

Group representations in indefinite metric spaces

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(Received 20 April 1983; accepted for publication 18 July 1983)

A group G of symmetry transformations of the rays of an indefinite metric space V with metric operator η leads to a projective representation U of G in V in terms of η -unitary, η -antiunitary, η -pseudounitary, and η -pseudoantiunitary operators. We investigate the restrictions which are put on the irreducible components of U by the metric, and examine to what extent it is possible to decompose V into a direct sum of indefinite metric spaces, each carrying a projective representation of G . Attention is restricted to the cases where the subgroup of G which is represented by η -unitary operators is of index 1 or 2.

PACS numbers: 02.20. + b

1. INTRODUCTION

Our main motivation for studying group representations in indefinite metric spaces is the fact that twistor space is an indefinite metric space which carries in a natural way a (generalized) representation of the conformal group.^{1,2} However, our results may be of interest in any field where indefinite metric spaces are used and symmetry arguments do play a role (Ref. 3 for references).

Let V be a n -dimensional complex vector space, $n \geq 3$, with a scalar product denoted by $(,)$ and let η be a Hermitian nonsingular linear operator on V . The indefinite metric of V is given by

$$\langle \phi, \psi \rangle = (\phi, \eta \psi), \quad (1.1)$$

and η is called the metric operator.

The rays of V are the one-dimensional linear subspaces of V . If $\psi \in V$ and $\psi \neq 0$, then ψ denotes the ray which contains ψ .

The rays ψ and ϕ are said to be orthogonal (denoted by $\langle \psi, \phi \rangle = 0$) if and only if $\langle \psi, \phi \rangle = 0$ for each $\psi \in \psi$ and each $\phi \in \phi$.

The rays of V form the projective space \mathbf{V} . Let T be a bijective mapping of \mathbf{V} onto \mathbf{V} which has the property

$$\langle \psi, \phi \rangle = 0 \Leftrightarrow \langle T\psi, T\phi \rangle = 0. \quad (1.2)$$

Then we have the following theorem:

Theorem 1: There exists a operator U on V such that

$$(i) \psi \in \psi \Rightarrow U\psi \in T\psi, \quad (1.3)$$

(ii) Either U is linear and satisfies

$$\langle U\phi, U\psi \rangle = \langle \phi, \psi \rangle \quad \forall \phi, \psi \in V, \quad (1.4)$$

or U is linear and satisfies

$$\langle U\phi, U\psi \rangle = -\langle \phi, \psi \rangle \quad \forall \phi, \psi \in V, \quad (1.5)$$

or U is antilinear and satisfies

$$\langle U\phi, U\psi \rangle = \langle \psi, \phi \rangle \quad \forall \phi, \psi \in V, \quad (1.6)$$

or U is antilinear and satisfies

$$\langle U\phi, U\psi \rangle = -\langle \psi, \phi \rangle \quad \forall \phi, \psi \in V. \quad (1.7)$$

Moreover, U is determined uniquely by T up to a factor of modulus 1.

This theorem, first given in Ref. 4, is a generalization of a theorem in Ref. 3, which in turn is the generalization of Wigner's theorem to indefinite spaces. The set of operators

enumerated under (ii) of the theorem are called η -unitary, η -pseudounitary, η -antiunitary, and η -pseudoantiunitary, respectively.

Let G be a finite group and suppose that for each $g \in G$ there is a bijective mapping $T(g)$ of \mathbf{V} onto \mathbf{V} such that Eq. (1.2) holds. Then the theorem above gives us for each $g \in G$ an operator $U(g)$ on V which is either η -unitary, or η -antiunitary, or η -pseudounitary, or η -pseudoantiunitary. Since $U(g)$ is determined by $T(g)$ up to a factor of modulus 1, we have

$$U(g)U(g') = \sigma(g, g')U(gg') \quad \forall g, g' \in G, \quad (1.8)$$

where the mapping σ of $G \times G$ into the complex numbers of modulus 1 is called a factor system of G .

