

# Period doubling solitons: yes or no?

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We investigate the possibility of period doubling transition to temporal chaos of a coherent structure in a finite dimensional Hamiltonian system with moderately small friction. A solitary wave in a periodically driven chain of particles provides a typical example. To this end the equations of motion are transformed to those of one driven dominant oscillator with one degree of freedom, weakly coupled to a set of oscillators representing the other degrees of freedom. The equations for the dominant oscillator alone confirm the possibility of such a transition to chaos for well chosen driving. However, taking into account coupling to the other degrees of freedom, one must conclude that the supposed infinite period doubling sequence for small or no damping survives only in a heavily damaged state. In practice a reproducible completed sequence cannot be expected unless the damping is strong enough.

## 1. Introduction

In numerical [1, 2] and real experiments [3] on systems which carry a coherent state, often the first few steps of a period doubling sequence and a resulting chaotic state are observed. This suggests that infinite period doubling sequences of such coherent states are possible. The fact that the motion remains spatially coherent indicates that the system behaves low dimensional in some sense. The systems in these experiments are dissipative, which may account for the contraction to a low dimensional manifold in phase space. This supposedly explains the low dimensional aspect of the phenomenon, and in particular the agreement with universal behavior as found in the logistic map [4].

In this paper we consider such a transition to chaos in a case in which the low dimensional behavior is not at all clear a priori because there is no or only small dissipation. In fact we investigate a solitary wave in a periodically driven chain of identical particles with friction, which yields a typical example of the present phenomenon. Our aim is twofold: to investigate the possibility of an infinite period doubling sequence by a theoretical analysis of the equations of motion and at the same time to clarify the observed low dimensional aspect.

The set up of this paper is as follows. In section 2 we briefly describe a numerical experiment showing the period doubling phenomenon. In this experiment a longitudinal soliton in a chain bounces back and forth between the two ends of a chain. One end is fixed. The other end moves periodically in time in a prescribed way. The particles are forced by a nearest neighbor interaction potential, which is time independent, and by friction linear in the momentum of each particle. The driving is realized by the prescribed periodic motion of one end of the chain, which depends on a “strength” parameter. From the behavior of the power spectrum we observe for increasing parameter a period doubling sequence, up to period  $2^5$ , and finally a chaotic state. Furthermore direct inspection of the motion shows that the solitary wave behavior remains intact. The period doublings and the resulting chaos are found e.g. in the times of collision at the fixed end. As a whole the motion is very similar to that of a ball bouncing between two walls (the so called Fermi experiment [5]).

In section 3 we obtain approximate equations of motion. The procedure to obtain them is based on the fact that the complete equations consist of three terms: one Hamiltonian nondriven term, one time dependent Hamiltonian part and one friction term. For the phenomenon and its expla-

nation it is essential that the first part has a solution family representing an undisturbed soliton which is spectrally stable. Translated in geometrical terms: the Hamiltonian flow corresponding to this first part has a two dimensional invariant surface which consists of periodic orbits smoothly connected to each other by for instance the energy of each orbit. Furthermore the trajectories we are interested in of the full system remain near this invariant surface. Such trajectories in the neighborhood of this surface are described by choosing a proper set of coordinates: two coordinates giving a projection on the invariant surface and  $2N - 2$  coordinates transversal to the surface ( $N$  being the number of particles in the chain, and thus equal to the number of degrees of freedom). In these coordinates the full equations of motion can be interpreted as those of a driven oscillator with one degree of freedom, representing the motion along the surface, coupled to a set of  $N - 1$  oscillators for the motion transversal to the surface. Assuming that the transversal oscillators are excited only slightly, the one degree of freedom oscillator is dominant which is consistent with the observed behavior.

In section 4 we interpret the observed period doubling in terms of these equations in more detail. The first observation is that neglecting the transversal oscillators the equations reduce to that of one driven damped nonlinear oscillator with one degree of freedom, which obviously can show a period doubling transition to chaos for properly chosen driving. This seems to be consistent with the observed phenomenon. At the same time, however, it becomes clear that for no or small enough friction the coupling to the transversal degrees of freedom will affect the period doubling process a great deal. This is the consequence of the Hamiltonian character (or near Hamiltonian, due to small friction) of the system.

The effect of coupling between dominant and transversal oscillators can be described with the aid of the formalism of Krein signatures for Hamiltonian systems [6, 7] which can be extended

to include friction [8]. This is done in section 5. Assuming that the dominant oscillator has a period doubling sequence, we show that the coupling inevitably leads to a heavily damaged sequence in the full systems.

Some practical consequences are summarized in section 6. The main conclusion is that the period doubling transition to chaos for increasing driving parameter can exist for small or no friction. However, an interval between two bifurcation points can have “holes” in which the period  $2^q$  orbit is unstable, and even the bifurcation from period  $2^q$  to period  $2^{q+1}$  can be affected. Such events will occur infinitely often in the sequence. Moreover, since the “holes” of instability are accompanied by Hopf bifurcation of tori, one consequence is that the sequence due to hysteresis can be invisible in a real experiment in which the driving parameter is changed adiabatically.

Concerning the low dimensionality of the observed behavior, the analysis shows that the some gross features of the phenomenon can be described indeed by a low dimensional model (the dominant oscillator). It remains open, however, whether the resulting chaotic state has a low dimension. The present representation of the equations of motion however seem to give a good starting point to investigate this problem.

## 2. An example: period doubling of a bouncing soliton in a chain

Consider a one dimensional chain of  $N + 2$  identical particles with nearest neighbor interaction Lennard-Jones type. At rest the particles are at unit distance. The deviation from the equilibrium position of the  $i$ th particle is denoted by  $q_i$  so that the equations of motion are

$$\frac{d}{dt} [p, q]^T = \left[ -\frac{\partial U}{\partial q}, p \right]^T - \lambda [p, 0]^T, \quad (2.1)$$

