Discrete Mathematics 44 (1983) 309–316
North-Holland Publishing Company

EXISTENCE OF D_λ-CYCLES AND D_λ-PATHS

H.J. VELDMAN

Department of Applied Mathematics, Twente University of Technology, Enschede, the Netherlands

Received 4 March 1982
Revised 28 May 1982

A cycle C of a graph G is called a D_λ-*cycle* if every component of $G - V(C)$ has order less than λ. A D_λ-*path* is defined analogously. In particular, a D_1-cycle is a hamiltonian cycle and a D_1-path is a hamiltonian path. Necessary conditions and sufficient conditions are derived for graphs to have a D_λ-cycle or D_λ-path. The results are generalizations of theorems in hamiltonian graph theory. Extensions of notions such as vertex degree and adjacency of vertices to subgraphs of order greater than 1 arise in a natural way

1. Introduction

We employ the terminology of Bondy and Murty [3] and consider only simple graphs.

In [2], Bondy stated a sufficient condition for a graph G to have a cycle C such that $G - V(C)$ contains no K_k. For $k = 1$, it coincides with Ore's condition for the existence of a hamiltonian cycle. Here we introduce another kind of generalized hamiltonian cycle. A cycle C of a graph G is a D_λ-*cycle* if all components of $G - V(C)$ have order less than λ. Alternatively, C is a D_λ-cycle of G if and only if every connected subgraph of order λ of G has at least one vertex with C in common. Thus a D_λ-cycle dominates all connected subgraphs of order λ. Analogously, a path P of G is a D_λ-*path* if every component of $G - V(P)$ has order less than λ. Graphs containing a D_λ-cycle (D_λ-path) will be called D_λ-*cyclic* (D_λ-*traceable*). A D_1-cycle (D_1-path) is the same as a hamiltonian cycle (hamiltonian path). D_2-cycles were studied in [6].

In subsequent sections, existence theorems for D_λ-cycles are proved. In [6], most of them were already proved for $\lambda = 2$. We will henceforth refrain from referring to these special cases, unless this is essential. Parallel results on D_λ-paths can be obtained, using the following obvious lemma.

Lemma 1. *A graph G is D_λ-traceable if and only if $G \vee K_1$ is D_λ-cyclic.*

The theorems derived are generalizations of known results in hamiltonian graph theory. A corresponding remark can be made about the proof techniques used. Some of the results in Section 3 are closely related to Bondy's work [2].

Extensions to subgraphs of order greater than 1 of concepts such as adjacency of vertices, independence number and vertex degree arise in correspondence with the generalization of hamiltonian cycles to D_λ-cycles.

2. A necessary condition in terms of cut sets

To start with, we generalize a necessary condition for the existence of a hamiltonian cycle.

Theorem A [3, Theorem 4.2]. *If a graph G is hamiltonian, then, for every nonempty proper subset S of $V(G)$,*

$$\omega(G-S)\le|S|.$$

Denote by $\omega_\lambda(G)$ the number of components of G of order at least λ. Theorem A is then a special case ($\lambda=1$) of

Theorem 1. *If a graph G is D_λ-cyclic, then, for every nonempty proper subset S of $V(G)$,*

$$\omega_\lambda(G-S)\le|S|.$$

The proof, being an easy extension of the proof of [6, Theorem 1], is omitted.

For future reference we denote by $\mathscr{K}_\lambda$ the class of graphs not satisfying the necessary condition of Theorem 1. Thus G is in $\mathscr{K}_\lambda$ iff, for some nonempty proper subset S of $V(G)$, $\omega_\lambda(G-S)>|S|$.

3. Sufficient conditions involving subgraph degrees

We now turn our attention to sufficient conditions for the existence of D_λ-cycles. One of the earliest results in hamiltonian graph theory to be generalized here is due to Dirac.

Theorem B [3, Theorem 4.3]. *If G is a graph with $\nu\ge3$ and $\delta\ge\frac{1}{2}\nu$, then G is hamiltonian.*

We also mention a result of Chvátal and Erdös.