Let G_0 be the normal subgroup of G consisting of those elements g of G for which $U(g)$ is η -unitary. Let a, b , and c be elements of G (if any exist) such that $U(a)$ is η -antiunitary, $U(b)$ is η -pseudounitary, and $U(c)$ is η -pseudoantiunitary. Then aG_0, bG_0 , and cG_0 denote the cosets of G with respect to G_0 which consist of the elements g of G for which $U(g)$ is η -antiunitary, η -pseudounitary, and η -pseudoantiunitary, respectively. For the coset decomposition of G with respect to G_0 there are the following five possibilities:

$$(I) G = G_0, \quad (1.9a)$$

$$(II) G = G_0 + aG_0, \quad (1.9b)$$

$$(III) G = G_0 + bG_0, \quad (1.9c)$$

$$(IV) G = G_0 + cG_0, \quad (1.9d)$$

$$(V) G = G_0 + aG_0 + bG_0 + cG_0. \quad (1.9e)$$

From Eqs. (1.3)–(1.6), it follows that

$$\langle U(g)\phi, U(g)\psi \rangle = (-)_g \langle \phi, \psi \rangle^g, \quad (1.10)$$

where $(-)_g$ is defined by

$$(-)_g = \begin{cases} +1 & \text{if } g \in G_0 + aG_0 \\ -1 & \text{if } g \notin G_0 + aG_0 \end{cases} \quad (1.11)$$

and, if $\lambda \in \mathbb{C}$, λ^g is defined by

$$\lambda^g = \begin{cases} \lambda & \text{if } g \in G_0 + bG_0 \\ \lambda^* & \text{if } g \notin G_0 + bG_0, \end{cases} \quad (1.12)$$

the asterisk denoting complex conjugation.

Now choose a basis $\{e_1, e_2, \dots, e_n\}$ of V which is orthonormal with respect to the ordinary scalar product and form the matrices of $U(g)$ and η via

$$U(g)e_i = \sum_{j=1}^n U(g)_{ji} e_j \quad (1.13)$$

and

$$\eta e_i = \sum_{j=1}^n \eta_{ji} e_j. \quad (1.14)$$

These matrices are again denoted by $U(g)$ and η ; they satisfy

$$U(g)U(g')^g = \sigma(g,g')U(gg'), \quad (1.15)$$

where the upper index g is defined as in Eq. (1.12), and

$$U^\dagger(g)\eta U(g) = (-)_g \eta^g, \quad (1.16)$$

where U^\dagger denotes the Hermitian conjugate of U .

Matrices $U(g)$ satisfying Eq. (1.15) form a projective linear-antilinear (PLA) representation of G . If all the matrices are unitary, they form a projective unitary-antiunitary (PUA) representation of G . For simplicity we call such representations PLA (resp. PUA) representations even if all operators are linear.

Two PLA representations U and U' of G are equivalent if there exists a nonsingular matrix A such that

$$U'(g) = A^{-1}U(g)A^g \quad \forall g \in G. \quad (1.17)$$

If U satisfies Eq. (1.16), then U' , defined by Eq. (1.17) for some nonsingular matrix A , satisfies

$$U'^\dagger(g)\eta' U'(g) = (-)_g \eta'^g, \quad (1.18)$$

where

$$\eta' = A^\dagger \eta A. \quad (1.19)$$

So the equivalence transformation given by A transforms the Hermitian matrix η into the Hermitian matrix η' , which has the same signature as η . U is said to be decomposable if there exists an equivalent PLA representation U' such that

$$U'(g) = \begin{pmatrix} U'_1(g) & 0 \\ 0 & U'_2(g) \end{pmatrix} \quad (1.20)$$

and

$$\eta' = \begin{pmatrix} \eta'_1 & 0 \\ 0 & \eta'_2 \end{pmatrix}. \quad (1.21)$$

Equation (1.18) then becomes

$$U'^\dagger_i(g)\eta'_i U'_i(g) = (-)_g \eta_i^g \quad (i = 1, 2). \quad (1.22)$$

The aim of this paper is twofold. Our first objective is to examine to what extent it is possible to decompose a PLA representation of G which satisfies Eq. (1.16). A decomposable PLA representation of G is reducible, but the reverse is not generally true, due to the restriction made in Eq. (1.21). Our second objective is to examine which restrictions are put on the irreducible components of U by Eq. (1.16). We will restrict ourselves to the case where G_0 is a subgroup of index 1 or 2 of G ; i.e., we consider the cases I, II, III, and IV of Eq. (1.9). The study of case V is left to a later paper.

