 of order λ . Let $x_1, \dots, x_m$ be the vertices in $N(X_0)$, occurring on $\vec{C}$ in the order of their indices. As in the proof of [11, Theorem 2] there exists, for $i = 1, \dots, m$, a $(\vec{C}, x_i^+, \lambda)$ -subgraph X_i such that $X_0, X_1, \dots, X_m$ are pairwise remote. Set $R = V(G) \setminus \bigcup_{i=0}^m V(X_i)$. Since G is k -regular and $X_0, X_1, \dots, X_m$ are pairwise remote, we have

$$\varepsilon(v, R) \geq k - \lambda + 1 \quad \text{for all } v \in \bigcup_{i=0}^m V(X_i). \quad (51)$$

We distinguish two cases. In the first case we reach a contradiction, while in the second we conclude that $G \in \mathcal{F}$.

Case 1. $2 \leq \lambda \leq k - 2$.

By (51), $m \geq k - \lambda + 1$ and

$$k|R| \geq \varepsilon\left(\bigcup_{i=0}^m V(X_i), R\right) \geq (m+1)\lambda(k - \lambda + 1) \geq (k - \lambda + 2)\lambda(k - \lambda + 1),$$

implying that

$$k(4k-3) \geq kn = k(m+1)\lambda + k|R| \geq \lambda(k-\lambda+2)(2k-\lambda+1). \quad (52)$$

Set $\alpha = \lambda - 2$ and $\beta = k - \lambda - 2$. Then $\lambda = \alpha + 2$ and $k = \alpha + \beta + 4$, so

$$\begin{aligned} & \lambda(k-\lambda+2)(2k-\lambda+1) - k(4k-3) \\ &= (\alpha+2)(\beta+4)(\alpha+2\beta+7) - (\alpha+\beta+4)(4\alpha+4\beta+13) \\ &= \alpha^2\beta + 2\alpha\beta^2 + 9\alpha\beta + 7\alpha + \beta + 4 > 0, \end{aligned}$$

contradicting (52).

Case 2. $\lambda \geq k - 1$.

Since $\{x_1, \dots, x_m\} \subseteq R$, we have

$$4k-3 \geq n \geq (m+1)\lambda + |R| \geq (m+1)\lambda + m. \quad (53)$$

If $m \geq 3$, then by (53), $4k-3 \geq 4(k-1) + 3$, a contradiction. Since G is 2-connected, we conclude that

$$m = 2. \quad (54)$$

If $\lambda = k - 1$, then by (51),

$$\begin{aligned} 2k &\geq \varepsilon(\{x_1, x_2\}, V(X_0) \cup \{x_1^-, x_1^+, x_2^-, x_2^+\}) \geq \varepsilon(V(X_0), \{x_1, x_2\}) + 4 \\ &\geq \lambda(k-\lambda+1) + 4 = 2k + 2, \end{aligned}$$

a contradiction. Hence

$$\lambda \geq k. \quad (55)$$

By (53), (54), and (55),

$$4k-3 \geq 3\lambda + 2 \geq \lambda + 2k + 2,$$

implying that $|V(X_0)| = \lambda \leq 2k - 5$. Since $\delta(X_0) \geq k - m = k - 2$, it follows that $\delta(X_0) \geq \frac{1}{2}(|V(X_0)| + 1)$. Hence X_0 is Hamilton-connected, i.e., every pair of vertices in X_0 is joined by a Hamilton path of X_0 (see, e.g., [1]). Thus there exists a cycle of G that properly contains all vertices of X_0 ; in each such cycle C' , the subgraph of C' induced by the vertices of X_0 is a path. Now set

