INSTABILITY PROBLEMS IN THE DYNAMIC LEONTIEF MODEL: AN ECONOMIC EXPLANATION

ABSTRACT In this note we present the outlines of a possible economic explanation of the instability problems of the dynamic Leontief model. Hereby we focus on the eigenvalues of a characteristic matrix.

1. Introduction

In his discussion of my approach to the stability issue in the dynamic Leontief model, A. R. G. Heesterman gives a number of comments on the model (Heesterman, 1990; Steenge, 1990). It is true, of course, that there is often cause to wonder about the model's interpretation and applicability. However, Mr Heesterman only makes some (loose) observations on the causality and finality aspects of the model. The form of (in)stability in the fully implemented models I have discussed depends on the eigenvalues of one specific matrix, symbolized by D in my article. If issues like causality or finality were to explain this type of instability, then there must be some sort of link with the empirically observed spectra of eigenvalues of matrices D . Such a link is not given, however, by Mr Heesterman in his article. Of course it is true that the introduction of, say, inequalities may sometimes result in a more realistic model. But the adoption of a new model framework does not explain what is going on in the more traditional forms, such as (1.1) and (1.2) in my article. And such an explanation is important because of the, in my view, fundamental nature of these forms.

The rank of the capital matrix is irrelevant here. Depending on the aggregation level, the capital matrix can have a much lower rank than the matrix of direct input coefficients. To solve models involving singular capital matrices requires nothing but a certain, relatively simple, technique. In my article, I have assumed a capital matrix of full rank. First, because some well-known empirical studies have a capital matrix of full rank. A second, and more important, reason is that the real problem here is the interpretation of the columns of D , whatever its rank. I shall argue below that under certain well-defined conditions all its columns must be near scalar multiples of the same vector. This implies that under such conditions an empirically implemented matrix D must closely approximate a rank 1 matrix. This, in my view, is the 'economic explanation' we need.

2. Integrated Capital Coefficients

The problem we want to discuss results from the fact that the closed dynamic Leontief model has properties in common both with an open and a closed static model, which means that there are two ways to look at indirect inputs and indirect costs. Realizing this is essential to understanding the economic background of the stability problem.

To illustrate the point, we shall first concentrate on the question which part of total output $x(t)$ is required, directly and indirectly, for the production of the capital that is needed for future expansion. Rewriting the standard forward-looking version

$$(2.1) \quad x(t) = Ax(t) + B[x(t+1) - x(t)],$$

we obtain, in obvious notation

$$(2.2) \quad (I - A)x(t) = B\Delta x(t).$$

The vector of newly produced capital $B[\Delta x(t)]$ thus can be looked upon as a vector of net outputs. We know that the j th column of B , $b_{.j}$, gives us the amounts of capital goods of each type required to produce one additional unit of the j th good in the next period. However, to produce the vector $b_{.j}$, flow inputs are needed in the amounts of $Ab_{.j}$. These, again, require inputs $A(Ab_{.j}) = A^2b_{.j}$, etc. Thus, total requirements $d_{.j}$ are given as:

$$(2.3) \quad d_{.j} = b_{.j} + Ab_{.j} + A^2b_{.j} + A^3b_{.j} + \dots$$

$$= (I - A)^{-1} b_{.j}.$$

So $d_{.j}$ is that part of total production $x(t)$ that must be imputed to a one unit increase in the production of the j th good. Total production $x(t)$ then is seen to be the weighted sum of the $d_{.j}$ values:

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The similarity with the output relation for the simple open static system is immediately clear; with b_j standing for a vector of 'direct' capital coefficients, d_j can be looked upon as a vector of 'gross' or 'integrated' capital coefficients. Regarding matrix D , we may note that traditional theory does not go beyond establishing that it is positive and has full rank, provided B has full rank, which we shall assume.

3. Another Look at Integrated Coefficients

However, closed Leontief models offer a second way to look at indirect inputs, which will give us further information. To see why, let us consider the closed static model

$$(3.1) \quad z = Mz$$

where M is the $n \times n$ matrix of direct input coefficients m_{ij} and z the total output vector. Assuming, for simplicity, that M is irreducible, output z can be written in terms of direct or first order inputs Mz . But, to produce the bundle Mz , 'second order' inputs $M(Mz) = M^2z$ are needed. However, $M^2z = Mz = z$, so we arrive at an economic interpretation of the second order system:

$$(3.2) \quad z = M^2z$$

We thus observe that the 'input' of good i required for the production of good j can be expressed by input coefficients differing from the traditional direct coefficients; the coefficients $m_{ij}^{(2)}$ of M^2 reflect an alternative way of imputing goods to the production of each good, clearly consistent with the rules of input-output accounting.