2. GENERAL REDUCTION

Due to a result of Murthy,⁵ U is equivalent to a PLA representation whose matrices are all unitary, i.e., there exists a nonsingular matrix A such that the matrices $U'(g)$ given by

$$U'(g) = A^{-1}U(g)A^g \quad (2.1)$$

form a PUA representation of G . Equation (1.16) then be-

comes

$$\eta' U'(g) = (-)_g U'(g) \eta'^g, \quad (2.2)$$

where

$$\eta' = A^\dagger \eta A. \quad (2.3)$$

There exists a unitary matrix X such that η'' , defined by

$$\eta'' = X^\dagger \eta' X, \quad (2.4)$$

is a real diagonal matrix. If

$$U''(g) = X^{-1}U'(g)X^g, \quad (2.5)$$

then U'' is a PUA representation of G which satisfies

$$\eta'' U''(g) = (-)_g U''(g) \eta''. \quad (2.6)$$

Writing out matrix elements gives

$$\eta''_{ii} U''(g)_{ij} = (-)_g U''(g)_{ij} \eta''_{jj}. \quad (2.7)$$

Now consider the cases I and II [Eqs. (1.9a) and (1.9b)]. In these cases $(-)_g = +1$ for each $g \in G$. Then $U''(g)_{ij} = 0$ whenever $\eta''_{ii} \neq \eta''_{jj}$. So by a suitable renumbering of rows and columns we have

$$U''(g) = \sum_i \oplus U''_i(g) \quad (2.8)$$

and

$$\eta'' = \sum_i \oplus \eta''_{ii}, \quad (2.9)$$

where each η''_{ii} is a multiple of the unit matrix.

So we have decomposed U'' as a direct sum of PUA representations U''_i of G and η'' in a corresponding direct sum of real multiples of the unit matrix. Each U''_i can now be reduced to a direct sum of irreducible PUA representations of G , and this equivalence transformation does not change η''_{ii} . So in the cases I and II our goal has been achieved; we have proved the following theorem.

Theorem 2: In the cases I and II there are no restrictions put by the metric on the irreducible components of a PLA representation U of G , and U is decomposable into irreducible PUA representations.

Now consider the cases where $g \in G$ exist with $(-)_g = -1$. Then we have from Eq. (2.7) that $U''(g)_{ij} = 0$ whenever $|\eta''_{ii}| \neq |\eta''_{jj}|$. So by a suitable renumbering of rows and columns we have the decomposition of Eqs. (2.8) and (2.9); but here each η''_{ii} is a real diagonal matrix with all diagonal elements equal up to a sign.

Each U''_i can now be studied separately; it satisfies

$$\eta''_i U''_i(g) = (-)_g U''_i(g) \eta''_i. \quad (2.10)$$

Let the diagonal elements of η''_i be a_i and $-a_i$, with $a_i > 0$. Since $g \in G$ exists with $(-)_g = -1$, it follows from Eq. (2.10) that $\text{Tr } \eta''_i = 0$. So a_i and $-a_i$ have equal multiplicity.

The equivalence transformation on U''_i and η''_i by the matrix A_i which is equal to $a_i^{-1/2}$ times the unit matrix leaves U''_i unchanged, but turns η''_i into a diagonal matrix with diagonal elements $+1$ and -1 with equal multiplicity. So, without loss of generality, we may assume that η''_i has eigenvalues $+1$ and -1 . In the next section we study case III and in Sec. 4 we treat case IV.

3. η -UNITARY AND η -PSEUDOUNITARY OPERATORS

In this section we consider case III [Eq. (1.9c)], i.e., the case where G is represented by η -unitary and η -pseudounitary operators. G_0 , the subgroup of G which is represented by η -unitary operators, has index 2. According to the theory of induced PUA representations, which in its most general form is given in Ref. 6, each irreducible PUA representation of G is either of type A or of type B.