$$\mu + 1 = \min\{i | G \text{ has a } D_i\text{-cycle containing } V(X_0)\}.$$

Clearly, $\mu \geq \lambda$. Among all $D_{\mu+1}$ -cycles containing $V(X_0)$, let $\vec{C}_1$ be one for which $\omega_\mu(G - V(C_1))$ is minimum. Let Y_0 be a component of $G - V(C_1)$ of order μ and $y_1, \dots, y_p$ the vertices of $N(Y_0)$, occurring on $\vec{C}_1$ in the order of their indices such that $V(X_0) \subseteq y_1^+ \vec{C}_1 y_2^-$. By the argument in the proof of [11, Theorem 2] there exists, for $i = 2, \dots, p$, a $(\vec{C}_1, y_i^+, \mu)$ -subgraph Y_i such that $Y_0, Y_2, \dots, Y_p$ are pairwise remote. Hence

$$4k-3 \geq n \geq |V(X_0)| + |V(Y_0)| + \left| \bigcup_{i=2}^p V(Y_i) \right| + |N(Y_0)| = \lambda + p\mu + p.$$

Arguing as for X_0 we obtain $|V(Y_0)| = \mu \leq 2k - 5$ and $\delta(Y_0) \geq k - p = k - 2$, implying that Y_0 is Hamilton-connected. Thus there exists a cycle of G containing $V(X_0) \cup V(Y_0)$. Set

$$\nu + 1 = \min\{i | G \text{ has a } D_i\text{-cycle containing } V(X_0) \cup V(Y_0)\}.$$

Clearly, $\nu \geq \mu$. Among all $D_{\nu+1}$ -cycles containing $V(X_0) \cup V(Y_0)$, let $\vec{C}_2$ be one for which $\omega_\nu(G - V(C_2))$ is minimum. Let Z_0 be a component of $G - V(C_2)$ of order ν and $z_1, \dots, z_q$ the vertices in $N(Z_0)$, occurring on $\vec{C}_2$ in the order of their indices such that $V(X_0) \subseteq z_1^+ \vec{C}_2 z_2^-$. If $z_i^+ \vec{C}_2 z_{i+1}^- \cap V(Y_0) = \emptyset$ for some $i \in \{2, \dots, q\}$ (indices mod q), then there exists a $(\vec{C}_2, z_i^+, \nu)$ -subgraph Z_i such that Z_0 and Z_i are remote, implying that

$$4k - 3 \geq n > |V(X_0)| + |V(Y_0)| + |V(Z_0)| + |V(Z_i)| = \lambda + \mu + 2\nu \geq 4k,$$

a contradiction. It follows that $q = 2$ and $V(Y_0) \subseteq z_2^+ \vec{C}_2 z_1^-$. Set $S = \{x_1, x_2, y_1, y_2, z_1, z_2\}$ and $T = V(G) \setminus (V(X_0) \cup V(Y_0) \cup V(Z_0) \cup S)$. (Note that $\{x_1, x_2\}, \{y_1, y_2\}, \{z_1, z_2\}$ need not be pairwise disjoint; the three sets may even coincide.) We have

$$|T| = n - \lambda - \mu - \nu - |S| \leq 4k - 3 - 3k - |S| = k - |S| - 3,$$

implying that

$$|T||S| \geq \varepsilon(T, S) = \varepsilon(T, V(G) \setminus T) \geq |T|(k - |T| + 1) \geq |T|(|S| + 4).$$

It follows that $T = \emptyset$. Since Z_0 , like X_0 and Y_0 , is Hamilton-connected while G is non-hamiltonian, we now easily conclude that $G \in \mathcal{F}$. ■

Remark. It follows from the remark after Theorem 3 in Section 1 that Theorem 7 is close to the best possible result. We did not attempt to increase the bound on $|V(G)|$ given in Theorem 7 beyond $4k - 3$ in this paper, since the present result is already more than is needed for our main result, Theorem 5, and its proof is relatively simple. Using additional and more complicated arguments, the fourth author has now convinced himself that, for k sufficiently large, the bound $4k - 3$ in Theorem 7 can be improved to $4k + 3$. The resulting theorem is best possible: for $k \geq 4$, the graph $\overline{K}_3 \vee (K_{k-2} + 3K_{k+1})$ of order $4k + 4$ has a (unique) 2-connected (but not 3-connected) k -regular spanning subgraph, which contains no dominating cycle and does not belong to $\mathcal{F}$.

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