$$q_0 = r_0(t), \quad q_{N+1} = 0,$$

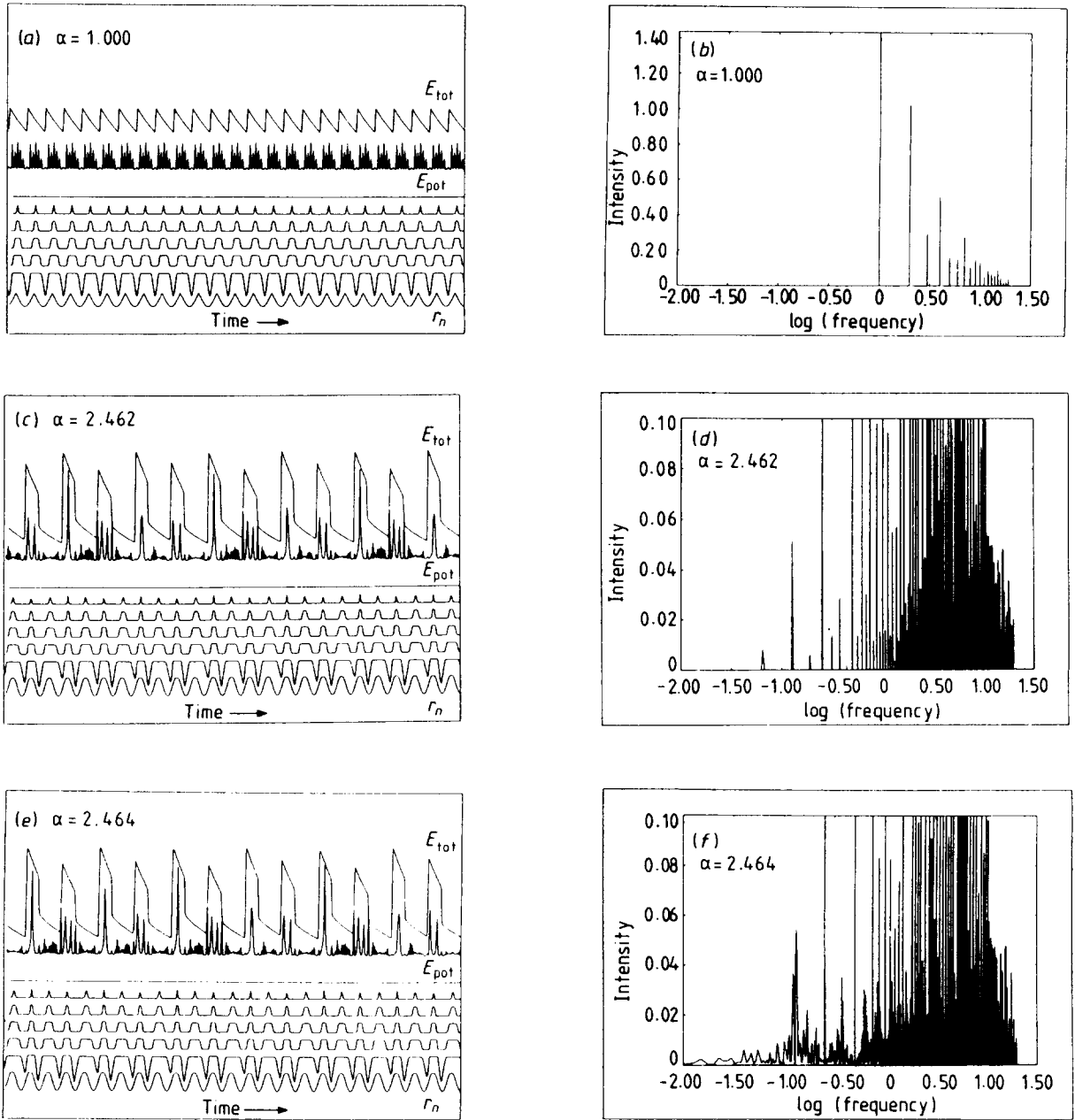


Fig. 3. Data from ref. [2] for a driven damped chain. The interaction potential is a 6-12 Lennard-Jones potential given by  $V(x) = (1+x)^{-12} - 2(1+x)^{-6}$ . The driving frequency and friction coefficient  $\nu = 10$  and  $\lambda = 0.3\nu$  respectively. Left:  $r_n = n + q_n$  versus time for different values of  $\alpha$ . Upper curves represent total and potential energy of the chain. Right: discrete Fourier spectra of the velocity of the first particle. Shown are a period one, a period 16 and a time chaotic motion. Observe in particular that the spatial coherence of the motion is preserved in the three cases. The graphs of the total energy show clearly the decrease due to friction and the increase due to the collision of the soliton with the moving end particle.

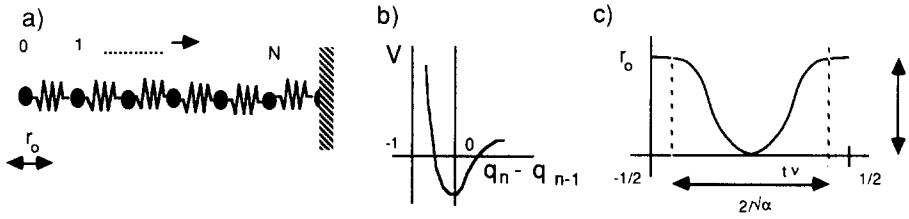


Fig. 1. (a) Schematic representation of the chain, for which we consider longitudinal motion only. In the nondriven system  $q_0 \equiv 0$ . (b) Sketch of the interaction potential. (c) In the driven system  $q_0$  is a prescribed periodic function  $r_0(t; \alpha, \nu)$  of time as shown.

$\lambda$  being the friction coefficient. Here  $q$  denotes the displacement vector from the equilibrium position, with elements  $q_i$ .  $r_0(t)$  is a given function of time, which is zero in the nondriven system. In the driven system  $r_0(t)$  has period  $\nu$  and is given by, on the interval  $|t\nu| \leq \frac{1}{2}$ ,

$$\begin{aligned}
 r_0(t; \alpha, \nu) &= \alpha(t\nu)^2 - \frac{1}{4}\alpha^2(t\nu)^4 \\
 &\text{if } |t\nu| \leq \sqrt{2}/\alpha, \\
 r_0(t) &= 1 \\
 &\text{else.}
 \end{aligned}
 \tag{2.2}$$

The parameter  $\alpha$  determines the second derivative of  $r_0$  and the width of the shape of  $r_0$ . Thus it gives effectively the strength of the forcing (cf. fig. 1).  $U$  is the sum of nearest neighbor contributions of Lennard-Jones type, so that the particles cannot pass one another, and is given by

$$U(q) = \sum_{n=1}^N V(q_n - q_{n-1}).
 \tag{2.3}$$

There is numerical and theoretical evidence that

the nondriven ( $q_0 \equiv 0$ ) nondissipative system has a solution family which represents a solitary wave bouncing back and forth between the two fixed walls (cf. also section 3). This solution family is the continuation [9] of the longitudinal standing wave with wavelength  $N + 1$  that bifurcates from the rest solution [10]. It is a compression wave whose center describes a path as is sketched in fig. 2a. It exists for a range of energy values, thus forming a family of solutions. At low energy it resembles the well known harmonic wave. At higher energy the interaction becomes more and more similar to that of hard particles. This leads to trajectories as displayed in fig. 3, with frequency increasing as a function of the energy of the wave (cf. ref. [11] for an analytically established result for waves on a ring).

In a system with friction such a wave obviously dies out. The intuitive idea now is that if a proper driving is added the bouncing solitary wave solution exists as an attracting solution with frequency of the driving, as a result of 1:1 resonance. In the case of no friction the similar result is expected to hold: in response to a driving with a given frequency two solutions resonate,

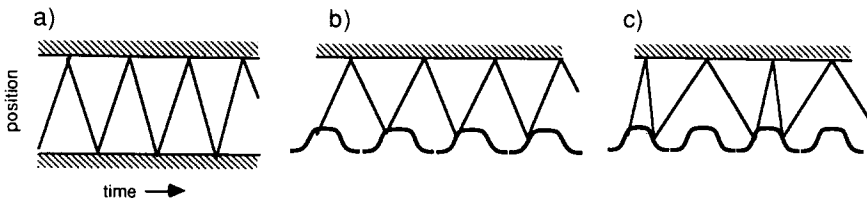


Fig. 2. Schematic representation of the trajectory of the center of the wave in period one and period two motion respectively. (a) A solitary wave bouncing back and forth in nondriven chain. (b) Similar in the driven system. (c) Similar, but now for a period doubled motion.

one of them being hyperbolic and the other one elliptic. A natural way to drive the system is obtained by prescribing a periodic motion for  $q_0 = r_0(t; \alpha, \nu)$ ,  $\nu$  being the frequency and  $\alpha$  a “strength” parameter. One might hope that for a well chosen  $r_0(t; \alpha, \nu)$ , i.e. adapted in some sense to the shape of the soliton, the wave goes through a period doubling sequence for increasing  $\alpha$ , maintaining its “solitary” character. This is observed indeed [2] as demonstrated in fig. 3.