A PUA representation of type A is equivalent to a PUA representation D , which can be written as

$$D(g) = \begin{pmatrix} \Delta(g) & 0 \\ 0 & \tilde{\Delta}(g) \end{pmatrix} \quad \forall g \in G_0, \quad (3.1)$$

$$D(b) = \begin{pmatrix} 0 & \sigma(b,b)\Delta(b^2) \\ \mathbf{1}_d & 0 \end{pmatrix}. \quad (3.2)$$

Here Δ is an irreducible PUA representation of G_0 ; $\tilde{\Delta}$ is an irreducible PUA representation of G_0 which is related to Δ by

$$\tilde{\Delta}(g) = \sigma(g,b)\sigma^*(b,b^{-1}gb)\Delta(b^{-1}gb) \quad (3.3)$$

and which is not equivalent to Δ . The dimension of Δ is d . $\mathbf{1}_d$ denotes the unit matrix of dimension d . D is determined up to equivalence by the equivalence classes of PUA representations of G_0 containing Δ and $\tilde{\Delta}$. A PUA representation D of type B has the property that its restriction $\Delta = D|_{G_0}$ to G_0 is irreducible. In this case Δ and $\tilde{\Delta}$ are equivalent. D is not uniquely determined by Δ : The PUA representation D' of G , given by

$$D'(g) = \begin{cases} D(g) & \text{if } g \in G_0 \\ -D(g) & \text{if } g \notin G_0, \end{cases} \quad (3.4)$$

has the same restriction to G_0 , but is not equivalent to D . However, the pair (D, D') is determined up to equivalence by the equivalence class of Δ . A pair of PUA representations of G of type B which are not equivalent, but whose restrictions to G_0 are equivalent, are said to be related. Let U be a PUA representation of G with

$$\eta U(g) = (-)_g U(g) \eta. \quad (3.5)$$

Due to the results of the previous section, we may assume that the Hermitian matrix η has eigenvalues $+1$ and -1 only, with equal multiplicity. We may now perform a unitary equivalence transformation

$$U'(g) = W^{-1} U(g) W, \quad (3.6)$$

$$\eta' = W^\dagger \eta W \quad (3.7)$$

such that the PUA representation U' of G has the following properties:

- (i) U' is a direct sum of irreducible PUA representations D_i .
- (ii) The components D_i of this direct sum are pairwise either equal or inequivalent.
- (iii) Components of type A have the form of Eqs. (3.1) and (3.2).
- (iv) Components of type B which are related satisfy Eq. (3.4).
- (v) The components are arranged into blocks U'_i :

$$U'(g) = \sum_i \oplus U'_i(g) \quad (3.8)$$

according to the following rules:

- (1) irreducible components of type A are in the same block if and only if they are equivalent (and thus equal);
- (2) irreducible components of type B are in the same block if and only if they are either equal or related.

Note that η can be chosen to be a diagonal matrix due to the results of the previous section, but this property is not inherited by η' . If the Hermitian matrix η' is divided into blocks in the same way as U' ,

$$\eta' = \begin{pmatrix} \eta'_{[1,1]} \cdots \eta'_{[1,p]} \\ \eta'_{[p,1]} \cdots \eta'_{[p,p]} \end{pmatrix}, \quad (3.9)$$

we find

$$\eta'_{[i,k]} U'_k(g) = (-)_g U'_i(g) \eta'_{[i,k]}. \quad (3.10)$$

If $i \neq k$, the blocks U'_k and U'_i , when restricted to G_0 , are PUA representations of G_0 which have no irreducible components in common. Due to Schur's lemma, we thus have

$$\eta'_{[i,k]} = 0 \quad (3.11)$$

if $i \neq k$. This means that we have found a decomposition of U' into blocks U'_i ; each block U'_i can now be studied further separately. Let V be some block U'_i , and let ζ be the Hermitian matrix $\eta'_{[i,i]}$. Then

$$\zeta V(g) = (-)_g V(g) \zeta. \quad (3.12)$$

ζ is Hermitian and has eigenvalues $+1$ and -1 with equal multiplicity. Consider first the case that V consists of irreducible PUA representations D of G type A. We may then write

$$V(g) = \mathbf{1}_n \otimes \begin{pmatrix} \Delta(g) & 0 \\ 0 & \tilde{\Delta}(g) \end{pmatrix} \quad \forall g \in G_0, \quad (3.13)$$

$$V(b) = \mathbf{1}_n \otimes \begin{pmatrix} 0 & \sigma(b,b)\Delta(b^2) \\ \mathbf{1}_d & 0 \end{pmatrix}. \quad (3.14)$$