Theorem C [4, Theorem 1]. *If G is a k-connected graph with $\nu\ge3$ and $\alpha\le k$, then G is hamiltonian.*

Bondy proved a common generalization of Theorems B and C.

Theorem D [2, Theorem 2]. *Let G be a k-connected graph with $\nu \geq 3$ such that, for every $k+1$ mutually nonadjacent vertices $u_0, u_1, \ldots, u_k$ of G,*

$$\sum_{i=0}^{k} d(u_i) > \tfrac{1}{2}(k+1)(\nu-1).$$

Then G is hamiltonian.

In order to extend Theorems B, C and D to results on D_λ-cycles for $\lambda > 1$ we need some additional definitions. As in [6], two subgraphs H_1 and H_2 of a graph G are said to be *close* in G if they are disjoint and there is an edge of G joining a vertex of H_1 and one of H_2; if no such edge exists in G, then H_1 and H_2, provided they are disjoint, are *remote* in G. Thus, if H_1 and H_2 both consist of exactly one vertex, H_1 and H_2 are close (remote) iff the corresponding vertices are adjacent (nonadjacent). By $\alpha_\lambda(G)$ (or just α_λ) we denote the maximum number of mutually remote connected subgraphs of order λ of G. Thus α_1 coincides with the independence number α. The *degree of a subgraph* H of G, denoted $d_G(H)$ or $d(H)$, is the number of vertices in $V(G) - V(H)$ adjacent to one or more vertices of H. In other words, considering vertices as subgraphs of order 1, $d(H)$ is the number of vertices of G close to H. If H consists of a single vertex, then $d(H)$ is just the degree of this vertex. The minimum degree of connected subgraphs of order λ will be denoted δ_λ, so that $\delta_1 = \delta$. If Q is an oriented cycle or path in a graph and u and v are vertices on Q, then $\vec{Q}[u, v]$ and $\overleftarrow{Q}[v, u]$ denote, respectively, the segment of Q from u to v and the reverse segment from v to u. Furthermore, $\vec{Q}(u, v] := \vec{Q}[u, v] - \{u\}$, $\vec{Q}[u, v) := \vec{Q}[u, v] - \{v\}$ and $\vec{Q}(u, v) := \vec{Q}[u, v] - \{u, v\}$. Three more defining relations are obtained by reversing the arrows in the previous sentence.

We are now ready to prove a generalization of Theorem C.

Theorem 2. *Let k and λ be positive integers such that either $k \geq 2$ or $k = 1$ and $\lambda \leq 2$. If G is a k-connected graph, other than a tree (in case $k = 1$), with $\alpha_\lambda \leq k$, then G is D_λ-cyclic.*

Proof. By contraposition. Let G be a k-connected non-D_λ-cyclic graph other than a tree. We will show that $\alpha_\lambda > k$. Put $t + 1 = \min\{i \mid G$ is D_i-cyclic$\}$, so that $t \geq \lambda$. Let C be a longest D_{t+1}-cycle among all D_{t+1}-cycles C' of G for which $\omega_t(G - V(C'))$ is minimum. As in the proof of [6, Theorem 3] one shows that C has length at least $k+1$. Fix an orientation on C. By assumption, C is a D_{t+1}-cycle, but not a D_t-cycle of G. Hence $G - V(C)$ has a component H_0 of order t. All vertices of G close to H_0 are on C and, since G is k-connected and $|V(C)| \geq k$, we have that $d(H_0) \geq k$. Let $v_1, \ldots, v_k$ be k vertices of C close to H_0. For $i = 1, \ldots, k$, let u_{0i} be a vertex of H_0 adjacent to v_i (for $i \neq j$, u_{0i} and u_{0j} may coincide). Assume that $v_1, \ldots, v_k$ occur on C in the order of their indices and let u_{i1} be the immediate successor of v_i on C ($i = 1, \ldots, k$). It will prove possible to choose, for $i=1,\ldots,k$, a subgraph H_i of G satisfying the following requirements:

(i) H_i is connected and has order t,

(ii) $H_i \cap C = \vec{C}[u_{i1}, u_{i2}]$, where u_{i2} is a vertex of $\vec{C}[u_{i1}, v_i)$ chosen in such a way that

(iii) The length of $\vec{C}[u_{i1}, u_{i2}]$ is minimum, i.e. if H is a connected subgraph of order t of G with $H \cap C = \vec{C}[u_{i1}, w]$, then $\vec{C}[u_{i1}, u_{i2}]$ is a subpath of $\vec{C}[u_{i1}, w]$. Note that u_{i1} and u_{i2} may coincide, in other words $\vec{C}[u_{i1}, u_{i2}]$ may have length 0.