**3. The equations of motion near the nonlinear mode surface**

Consider a Hamiltonian system with phase space  $\mathbb{F} = \mathbb{R}^N \oplus \mathbb{R}^N$ , whose elements are denoted with  $v = [p, q]^T$ , with the convention that  $[ \quad ]$  denotes a row vector. The systems we consider have Hamiltonians of the form ( $\nu$  is the driving frequency,  $\alpha$  is “strength” parameter)

$$H(v, t) = H_0(v) + F(v; \nu t, \alpha),$$

$$v = [p, q]^T \in \mathbb{F}. \tag{3.1}$$

The splitting in two terms is not unique. For the chain a natural choice is

$$H_0 = \frac{1}{2}(p, p) + U_0(q),$$

$$U_0(q) = V(q_1) + V(q_2 - q_1) + \dots + V(-q_N), \tag{3.2a}$$

$$F = V(q_1 - q_0(t)) - V(q_1), \tag{3.2b}$$

so that  $H_0$  describes the chain with fixed ends. We first concentrate on periodic orbits of the nondriven part with Hamiltonian  $H_0(v)$  with equations of motion

$$v = J_N \nabla H_0 \tag{3.3a}$$

with  $J_N$  the symplectic matrix

$$J_N^{\text{def}} = \begin{bmatrix} 0 & -I_N \\ I_N & 0 \end{bmatrix},$$

$I_N$  the  $N$ -dimensional identity matrix.  $\tag{3.3b}$

Such orbits are related easily to a variational problem, in fact to the problem of least action [12], as follows. Consider the set of closed curves in phase space, and parameterize them with an angle  $\phi$ ,  $0 \leq \phi < 2\pi$ . Define functionals  $H_0$  and  $A$  on this set:

$$H_0 \equiv (2\pi)^{-1} \int_0^{2\pi} H_0(v) d\phi,$$

$$A \equiv -(2\pi)^{-1} \int_0^{2\pi} \frac{1}{2}(v, J_N \partial_\phi v) d\phi, \tag{3.4}$$

and consider the critical point problem:

Find extrema of  $H_0$  for constant  $A$ .  $\tag{3.5}$

Solutions satisfy the equation [10]

$$\omega \text{grad } A = \text{grad } H_0, \tag{3.6a}$$

$\omega$  being a Lagrange multiplier. Evaluating these gradients one finds, using  $J_N^T = -J$  so that  $J_N \partial_\phi$  is a symmetric operator on the function space on which  $A$  and  $H_0$  are defined:

$$-\omega J_N \partial_\phi v = \nabla H_0. \tag{3.6b}$$

Now let  $\tilde{r}(\phi)$  be a solution of (3.6), then we conclude immediately, identifying  $\phi$  with  $\omega t$ , that each solution of the variational problem (3.5) corresponds to a periodic orbit of (3.3) with frequency  $\omega$  and vice versa.

In fact a variant of this variational problem was used to prove existence of solitary waves in systems with discrete translational symmetry such as a closed chain [11], and it proves to be useful in numerically finding solitary wave solutions [13].

For the present problem it is relevant to observe that a solution  $\{\omega(A), \tilde{v}(A, \phi)\}$  of the variational problem typically is a member of a solution family, with the value of  $A$  from the variational problem, called  $\mathcal{A}$ , as the parameter. With  $R_{\mathcal{A}}$  denoting some range of  $A$ , this family spans what

we call the nonlinear mode surface (NMS)

$$V \equiv \{v = \tilde{v}(A, \phi) | 0 \leq \phi < 2\pi, A \in \mathbb{R}_A\}, \quad (3.7)$$

which is a two dimensional plane in the phase space. For fixed  $A$  a solution  $\tilde{v}(A, \phi)$  is a closed curve on  $V$  which corresponds precisely to a periodic orbit with frequency  $\omega(A)$ . Consequently  $V$  is invariant for the flow of  $H_0$ . Moreover if we define

$$H_r(A) \equiv H_0(\tilde{v}(A, \phi)), \quad (3.8)$$

one readily shows that the restriction of the flow to  $V$  is Hamiltonian with  $H_r(A)$  as Hamiltonian and  $[A, \phi]$  as action angle variables. To see this observe that  $H_r(A)$  indeed does not depend on  $\phi$  since  $H_0$  is a constant of the motion for the  $H_0$ -flow. This implies  $H_r(A) = H_0(\tilde{v}(A, \phi)) = H_0(\tilde{v})$ , so that

$$\begin{aligned} \frac{dH_r}{dA} &= \left( \text{grad } H(\tilde{v}), \frac{\partial \tilde{v}}{\partial A} \right) \\ &= \omega \left( \text{grad } A(\tilde{v}), \frac{\partial \tilde{v}}{\partial A} \right) = \omega \frac{dA(\tilde{v})}{dA} = \omega. \end{aligned} \quad (3.9)$$

Equations of motion of the  $H_0$ -system restricted to  $V$  then are

$$\frac{d}{dt} A = 0 = - \frac{\partial H_r(A)}{\partial \phi}, \quad \frac{d}{dt} \phi = \omega(A) = \frac{dH_r}{dA}, \quad (3.10)$$

which equations are canonical indeed.

In general solutions of the variational problem (3.5) are not easily obtained analytically, although one can find them numerically. In some cases one knows however that they exist. Clearly any nonlinear periodic solution bifurcating from the origin is an example. In the chain with two fixed walls we have for instance the standing wave with half wavelength equal to  $N + 1$ . Near the origin it

can be written as (here  $A$  is small)

$$q_n(t) \cong \omega_0^{-1} A^{1/2} \sin(\omega_0 t) \sin(n\pi/(N + 1)) c_N, \quad (3.11)$$

with  $\omega_0$  the harmonic frequency and  $c_N$  a factor found with definition (3.4). Analytical considerations [9] and numerical evidence show that this bifurcating branch can be continued to larger values of  $A$ . This results in the family meant in section 2. Due to the vertical asymptote in the interaction potential  $V$ , the shape of the wave steepens for larger values of  $A$ , resulting in the bouncing soliton profile [11]. This property may seem contradictory to (3.11). One must realize, however, that also in this expression, representing a longitudinal wave, the center of mass bounces back and forth.