Here n is the multiplicity of D in V . Divide ζ into blocks in the same way as V :

$$\zeta = \begin{pmatrix} \zeta_{[1,1]} \zeta_{[1,2]} \cdots \zeta_{[1,n]} \\ \zeta_{[n,1]} \cdots \zeta_{[n,n]} \end{pmatrix}. \quad (3.15)$$

From Eqs. (3.12)–(3.14) and Schur's lemma it follows that each $\zeta_{[ij]}$ satisfies

$$\zeta_{[ij]} = \begin{pmatrix} a_{ij} \mathbf{1}_d & 0 \\ 0 & -a_{ij} \mathbf{1}_d \end{pmatrix} \quad (3.16)$$

for some $a_{ij} \in \mathbb{C}$. So we can write

$$\zeta = A \otimes \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix}, \quad (3.17)$$

where A is a Hermitian $n \times n$ matrix. There exists a unitary matrix X such that A' , defined by

$$A' = X^\dagger A X \quad (3.18)$$

is a diagonal matrix; its diagonal elements are equal to $+1$ or -1 . Define the unitary matrix Y by

$$Y = X \otimes \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & \mathbf{1}_d \end{pmatrix} \quad (3.19)$$

and perform the equivalence transformation on V and ζ by

Y . Then V is left unchanged, but ζ turns into ζ' , where

$$\zeta' = A' \otimes \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix}. \quad (3.20)$$

So ζ' is a diagonal matrix with diagonal elements $+1$ and -1 with equal multiplicity. This means that V has been decomposed to a direct sum of irreducible PUA representations of G .

Now consider the case that V consists of irreducible PUA representations of G of type B; suppose the multiplicity of D in V is n and the multiplicity of D' in V is m . With a suitable numbering of rows and columns we have

$$V(g) = \begin{pmatrix} \mathbf{1}_n \otimes D(g) & 0 \\ 0 & \mathbf{1}_m \otimes D(g) \end{pmatrix} \quad \forall g \in G_0 \quad (3.21)$$

and

$$V(b) = \begin{pmatrix} \mathbf{1}_n \otimes D(b) & 0 \\ 0 & -\mathbf{1}_m \otimes D(b) \end{pmatrix}. \quad (3.22)$$

Let ζ be divided accordingly:

$$\zeta = \begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{pmatrix}. \quad (3.23)$$

From Eqs. (3.12) and (3.21) and Schur's lemma it follows that $\zeta_{11} = 0$ and $\zeta_{22} = 0$. Since $\det \zeta \neq 0$, the dimension of ζ_{11} and ζ_{22} must be equal. Thus $n = m$, and we arrive at the following restriction put on the irreducible components of U by the metric:

Theorem 3: In a PLA representation of G belonging to case III related irreducible PUA representations have the same multiplicity.

From Eq. (3.12), Schur's lemma, and the Hermiticity of ζ , it follows that

$$\zeta_{12} = \chi \otimes \mathbf{1}_d \quad (3.24)$$

and

$$\zeta_{21} = \chi^\dagger \otimes \mathbf{1}_d \quad (3.25)$$

for some $n \times n$ matrix χ . Since ζ is Hermitian and has eigenvalues $+1$ and -1 only, the matrix ζ^2 equals $\mathbf{1}_n$. This implies that χ is unitary. Define the unitary matrix Y by

$$Y = \begin{pmatrix} \mathbf{1}_{nd} & 0 \\ 0 & \chi^\dagger \otimes \mathbf{1}_d \end{pmatrix} \quad (3.26)$$

and perform the equivalence transformation on V and ζ by Y . Then V is left unchanged, but ζ turns into ζ' , where

$$\zeta' = \begin{pmatrix} 0 & \mathbf{1}_{nd} \\ \mathbf{1}_{nd} & 0 \end{pmatrix}. \quad (3.27)$$

By a suitable reordering of rows and columns, we may now write V and ζ' as follows:

$$V(g) = \mathbf{1}_n \otimes \begin{pmatrix} D(g) & 0 \\ 0 & D(g) \end{pmatrix} \quad \forall g \in G_0, \quad (3.28)$$

$$V(b) = \mathbf{1}_n \otimes \begin{pmatrix} D(b) & 0 \\ 0 & -D(b) \end{pmatrix}, \quad (3.29)$$

$$\zeta' = \mathbf{1}_n \otimes \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}. \quad (3.30)$$

So V has been decomposed into components which are pairs of an irreducible PUA representation and its related PUA

representation. A further decomposition of these components is not possible. This follows from the fact that if V is an irreducible PUA representation of G of type B, there exists no Hermitian matrix ζ which satisfies Eq. (3.12). So we have derived the following theorem:

Theorem 4: Any PLA representation of G belonging to case III is decomposable into irreducible PUA representations of type A and pairs of related irreducible PUA representations of type B.