Evidently an entire class of such input matrices can be defined, each member of which incorporates more indirect inputs than the previous one. The limit case is especially interesting. If M is assumed to be primitive (which does not seem a heroic assumption), we have

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where z and p are normalized ($pz = 1$) right- and left-hand Frobenius eigenvectors of M . The limit of the sequence $M, M^2, M^3, \dots$, now gives us 'infinite order' input coefficients for our closed system. Here 'infinite' means that, given our rules of imputation, all direct and indirect inputs have been incorporated. The elements of M Infinity are a new type of 'all-in' production coefficients, as becomes clear by writing output z as

$$(3.3) \quad z = M \text{ Infinity } z.$$

(As a dual system, we have here $p = pM \text{ Infinity}$, with an analogous interpretation.) Written out, we have

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We see that each column of $M \text{ Infinity}$ is proportional to z , the total output vector. In fact, for $m^{(\text{Infinity})}_{\cdot j}$ the j th column, we have $m^{(\text{Infinity})}_{\cdot j} = p_j z$, with p_j the j th element of the standardized price vector p . (Note that each element of $M \text{ Infinity}$ has the dimension 'quantity'.) Thus, if we wish to analyse production in terms of all indirect effects, we must impute to the production of each good a scalar multiple of the output vector z . This, naturally, is equivalent to saying that the bundle i is a numeraire. (We may wish to call such a numeraire-bundle a composite numeraire.)

4. Integrated Capital Coefficients Revisited

The approach described in the previous section can be straightforwardly applied to the dynamic model in a state of balanced growth. If

$$(4.1) \quad x(t+1) = (1 + [1/Mu_1])x(t),$$

with Mu_1 the Frobenius eigenvalue of $(I - A)^{-1}B$, then, dropping the time index, we may rewrite (4.1) to

$$(4.2) \ x = Cx$$

with $C = A + [1/\mu_1]B$. The columns of matrix C stand for the current inputs needed for the production of the vector x , supplemented with the outlays on capital. By assumption, each good being produced by the economy can be used either as an intermediate input, or as a capital input. This implies that higher order input coefficients of the type introduced in the previous section can be obtained for the closed dynamic model just as easily as for the closed static model; the model's underlying assumptions guarantee that we may treat system (4.1) in the same way as we have treated system (3.1) above. Thus, if C also is assumed to be primitive, matrix

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exists, and has rank 1. Therefore, by the same argument as used in Section 3, we find that the amounts of each good i to be imputed to the production of good j (whether used as an intermediate input or as a capital good), when taking into account all indirect inputs, are given by the j th column of C Infinity]. Thus, exactly as in the static case, also for the dynamic Leontief model, total output x can be analysed in terms of a Markovian limit matrix:

$$(4.3) \ x = C \text{ Infinity } x,$$

with $C \text{ Infinity} = xp$, x and p being the vectors of standardized outputs and prices. (Because no confusion will arise, we use the symbol p also to denote the standardized price vector of the dynamic model.)

Following the above 'Markovian' imputation approach, it is not difficult to see which part of total production x is required, directly and indirectly, for the production of the capital needed for a unit increase in the production of the j th good. After some manipulation we find

$$(4.4) \ x [I - A]^{-1}B \text{Deltax}.$$

with $\text{Deltax} = (1/\mu_1)x$, and

$$(4.5) \ A = [\alpha_1x, \alpha_2x, \dots, \alpha_nx],$$

with $\alpha_j = \sum_i a_{ij}p_i$, and

$$(4.6) \ B = (\beta_1x, \beta_2x, \dots, \beta_nx),$$

with β_j equivalent to $\sum_i b_{ij}p_i$. These matrices have been introduced in equations (2.7) and (2.8) in Steenge (1990), where all entries were in nominal values. Writing D equivalent to $(I - A)^{-1}B$, we have that D has rank 1, each column $d \ x_j$ being proportional to x , our numeraire. All columns of D essentially give the same information, their only difference being a scalar multiple.

5. Conclusion

Stability and price problems have always been stumbling-blocks in the study of the dynamic Leontief model. Of course, these problems have been dealt with in all kinds of sophisticated ways. However, we do not feel that they have ever really been solved satisfactorily. Therefore we feel that it is essential to return to basics.

What has been lacking is the realization that indirect inputs can often be described not only in terms of Leontief inverses, but also in terms of limit matrices of the type we have discussed. As we have seen, both approaches provide an answer to a specific question (i.e. which part of the total output should be imputed to the production of the capital needed for the next period's increase in production). Here the 'traditional' approach directly indicates a matrix of full rank, the 'Markovian' one indicates a rank 1 matrix.

For want of a theory encompassing both approaches, the very existence of both of them gives us further insight into the model's interpretation. The sets of columns of matrices D and D , based on these two approaches, have a specific interpretation, as we have seen. If indeed it is realistic to describe an economy in terms of only an intermediate inputs and a capital coefficients matrix, and if the economy is in a state of balanced growth, then both interpretations should coincide. From a purely mathematical point of view this is an impossibility, of course (one matrix having full rank, the other rank 1). On the other hand we should not forget that we are dealing with models, i.e.

`only' approximations of reality. Therefore, if the difference between the two outcomes is not too large, we should be satisfied and be confident that a fully integrated theory is within reach. Naturally this would imply that D can be closely approximated by a rank 1 matrix. In such a case, empirical work will have to decide whether our ideas have potential.

If our approach is correct, and if the economy is in a situation of (near) balanced growth, calculating matrices A and B thus may be seen to provide an answer to the following problem. Divide economic transactions into an intermediate and a capital output part such that matrix D is `close enough' to a rank 1 matrix. Given the data now available, statisticians have done an excellent job here.

References

Steenge, A.E. (1990) On the complete instability of empirically implemented dynamic Leontief models, *Economic Systems Research*, 2, pp. 3-16.

Heesterman, A.R.G. (1990) The instability of the dynamic input-output model, *Economic Systems Research*, 2, pp. 351-355.

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