We are interested in motions of the full system

$$\dot{v} = J_N \nabla \{H_0 + F\} - \lambda [p, 0]^T \quad (3.12)$$

that resemble the periodic solutions of the Hamiltonian  $H_0$ -flow. The essential observation is that such trajectories remain in the neighborhood of the nonlinear mode surface  $V$ . In the dissipative case one looks for an attracting orbit, so that trajectories have this property forever. Without friction one must be satisfied with a long time scale. To describe such trajectories one can fruitfully rewrite the equations of motion so that they can be seen as representing a driven oscillator of one degree of freedom tangential to the NMS, which is coupled to the transversal motion. One way to obtain such equations is to use a projection method [14]. A similar result, more suitable for the present purpose, can be found with use of the canonical formalism.

Introduce *canonical coordinates*  $[A, \xi, \phi, \eta]$  near  $V$ :

$$\begin{aligned} [A, \xi, \phi, \eta] \\ \xi = [\xi_1, \dots, \xi_{N-1}], \quad \eta = [\eta_1, \dots, \eta_{N-1}], \end{aligned} \quad (3.13)$$

in such a way that the transformation has the form ( $\zeta$  collects the  $\xi$  and  $\eta$  variables)

$$v = \bar{v}(A, \phi; \zeta) \equiv \bar{v}(A, \phi) + \bar{z}(A, \phi; \zeta),$$

$$\bar{z}(A, \phi; 0) = Z(A, \phi)\zeta + \mathcal{O}(\zeta^2), \quad (3.14)$$

where  $Z(A, \phi)$  is a  $2N \times (2N - 2)$  matrix with columns denoted by  $e_i$ ,  $i = 3, \dots, 2N$ . The  $2N - 2$  coordinates  $\zeta_i$  denote directions transversal to the NMS. The condition that this transformation is canonical is  $D\bar{v}^T J_N D\bar{v} = J_N$ . Analysis of this condition yields a solution for  $\bar{z}$  which is not unique. Making a Taylor series with respect to  $\zeta$  of the left hand side, one can determine the Taylor coefficients of  $\bar{z}$  by comparison of the homogeneous terms in  $\zeta$ . For the lowest order term this procedure yields for the  $2N$  vectors  $e_i$  defined by

$$e_1 \equiv \frac{\partial \bar{v}}{\partial A}, \quad e_{1+i} \equiv \left( \frac{\partial \bar{z}}{\partial \xi_i} \right)_{\zeta=0}, \quad e_{N+1} \equiv \frac{\partial \bar{v}}{\partial \phi},$$

$$e_{1+N+i} \equiv \left( \frac{\partial \bar{z}}{\partial \eta_i} \right)_{\zeta=0}, \quad i = 1, \dots, N-1, \quad (3.15)$$

the conditions

$$(e_i, J_N e_{i+N}) = 1, \quad i = 1, \dots, N,$$

$$(e_k, J_N e_m) = 0$$

$$\text{for all other combinations } k \text{ and } m. \quad (3.16)$$

For  $i = 1$  the first relation can be derived from (3.6). The other relations can be met by choosing vectors  $e_i(A, \phi)$  properly. One possibility is to choose the eigenvectors of the variational (Floquet) problem around a periodic orbit  $\bar{v}(A, \omega(A)t)$  [6]. Here it is relevant to observe that  $e_1$  and  $e_{N+1}$  are tangent to  $V$  whereas the other  $e_i$  are transversal in the sense of the “ $J_N$ -orthogonality” relations (3.16). Together they span the space tangent to the phase space (cf. ref. [14]).

To find equations of motion in these new variables we refer to a result from ref. [15] which says how a dissipative equation as (3.12) transforms

under a canonical transformation: this result states that the dissipative term splits in two. One part remains proportional to the new momenta, and the other part is Hamiltonian and can be expressed with the generating function of the transformation. Observe in particular that the form (3.12) is left invariant. In fact the equations of motion after transformation read

$$\frac{d}{dt} [A, \xi, \phi, \eta]^T = J_N \nabla K - \lambda [A, \xi, 0, 0]^T$$

$$K = H_0(\bar{v} + \bar{z}) + F(\bar{v} + \bar{z}, \nu t, \alpha)$$

$$+ \lambda S(A, \phi; \zeta). \quad (3.17)$$

Here  $S$  is immediately related to the generating function of the transformation. These equations are correct in a region near  $V$  where the coordinate transformation is defined.

Since we want to consider trajectories of the full system near  $V$  we now order the equations of motion with respect to powers of  $\zeta$ . This can now be done easily by taking only linear and quadratic terms in  $\zeta$  the Hamiltonian  $K$  in (3.17). Writing the generating function of the transformation as

$$S(A, \phi; \zeta) = S_0(A, \phi) + (S_1(A, \phi), \zeta)$$

$$+ \frac{1}{2}(\zeta, S_2(A, \phi)\zeta) + \mathcal{O}(\zeta^3), \quad (3.18)$$

one finds a Hamiltonian (t and c denote “transversal” and “coupling” respectively)

$$H = K_0 + K_t + K_c + \mathcal{O}(\zeta^3) \quad (3.19)$$

with

$$K_0 = H_r(A) + F_r(A, \phi; \nu t, \alpha) + \lambda S_0(A, \phi), \quad (3.20a)$$

$$K_t = \frac{1}{2}(\zeta, M(A, \phi; \nu t, \alpha)\zeta),$$

$$M = M_0(A, \phi) + M_F(A, \phi; \nu t, \alpha)$$

$$+ \lambda S_2(A, \phi), \quad (3.20b)$$

$$K_c = (b(A, \phi; \nu t, \alpha), \zeta),$$

$$b = Z^T \nabla F(\bar{v}; \nu t, \alpha) + \lambda S_1(A, \phi). \quad (3.20c)$$

$M_0$  and  $M_F$  arise from the quadratic terms of  $H_0(\tilde{v} + \tilde{z})$  and  $F(\tilde{v} + \tilde{z})$ . Important is that the linear term from  $H_0$  appears to be zero because of  $(e_{N+1}, J_N e_n) = 0, n \neq 1$ , cf. (3.15). The resulting equations of motion are

$$\begin{aligned} \frac{d}{dt} [A, \phi]^T &= J_1 \nabla_{A, \phi} K_0 - \lambda [A, 0]^T \\ &+ J_1 \nabla_{A, \phi} (b, \zeta) + \mathcal{O}(\zeta^2), \end{aligned} \quad (3.21a)$$

$$\begin{aligned} \frac{d}{dt} [\xi, \eta]^T &= J_{N-1} M \zeta - \lambda [\xi, 0]^T + J_{N-1} b \\ &+ \mathcal{O}(\zeta^2). \end{aligned} \quad (3.21b)$$

Although we do not know all terms in this equation explicitly, its form nevertheless will clarify the observed low dimensionality of the phenomena considered to a certain extent. It was remarked already that phenomena as described in section 2 suggest that the trajectories of the full system remain near  $V$ , i.e. the coordinates  $\zeta$  remain small. Let this be the case, then one may expect that the first two terms on the right of (3.21a) dominate, i.e. the motion is dominantly described by the one degree of freedom driven dissipative oscillator with Hamiltonian  $K_0$ . In fact these terms should describe the projection of the trajectory on  $V$ , equivalently said its tangential component. This oscillator drives parametrically a set of linear oscillators, Hamiltonian  $K_1$  which represent the transversal component. The two are coupled via the term with  $b$  and the higher order terms. In the next section we will consider in more detail the effect of the coupling terms on a supposed period doubling sequence of the dominant oscillator.