4. η -UNITARY AND η -PSEUDOANTIUNITARY OPERATORS

In this section we consider case IV [Eq. (1.9d)], i.e., the case where G is represented by η -unitary and η -pseudoantiunitary operators. G_0 , the subgroup of G which is represented by η -unitary operators had index 2. Each irreducible PUA representation of G is either of type A, or of type B or of type C.

A PUA representation D of type A has the property that its restriction $\Delta = D \downarrow G_0$ to G_0 is irreducible. D is determined up to equivalence by the equivalence class of Δ . A PUA representation of type B is equivalent to a PUA representation D , which can be written as

$$D(g) = \begin{pmatrix} \Delta(g) & 0 \\ 0 & \Delta(g) \end{pmatrix} \quad \forall g \in G_0, \quad (4.1)$$

$$D(c) = \begin{pmatrix} 0 & \mathcal{D} \\ -\mathcal{D} & 0 \end{pmatrix}. \quad (4.2)$$

D is determined up to equivalence by the equivalence class of Δ . A PUA representation of type C is equivalent to a PUA representation D , which can be written as

$$D(g) = \begin{pmatrix} \Delta(g) & 0 \\ 0 & \bar{\Delta}(g) \end{pmatrix} \quad \forall g \in G_0, \quad (4.3)$$

$$D(c) = \begin{pmatrix} 0 & \sigma(c,c)\Delta(c^2) \\ \mathbf{1}_d & 0 \end{pmatrix}. \quad (4.4)$$

Δ and $\bar{\Delta}$ are irreducible d -dimensional PUA representations of G_0 , which are not equivalent to each other and are related by

$$\bar{\Delta}(g) = \sigma(g,c)\sigma^*(c,c^{-1}gc)\Delta^*(c^{-1}gc). \quad (4.5)$$

D is determined up to equivalence by the equivalence classes of Δ and $\bar{\Delta}$. Let U be a PUA representation of G with

$$\eta U(g) = (-) U(g) \eta^g. \quad (4.6)$$

Due to the results of Sec. 2, we may assume that the Hermitian matrix η has eigenvalues $+1$ and -1 only, with equal multiplicity. We may now perform a unitary equivalence transformation

$$U'(g) = W^{-1}U(g)W^g, \quad (4.7)$$

$$\eta' = W^\dagger \eta W \quad (4.8)$$

such that the PUA representation U' of G has the following properties:

- (i) U' is a direct sum of irreducible PUA representations D_i .
- (ii) The components D_i of this direct sum are pairwise either equal or inequivalent.
- (iii) Components of type B have the form of Eqs. (4.1)

and (4.2).

(iv) Components of type C have the form of Eqs. (4.3)

and (4.4).

(v) The components are arranged into blocks U'_i :

$$U'(g) = \sum_i \oplus U'_i(g) \quad (4.9)$$

such that irreducible components of U' are in the same block if and only if they are equivalent (and thus equal).

Equation (4.9) is a decomposition of U' . This is proved in the same way as the corresponding decomposition in the previous section. So each block U'_i can now be studied separately.

Let V be some block U'_i , and let ζ be the corresponding block of η' . Then

$$\zeta V(g) = (-)_g V(g) \zeta^g. \quad (4.10)$$

ζ is Hermitian and has eigenvalues $+1$ and -1 with equal multiplicity. Consider first the case that V consists of irreducible PUA representations D of G of type A. Then we may write

$$V(g) = \mathbf{1}_n \otimes \Delta(g) \quad \forall g \in G_0 \quad (4.11)$$

and

$$V(c) = \mathbf{1}_n \otimes D(c). \quad (4.12)$$

Here n is the multiplicity of D in V . Divide ζ into blocks in the same way as V [Eq. (3.15)]. Then it follows from Eqs. (4.10) and (4.11) and Schur's lemma that

$$\zeta_{\{i,j\}} = a_{ij} \mathbf{1}_d \quad (4.13)$$

for some $a_{ij} \in \mathbb{C}$. We thus have

$$\zeta = A \otimes \mathbf{1}_d \quad (4.14)$$

for some Hermitian $n \times n$ matrix A . From Eqs. (4.10) and (4.12) it now follows that

$$A = -A^*. \quad (4.15)$$

Thus the dimension of A is even, and thus we have arrived at the following restriction put by the metric on the irreducible components of U :

Theorem 5: In a PLA representation of G belonging to case IV the irreducible PUA representations of type A have even multiplicity.