This interpretation make sense only if  $\zeta$  remains small. This requirement can be frustrated in at least two ways: The homogeneous part of the transversal oscillator can have multipliers with positive real part so that  $\zeta$  grows exponentially. This certainly will occur if the periodic orbits  $\tilde{v}$  of the  $H_0$ -flow are hyperbolic. So ellipticity of these orbits appears to be a prerequisite for the usefulness of this approach. From the next section it

will become clear that also nondegeneracy of their multipliers is helpful condition. Furthermore, the coupling term  $b$  can be so large that  $\zeta$ , as it follows from the linear equation (3.21b), becomes so big that one must take into account the nonlinear terms. The same can happen if the driving frequency is close to resonance or if the contribution of  $F$  to the homogeneous part of the transversal term  $K_1$  is so large that it destabilizes this transversal oscillators. Also in these cases the usefulness of these approximate equations is doubtful. It is difficult to formulate precisely under what conditions these requirements are met. They seem to be satisfied, however, for the calculations on the chain as follows from explicit calculations of the multipliers [8]. In a rough approach one can say that the tangential component of the force resulting from  $F$  can be large, but that its transversal component must be small. The analysis in the next section gives more details.

#### 4. Period doubling in the equations of motion

We investigate to what extent and under what conditions eqs. (3.21) can describe a period doubling transition to chaos, dominantly described by the  $[A, \phi]$  variables.

Consider the one-degree-of-freedom dissipative oscillator with Hamiltonian  $K_0$  separately which has equations of motion

$$\begin{aligned} \frac{d}{dt} [A, \phi]^T &= [0, \omega(A)]^T \\ &+ J_1 \nabla_{A, \phi} [F_r(A, \phi; \nu t, \alpha) + \lambda S_0(A, \phi)] \\ &- \lambda [A, 0]^T. \end{aligned} \quad (4.1)$$

The main observation is that such a system indeed can show a period doubling sequence for changing  $\alpha$  if  $F_r$  is chosen appropriately. This is confirmed by many (numerical) experiments [16]. In conclusion, if one is allowed to discard the transversal oscillators, one should be able to find



period doubling bifurcations in the full system for suitable driving.

In the case of the section 2,  $F_r$  is given by (cf. (3.2))

$$F_r = V(\tilde{q}_1(A, \phi) - r_0(t)) - V(\tilde{q}_1(A, \phi)). \quad (4.2)$$

For each fixed  $t$  this expression is periodic in  $\phi$ . The behavior of  $\tilde{q}_1(A, \phi)$  can be read from fig. 3 approximately: in the nondriven system  $\tilde{q}_1(A, \phi) = -\tilde{q}_N(A, \phi + \pi)$ . Since in the driven system the soliton reflects more or less undisturbed against the fixed wall, the latter function is approximately given by the corresponding curve of  $r_N(t)$  in fig. 3. Together with the shape of the potential this shows that  $F_r$  is not a small perturbation, but has sharp high peaks as a function of  $\phi$ . These peaks vary in height periodically in time. This makes (4.1) in a certain sense analogous to the equations of motion of a particle on a line moving in a potential that is periodic both in space and in time: since  $\omega(A)$  appears (numerically) to be an increasing function of  $A$ ,  $H_r(A)$  can be seen as the kinetic energy of a free particle with linear momentum  $A$ . Correspondingly  $\phi$  is the linear coordinate and  $F_r$  can be seen as the ‘‘potential’’, although it depends on  $A$ . Thus (4.1) is analogous to the equations of motion of a damped particle with Hamiltonian  $\frac{1}{2}p^2 + V(q, t)$ ,  $V(q, t)$  periodic both in  $q$  and in  $t$ . Thus its period doubling behavior is comparable e.g. to that of the Chirikov (standard) map [4].

Before we proceed consider a supposed period doubling sequence of (4.1) for increasing  $\alpha$  in more detail. In that case there are bifurcation points  $\alpha_q < \alpha_{q+1}$  between which (4.1) has a stable period  $\nu^{-1}2^q$  orbit which is born at  $\alpha_q$  and turns unstable at  $\alpha_{q+1}$ , at which point a stable period  $\nu^{-1}2^{q+1}$  orbit bifurcates. For the discussion of the period doubling in the full system it is useful to recall the behavior of the multipliers of the period orbit. In the conservative case these eigenvalues move along the unit circle from  $\pm 1$  to  $-1$  for increasing  $\alpha$  in an interval  $[\alpha_q, \alpha_{q+1}]$ . Then a bifurcation takes place and the process starts

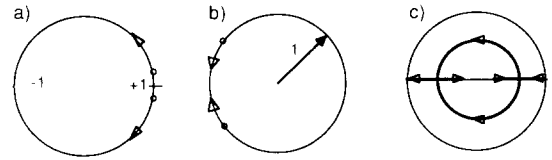


Fig. 4. Behavior of eigenvalues of a stable period during the period doubling process for a Hamiltonian one-degree-of-freedom oscillator with friction. (a) No friction: for changing bifurcation parameter the eigenvalues of a period  $2^q$  orbit move along the unit circle towards  $-1$ , where a period  $2^{q-1}$  orbit is born. The process repeats itself at each period. (b) With friction: The multipliers now move along a circle with radius  $\exp[-\lambda(\frac{1}{2}\nu^{-1})2^q]$ .

anew, as is sketched in fig. 4. If there is friction the behavior is similar. Such a period doubling sequence is essential for the further analysis and we make

*Assumption 4.1.* The driving  $F(v; \nu t, \alpha)$  is such that that (4.1) has a period doubling sequence for changing  $\alpha$  in which the multipliers for each period  $2^q$  behave as described in fig. 4. □

It is difficult, or even impossible, to determine analytically a function  $F$  that will do. One reason is that  $\tilde{r}$  and  $S$  are not known explicitly. In the work described in section 2 it appeared to be a matter of trial and error to find an  $F$  that produced the observed stable periodic orbits.

We are able now to formulate our problem somewhat more precisely.