Now there exists an orthogonal matrix B such that

$$B^\dagger A B = \mathbf{1}_{n/2} \otimes \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \quad (4.16)$$

This follows from §13 of Chap. IX of Ref. 7, since iA is real, skew-symmetric, and has eigenvalues $+i$ and $-i$ with equal multiplicity. Define

$$C = B \otimes \mathbf{1}_d \quad (4.17)$$

and perform the equivalence transformation on V and ζ by C . Then V is left unchanged, and ζ is transformed into

$$\zeta' = \mathbf{1}_{q/2} \otimes \begin{pmatrix} 0 & i\mathbf{1}_d \\ -i\mathbf{1}_d & 0 \end{pmatrix}. \quad (4.18)$$

It is seen that we have decomposed V into blocks containing two equal irreducible PUA representations of type A.

Further decomposition is not possible, due to Theorem 5.

Next consider the case where V consists of irreducible

PUA representations D of type B. With a suitable numbering of rows and columns, we may write

$$V(g) = \begin{pmatrix} \mathbf{1}_n \otimes \Delta(g) & 0 \\ 0 & \mathbf{1}_n \otimes \bar{\Delta}(g) \end{pmatrix} \quad \forall g \in G_0 \quad (4.19)$$

and

$$V(c) = \begin{pmatrix} 0 & \mathbf{1}_n \otimes \mathcal{D} \\ -\mathbf{1}_n \otimes \mathcal{D} & 0 \end{pmatrix}. \quad (4.20)$$

From Eqs. (4.10), (4.19), and (4.20) and Schur's lemma it follows that

$$\zeta = \chi \otimes \mathbf{1}_d, \quad (4.21)$$

where χ is a Hermitian $2n \times 2n$ matrix which is partitioned into square blocks as follows:

$$\chi = \begin{pmatrix} \chi_1 & \chi_2 \\ \chi_2^* & -\chi_1^* \end{pmatrix}. \quad (4.22)$$

Due to a lemma, which we prove in the Appendix, there exists a unitary $2n \times 2n$ matrix U which is partitioned into square blocks as follows:

$$U = \begin{pmatrix} U_1 & U_2 \\ -U_2^* & U_1^* \end{pmatrix} \quad (4.23)$$

and which satisfies

$$U^\dagger \chi U = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & -\mathbf{1}_n \end{pmatrix}. \quad (4.24)$$

Define the unitary matrix Y by

$$Y = U \otimes \mathbf{1}_d \quad (4.25)$$

and perform the equivalence transformation on V and ζ by Y . Then V is left unchanged and ζ is transformed into

$$\zeta' = U^\dagger \chi U \otimes \mathbf{1}_d = \begin{pmatrix} \mathbf{1}_{nd} & 0 \\ 0 & -\mathbf{1}_{nd} \end{pmatrix}. \quad (4.26)$$

This means that V has been decomposed to a direct sum of irreducible PUA representations of G .

Finally consider the case that V consists of irreducible PUA representations D of type C. With a suitable numbering of rows and columns, we may write

$$V(g) = \begin{pmatrix} \mathbf{1}_n \otimes \Delta(g) & 0 \\ 0 & \mathbf{1}_n \otimes \bar{\Delta}(g) \end{pmatrix} \quad \forall g \in G_0 \quad (4.27)$$

and

$$V(c) = \begin{pmatrix} 0 & \sigma(c,c) \mathbf{1}_n \otimes \Delta(c^2) \\ \mathbf{1}_{nd} & 0 \end{pmatrix}. \quad (4.28)$$