*Problem.* Under assumption 4.1, do the equations of motion (3.21) have an infinite period doubling sequence for changing  $\alpha$  in which the periodic orbits are given by  $v(t) = \tilde{r}(A, \phi) + \tilde{z}(A, \phi; \zeta)$  with  $[A(t), \phi(t)]$  and the bifurcation values  $\alpha_q$  approximately predicted by (4.1) and with  $\tilde{z}$  small with respect to  $\tilde{r}$ ? □

To find an answer we begin with an approximate set of equations, i.e. we include the transversal degrees of freedom, but omit the explicit coupling term due to  $K_c$  and higher order terms in the Hamiltonian. The truncated Hamil-

tonian then reads

$$H_r(A) + F_r(A, \phi; \nu t, \alpha) + \lambda S_0(A, \phi) + \frac{1}{2}(\zeta, M(A, \phi; \nu t, \alpha)\zeta) \tag{4.3}$$

and the resulting equations are

$$\frac{d}{dt}[A, \phi]^T = J_1 \nabla_{A, \phi} K_0 - \lambda[A, 0]^T + \frac{1}{2} J_1 \nabla_{A, \phi}(\zeta, M\zeta) \tag{4.4a}$$

$$\frac{d}{dt}[\xi, \eta]^T = J_{N-1}[M_0(A, \phi) + M_F(A, \phi; \nu t, \alpha) + \lambda S_2(A, \phi)] \times [\xi, \eta]^T - \lambda[\xi, 0]^T. \tag{4.4b}$$

Eq. (4.4a) is just (4.1) with an additional term. (4.4b) is a linear homogeneous equation. For a given  $[A(t), \phi(t)]$  it is itself a nonautonomous Hamiltonian system with  $N - 1$  degrees of freedom.

We consider periodic orbits of truncated system and their stability. Obviously the homogeneity of (4.4b) and of the last term of (4.4a) leads immediately to

*Proposition 4.2.* If (4.1) has a periodic solution  $[A(t), \phi(t)]$  then  $[A(t), \phi(t), \zeta(t) = 0]$  is a periodic solution of the system (4.4). The Floquet exponents of this solution are the two of (4.1) and the  $2(N - 1)$  of the zero solution of (4.4b).  $\square$

With assumption 1 this proposition implies that (4.4) has a period doubling sequence. To investigate the stability of such a solution we consider first somewhat formally the period  $\nu^{-1}2^q$  solution of (4.4), in the case that there is no driving and damping. To start with we suppose

*Assumption 4.3.* Consider  $A$  such that  $\omega(A) = 2\pi\nu$ . Let the periodic solution  $\tilde{r}(A, \omega(A)t)$  of the  $H_0$ -flow have one double Floquet exponent at 0 and otherwise  $2(N - 1)$  different purely imaginary exponents  $\pm i\mu_n(A)$ . Furthermore let the multipliers with respect to period  $\nu^{-1}$  corre-

sponding to the latter exponents

$$\beta_{0, \pm n} = \exp[\pm i\mu_n(A)\nu^{-1}], \tag{4.5a}$$

$n = 1, \dots, N - 1$

be different and unequal to  $+1$  or  $-1$ .  $\square$

This is the least degenerate case since there is always a  $q$ -fold zero exponent with eigenspace spanned by  $\partial\tilde{r}/\partial A$  and  $\partial\tilde{r}/\partial\phi$ . If we consider  $\tilde{r}$  as a solution with period  $\nu^{-1}2^q$  the multipliers with respect to this period are zero and the appropriate powers of  $\beta_{0, \pm n}$

$$\beta_{q, +n} = (\beta_{0, +n})^{2^q}, \quad \beta_{q, -n} = \beta_{q, +n}^*, \tag{4.5b}$$

$n = 1, \dots, N - 1$ .

To see this recall that the multipliers of a periodic orbit with period  $P$  are the eigenvalues of the derivatives of the discrete time- $P$  mapping at the corresponding fixed point of this mapping. The multipliers of the same orbit considered as an  $nP$  periodic solution are the eigenvalues of the  $n$ th power of this derivative. Furthermore the quantities (4.5) are the multipliers of the zero solution of (4.4b) if considered as an independent equation.

For  $q = 0$  these multipliers are different and unequal to  $+1$  or  $-1$  by assumption. However for other values of  $q$  this nondegeneracy property can and will be disturbed. This is a consequence of

*Lemma 4.4.* Let  $a_{n, s_n}$  denote a finite set of  $q$ -values  $[q_n, q_{n+1}, \dots, q_{n+s_n}]$ . Let  $\rho, \rho_1, \rho_2$  be complex numbers with unit length. Then there is  $\epsilon_0 > 0$  so that

(i) for almost all  $\rho$  and for all  $0 < \epsilon \leq \epsilon_0$  there is an infinite set of nonoverlapping intervals  $a_{n, s_n}$  such that  $\rho^{2^q}$  is  $\epsilon$ -close to  $-1$  for each  $q$  in the union of these sets.

(ii) for almost all  $\rho_1, \rho_2$  and for all  $0 < \epsilon \leq \epsilon_0$  there is an infinite set of nonoverlapping intervals  $a_{n, s_n}$  such that  $\rho_1^{2^q}$  is  $\epsilon$ -close to  $\rho_2^{2^q}$  for each  $q$  in the union of these sets.

The actual sets  $a_{n,s_n}$  depend sensitively on the values of  $\rho$ ,  $\rho_1$  and  $\rho_2$ .

*Proof.* Any sequence  $\rho^{2^q}$ ,  $q = 0, 1, \dots$ , is a trajectory of the quadratic map on the circle. The chaotic properties of this map [17], especially topological transitivity and sensitive dependence on initial conditions, lead immediately to the result (i). Since  $\rho_1^{2^q} - \rho_2^{2^q} = \rho_1^{2^q}[1 - (\rho_2/\rho_1)^{2^q}]$ , the argument of (i) applied on  $\rho_2/\rho_1$  yields (ii).  $\square$

It is good to have in mind what actually happens for a typical value of  $\rho$ . Take  $\rho$  on the unit circle and consider the behavior of  $\rho^{2^q}$  with increasing  $q$ . The phase angle of  $\rho^{2^q}$  doubles at each step. For some  $q$  it arrives  $\varepsilon$  close to  $-1$ . The next step brings it close to  $+1$  from which it wanders away with increasing  $q$ .  $\rho^{2^q}$  can stay in the  $\varepsilon$ -neighborhood for some iterations depending on how close to  $+1$  it starts. Finally it leaves the  $\varepsilon$ -neighborhood of  $+1$  and wanders around the circle. There is a next  $q$  where this process starts anew. This process repeats an infinite number of times. Exceptional values for which the lemma does not apply are for instance eventually periodic trajectories, with a trajectory that falls upon  $+1$  as a special example. The similar things happens with two different starting values  $\rho_1$  and  $\rho_2$ .

Applying this lemma to the multipliers (4.5b) which are the  $2^q$  powers of the quantities in (4.5a) implies immediately

*Proposition 4.5.* The multipliers of the period  $\nu^{-1}2^q$  solution in the nondriven nondamped case are on the unit circle. They are “nearly degenerate” in sets of  $q$  values as formulated in lemma 4.4.  $\square$

These “near degeneracies” include e.g. one multiplier that is close to  $-1$  and to its complex conjugate and two multipliers that are close to one another somewhere on the unit circle. Obviously other multiple “near degeneracies” can occur too, although less frequently. Remark that

this proposition holds even if one assumes that the multipliers of the period- $\nu^{-1}$  solution are (nondegenerate and) spread out evenly on the circle. Furthermore the  $q$ -intervals depend sensitively on the values of  $\beta_0$  and thus for instance on the frequency  $\nu$ .

Now we consider the effect of driving and friction on the stability of periodic orbits as meant in proposition 4.2, assuming that (4.1) has a period doubling sequence. The multipliers are the one pair of (4.1) which move for  $q = 0$  as function of  $\alpha$  on a circle as shown in fig. 4. The other multipliers are those of (4.4b) with a solution  $[A(t), \phi(t)]$  substituted. To determine them we rewrite this equation after scaling of  $\zeta$  as

$$\begin{aligned} \frac{d}{dt} \hat{\zeta} &= \frac{1}{2} J_{N-1} \nabla \left\{ (\hat{\zeta}, M \hat{\zeta}) + \frac{1}{2} \lambda (\hat{\xi}, \hat{\eta}) \right\}, \\ \hat{\zeta} &= e^{i\lambda/2} \zeta, \end{aligned} \tag{4.6}$$

which can be verified easily. Important is that this equation is Hamiltonian again. Because of the scaling the multipliers of (4.4b) are equal to those of (4.6) multiplied by a factor  $(b^{1/2})^{2^q}$  with  $b$  defined by

$$b \equiv e^{-\lambda \nu^{-1}}. \tag{4.7}$$

These multipliers can be considered as those of proposition 4.5, but affected by a perturbation. The perturbation is the combined effect of  $M_F, S_2$ , the extra term with  $\lambda$  in the Hamiltonian in (4.6) and of the fact that  $A(t), \phi(t)$  differ from the nondriven case. Furthermore they depend on  $\alpha$  via these terms. To be able to say something we must assume that *all these effects can be considered as small*. Then, for  $q = 0$ , where the unperturbed multipliers are far apart and on the unit circle, the multipliers will still be nondegenerate but on the  $b^{1/2}$  circle. They will change their position a little as  $\alpha$  varies. For those values of  $q$  however where multipliers of the nondriven soliton are close to one another, or are close to  $+1$  or  $-1$ , these multipliers (of (4.6)) can leave the unit circle, with the similar result for multipliers

of (4.4b) with respect to the  $b^{1/2}$  circle. These perturbation processes follow from general theory for perturbations of (nearly) degenerate Hamiltonian systems, which is pointed out in the next section. In summary the formulated problem for (4.4) is answered by

*Proposition 4.6.* Under assumption 4.1 the system (4.4) has a period doubling sequence with periodic orbits given by  $[A(t), \phi(t), \zeta(t) = 0]$ , due to bifurcations described by (4.1). One pair of multipliers, the “tangential” ones from (4.1), move along the  $(b^{1/2})^{2^q}$ -circle as in fig. 4, passing the other “transversal” multipliers. Assuming that the effect of the driving and friction can be considered as a perturbation, the transversal multipliers are near those mentioned in proposition 4.5 and they vary slightly as a function of  $\alpha$ . If this perturbation is small enough the transversal multipliers are on the  $(b^{1/2})^{2^q}$ -circle for  $q = 0$ , and the same will be the case for many other unperturbed multipliers that are far enough apart. There are an infinite number of values of  $q$  however for which there are “nearly degenerate” unperturbed multipliers that are so close that their perturbed counterparts can be outside this circle (fig. 5).  $\square$

If there is no friction everything happens with respect to the unit circle. Consequently the period doubling sequence can have unstable periodic orbits. In the next section it will be shown

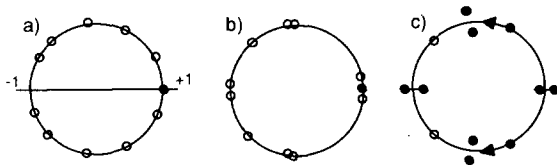


Fig. 5. Sketch of possibilities for multipliers in the period doubling sequence in the truncated model. (a) Multipliers of the nondriven period  $\nu^{-1}$  mode. (b) Several possibilities of near degeneracy of the period  $\nu^{-1}$  mode. (c) Possible instabilities for multipliers in the period doubling sequence in the truncated model. The effective interaction is supposed to be small and there is no friction. Tangential multipliers pass the transversal multipliers in all cases.

that one must expect that such instabilities occur. For  $\lambda$  unequal to zero stability depends on competition between  $\lambda$ , which brings transversal multipliers within the unit circle, and the effective interaction between (4.4a) and (4.4b) which can bring nearly degenerate multipliers outside this circle.

These results hold for the truncated system. For the full system one has to consider also the effect of  $K_c$  and of the higher order terms in (3.21), which are another source of disturbance of the period doubling sequence. Assume again that these terms can be considered as small. There are two extra effects. First, if the truncated system has multipliers near +1 it is not clear a priori that the periodic orbit as such survives. The reason is that the usual way to prove existence, via the implicit function theorem, is not applicable for a given perturbation if multipliers are too close to +1. Thus there can be “holes” in the period doubling sequence. Second, the tangential eigenvalues from (4.1) will typically not pass the transversal ones as in the truncated model (4.4). Interaction can cause that they leave the circle when they meet one another. The latter effect is discussed in some detail in the next section.

### 5. Behavior of multipliers under perturbation

In a situation in which multipliers of periodic orbits of Hamiltonian flows (nearly) coincide, Krein signature determines largely if a perturbation leads to instability or not [7]. This theory can be extended to Hamiltonian systems with friction [8]. Here we describe shortly some relevant aspects of this theory and apply it to the near degeneracies in the period doubling sequence.

Consider equations of motion of the form  $v = J\nabla H - \lambda[p, 0]^T$ . The flow of such a system has properties that are very much like symplectic flows, as is pointed out in ref. [8]. For that reason such a flow is called semi-symplectic. One of the properties is that the multipliers of periodic orbits, period  $P$ , are symmetric with respect to a

circle with radius  $b^{1/2}$ , with  $b \equiv \exp(-\lambda P)$ , called the symmetry circle (s-circle). In particular, if  $\beta$  is a multiplier of a periodic orbit of such a flow then so are  $\beta^*$ ,  $b\beta^{-1}$  and  $b\beta^{*-1}$ , not all necessarily different. This restricts the possible values for their behavior under perturbations largely. For our purpose the two relevant aspects are:

(i) Let  $\beta$  (and its complex conjugate  $\beta^*$ ) be nondegenerate, i.e. simple, complex multipliers on the s-circle. If the flow is perturbed the multiplier can change position but remains on the s-circle.

(ii) Let  $\beta$  be a twofold complex eigenvalue on the s-circle. If the flow is perturbed the eigenvalue typically will split in two nondegenerate ones: either both on the s-circle or one,  $\beta_s$ , inside and another one,  $\beta_u$ , outside the circle. The two are related by  $\beta_s^* = b\beta_u^{-1}$ .

Which of the two possibilities actually happens in case (ii) is determined by theorems of Krein and Moser. To formulate the consequences of these theorems the concept of Krein signature is of great help.

(iii) A pair of nonreal simple (i.e. nondegenerate) eigenvalues on the s-circle can be assigned uniquely a so called Krein signature, plus or minus. This signature does not change under perturbations as long as the eigenvalue remains simple.