From Eqs. (4.10), (4.27), and (4.28) and Schur's lemma it follows that

$$\zeta = \chi \otimes \mathbf{1}_d, \quad (4.29)$$

where χ is a Hermitian $2n \times 2n$ matrix which is partitioned into square blocks as follows:

$$\chi = \begin{pmatrix} \chi_1 & 0 \\ 0 & -\chi_1^* \end{pmatrix}. \quad (4.30)$$

Let U be a unitary $n \times n$ matrix such that $U^\dagger \chi_1 U$ is on diagonal form. Define the unitary matrix Y by

$$Y = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} \otimes \mathbf{1}_d \quad (4.31)$$

and perform the equivalence transformation on V and ξ by Y . Then V is left unchanged and ξ is transformed into

$$\xi' = \begin{pmatrix} U^\dagger \chi_1 U & 0 \\ 0 & -(U^\dagger \chi_1 U)^* \end{pmatrix} \otimes \mathbb{1}_d, \quad (4.32)$$

which is on diagonal form. This means that V has been decomposed into a direct sum of irreducible PUA representations of G . So we have derived the following theorem:

Theorem 6: Any PLA representation of G belonging to case IV is decomposable into a direct sum of irreducible PUA representations of type B, irreducible PUA representations of type C, and pairs of irreducible PUA representations of type A.

APPENDIX

This appendix is devoted to the proof of the following lemma:

Lemma: Let H be a $2n \times 2n$ Hermitian matrix which is partitioned into square blocks as follows:

$$H = \begin{pmatrix} H_1 & H_2 \\ H_2^* & -H_1^* \end{pmatrix} \quad (A1)$$

and which has only eigenvalues $+1$ and -1 . Let I be defined by

$$I = \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & -\mathbb{1}_n \end{pmatrix}. \quad (A2)$$

There exists a unitary matrix U which is partitioned into square blocks as follows:

$$U = \begin{pmatrix} U_1 & -U_2^* \\ U_2 & U_1^* \end{pmatrix}. \quad (A3)$$

and which has the property that

$$U^\dagger H U = I. \quad (A4)$$

Proof: Since $\text{Tr } H = 0$ the eigenvalues $+1$ and -1 have equal multiplicity. Thus there exists a unitary matrix V such that

$$V^\dagger H V = I. \quad (A5)$$

This implies that

$$H V = V I. \quad (A6)$$

If V is partitioned into square blocks as follows,

$$V = \begin{pmatrix} V_1 & V_3 \\ V_2 & V_4 \end{pmatrix}, \quad (A7)$$

then Eq. (A6) can be written as

$$H_1 V_1 + H_2 V_2 = V_1, \quad (A8)$$

$$H_1 V_3 + H_2 V_4 = -V_3, \quad (A9)$$

$$H_2^* V_1 - H_1^* V_2 = V_2, \quad (A10)$$

$$H_2 V_3^* - H_1^* V_4 = -V_4. \quad (A11)$$

Now define the matrix U by

$$U = \begin{pmatrix} U_1 & U_3 \\ U_2 & U_4 \end{pmatrix} = \begin{pmatrix} V_1 & -V_2^* \\ V_2 & V_1^* \end{pmatrix}. \quad (A12)$$

Then U satisfies equations (A8)–(A11) if V_i in these equations is replaced by U_i . Consequently,

$$H U = U I. \quad (A13)$$

Since U has the form prescribed by Eq. (A3), the lemma is proved if we show that U is unitary. Since V is unitary, we have

$$V_1^\dagger V_1 + V_2^\dagger V_2 = \mathbb{1}. \quad (A14)$$

Expressing $U^\dagger U$ in terms of V_1 and V_2 gives, with Eq. (A14),

$$U^\dagger U = \begin{pmatrix} \mathbb{1} & -V_1^\dagger V_2^* + V_2^\dagger V_1^* \\ -V_2^{*\dagger} V_1 + V_1^{*\dagger} V_2 & \mathbb{1} \end{pmatrix}. \quad (A15)$$

Equation (A13) implies

$$U^\dagger H U = U^\dagger U I. \quad (A16)$$

Taking the Hermitian adjoint of equation (A13) and multiplying with U from the right gives

$$U^\dagger H U = I U^\dagger U. \quad (A17)$$

From Eqs. (A16) and (A17), it follows that $U^\dagger U$ commutes with I . This fact, together with Eq. (A15), implies that $U^\dagger U$ is the unit matrix. Thus U is unitary, which proves the lemma.

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