We now apply these concepts to the multipliers related to (4.4) as they occur in propositions 4.5 and 4.6. Clearly the complex multipliers of the unperturbed period- $\nu^{-1}$  orbit of the  $H_0$ -flow are well-defined because of their supposed nondegeneracy (assumption (4.3)). These multipliers are the eigenvalues of the derivative of the period- $\nu^{-1}$  map of the flow. Similarly the multipliers of the same orbit but considered as having period  $\nu^{-1}2^q$  are the eigenvalues of the  $2^q$  power of this period- $\nu^{-1}$  map. In almost all cases their signature can be defined too, irrespective if they are degenerate or not. In particular we have from ref. [8]

*Lemma 5.1.* Let  $\beta$  be a nonreal simple eigenvalue on the s-circle of a semi-symplectic map-

ping  $B$ . Then for almost all values of  $\beta$ , the signatures of the eigenvalues  $\beta^{2^q}$  or  $B^{2^q}$  are well-defined and form an uncorrelated sequence. Furthermore this sequence depends sensitively on the “initial value”  $\beta$  and plus and minus occurs equally often on the average.  $\square$

The proof is given in ref. [8]. The chaotic aspect in this lemma is based on the fact that there is a one-one correspondence of the sequence  $\beta^{2^q}$  to trajectories of the tent map. One consequence is that the signatures of two “nearly degenerate” transversal multipliers as in proposition 4.5 can be both equal and different. Which combination actually occurs is in fact unpredictable (it depends for instance sensitively on the driving frequency  $\nu$  via  $\beta_{0,n}$ ). Furthermore both combinations occur equally often on the average.

The effect of perturbations is determined by the Krein signatures as the following consequences of the theorems of Krein and Moser respectively show. Let two nonreal multipliers on the s-circle with different signature be sufficiently close to one another then the effect of almost all Hamiltonian perturbations of the system is that (as a function of the perturbation strength)

(iv) they come together and leave the circle if the signatures are different,

(v) they push one another away and remain on the circle if the signatures are equal.

These results are a consequence of general theorems of Moser and Krein respectively [7] and of additional explicit calculations [8]. We apply them on the effect of perturbation of the transversal multipliers in proposition 4.5 to obtain

*Proposition 5.2.* Consider a given “typical” period doubling sequence. In half of the cases of “near degeneracy” as mentioned in proposition 4.5, the two nearly equal multipliers have equal signature. The corresponding perturbed multipliers, in proposition 4.6, are on the s-circle. Those with different signature can be outside this circle, de-

pending on the effective interaction and their distance.  $\square$

Now we turn to the possible instabilities caused by the interaction, due to  $K_c$  and the higher order terms in (3.21), between the tangential multipliers and the transversal multipliers mentioned in proposition (4.6). The following result, again from ref. [8], is essential:

*Lemma 5.3.* The signatures of the two multipliers of the period doubling sequence of (4.1) alternate with  $q$ .

Applying the two lemmas we see that in one interval  $[\alpha_q, \alpha_{q+1}]$ , in which a tangential eigenvalue moves from  $+1$  to  $-1$ , we are dealing with several different events. The most frequently occurring is that the tangential multiplier meets a transversal one on the s-circle. Because of the randomness of the signatures of the transversal eigenvalues the signatures can be equal and different, both events happening equally often. In the first case they will push one another away. If the interaction term is small enough they will push one another away like colliding particles with a short range repelling potential. If the signatures are different a bubble trajectory will be formed (fig. 6a). A second event is that the tangential multiplier meets a transversal one near  $-1$ . Then, if the signatures are different, the eigenvalues can follow a path as sketched in fig. 6b, depending of course on the effective interaction strength and the distance to  $-1$  of the

unperturbed transversal multipliers. Detailed investigation of a simpler but similar system than the present one [8] indicates that this happens an infinite number of times. We collect the results in

*Proposition 5.4.* Consider a given “typical” period doubling sequence. In half of the encounters of tangential and transversal multipliers in period doubling sequence as described in proposition 4.6 a “bubble” is formed as shown in fig. 6a. One must expect an infinite number of events like in fig. 6b, where the period doubling bifurcation itself is affected.  $\square$

### 6. Discussion

We are able now to give an answer to the problem formulated in the beginning of section 3. The answer is moderately affirmative: If  $F$  is chosen carefully there is a good chance that such an infinite period doubling exists. Carefully here means that the part  $F_r$  occurring in (4.1) must give rise to a period doubling sequence. But also the components in the transversal directions, acting via the coupling term  $K_c$  must be small. However, this period doubling sequence is damaged in several ways:

(i) on the interval in  $\alpha$  there can be subintervals where no period doubled orbit exists at all. This is due to the fact that transversal multipliers of the period  $2^q \nu^{-1}$  orbits can be near  $+1$ .

(ii) If there is no friction the period  $2^q$  orbit will be unstable in (parts) of an interval  $[\alpha_q, \alpha_{q+1}]$ , because multipliers are outside the unit circle. If there is friction the same will hold if the friction is sufficiently small. In general it depends on the radius of the s-circle, compared to the Hamiltonian effective interaction that brings the nearly degenerate multipliers outside the s-circle.

To appreciate these statements it seems relevant to observe that what actually happens is the combined effect of the forcing  $F$ , which one has

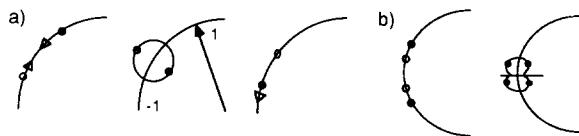


Fig. 6. Effect of a perturbation on trajectories of the multipliers with different signature. (a) The multipliers follow a “bubble” like path in the complex plane. (b) If the multipliers meet close enough to  $-1$  the period doubling bifurcation itself is affected.

in hand, and properties of the unperturbed nonlinear mode surface  $V$ . The latter is more difficult to control. One must also have in mind that the actual positions of instabilities depend sensitively on the driving frequency and on control parameters of  $V$ . In practice a reproducible completed sequence, stable or not, cannot be expected unless the damping is strong enough.

To conclude, consider some practical consequences of the present analysis for (numerical) experiments and their interpretation.

(i) Typically in an experiment one finds only the first few period doubling bifurcations. The analysis shows that it is very well possible that these are unaffected. One must be careful however to conclude to the existence of a nice and completed period doubling sequence.

(ii) In a system with dissipation a period doubling sequence is usually found by slow adiabatic change of the control parameter. However, instabilities are accompanied by a Hopf bifurcation of a stable torus (e.g. when the trajectory of a bubble passes the unit circle). Consequently it is impossible to find the sequence this way.

(iii) A special case occurs if the instability is caused by a transversal multiplier that passes  $-1$ . Then a period doubling bifurcation can take place but in a transversal direction, in contrast with the bifurcations of  $[A, \phi]$  variables which are parallel to the NMS. Such an event was observed in ref. [18]: inspection of the  $r-t$  diagram (cf. fig. 3) made clear that a spatial modulation of the wave was responsible for the period doubling, rather than a change in the time behavior of  $A$  and  $\phi$ .

(iv) Although the sequence can be damaged, there is nevertheless no objection against a resulting chaotic state, approximately described by the one degree of freedom oscillator (4.1). The process and the resulting state may be called low dimensional since they are described to a certain extent by a low dimensional model. However, this says nothing about the dimensionality of the chaotic attractor. The present choice of variables, leading to eqs. (3.21), seems to be appropriate for investigation of this point.

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