If $k=1$, then C may have length 3 and the existence of a subgraph H_1 with the above properties is guaranteed only if $t \leqslant 2$.

If $k \geqslant 2$, then, for $1 \leqslant i \leqslant k$,

(a) a subgraph H_i with the mentioned properties exists, and

(b) v_{i+1} does not belong to $\vec{C}[u_{i1}, u_{i2}]$ (indices mod k).

Assuming the contrary to (a) or (b), consider the cycle

$$C' = v_i u_{0i} \vec{P}[u_{0i}, u_{0,i+1}] u_{0,i+1} v_{i+1} \vec{C}[v_{i+1}, v_i],$$

where P is a $u_{0i}u_{0,i+1}$-path within H_0 (degenerate if $u_{0i} = u_{0,i+1}$). By assumption, $\vec{C}(v_i, v_{i+1})$ is not contained in a component of order at least t of $G - V(C')$. Since, moreover, $|H_0 - V(C')| < t$, it follows that C' is a D_{t+1}-cycle of G with $\omega_t(G - V(C')) < \omega_t(G - V(C))$, contradicting the choice of C.

Thus we have shown that, for $1 \leqslant i \leqslant k$, a subgraph H_i satisfying the requirements (i), (ii) and (iii) indeed exists, provided $t \leqslant 2$ in case $k=1$. Following an analogous reasoning one proves that H_0 and H_i are disjoint and, a fortiori, remote.

Next we prove by contradiction that, for $1 \leqslant i < j \leqslant k$, the subgraphs H_i and H_j are remote. Assume that H_i and H_j are close or non-disjoint. Then a $u_{i2}u_{j2}$-path P' can be found such that

(1) $P' \cap C = \overleftarrow{C}[u_{i2}, w_i] \cup \vec{C}[w_j, u_{j2}]$, where w_i and w_j are vertices of $\overleftarrow{C}[u_{i2}, u_{i1}]$ and $\vec{C}[u_{j1}, u_{j2}]$, respectively,

(2) no vertex of $V(P') - V(C)$ is in H_0,

(3) the sum of the lengths of $\overleftarrow{C}[u_{i2}, w_i]$ and $\vec{C}[w_j, u_{j2}]$ is maximum, i.e. no $u_{i2}u_{j2}$-path satisfying (1) and (2) has more vertices with C in common than P'.

Now consider the cycle

$$C'' = v_i u_{0i} \vec{P}''[u_{0i}, u_{0j}] u_{0j} v_j \overleftarrow{C}[v_j, u_{i2}] \vec{P}'[u_{i2}, u_{j2}] \overleftarrow{C}[u_{j2}, v_i],$$

where P'' is a $u_{0i}u_{0j}$-path in H_0. In Fig. 1 the cycle C'' is indicated by arrows.

Denote by L_i and L_j the components of $G - V(C'')$ containing the vertices (if any) of $\vec{C}[u_{i1}, w_i)$ and $\vec{C}[u_{j1}, w_j)$, respectively. If L_i and L_j would coincide, then a $u_{i2}u_{j2}$-path satisfying (1) and (2) could be indicated having more vertices with C in common than P', a contradiction with the choice of P'. Thus L_i and L_j are distinct. Moreover, by the way H_i and H_j were chosen, both L_i and L_j have order less than t (otherwise (iii) would be violated). But then C'' is a D_{t+1}-cycle with $\omega_t(G - V(C'')) < \omega_t(G - V(C))$, contradicting the choice of C.

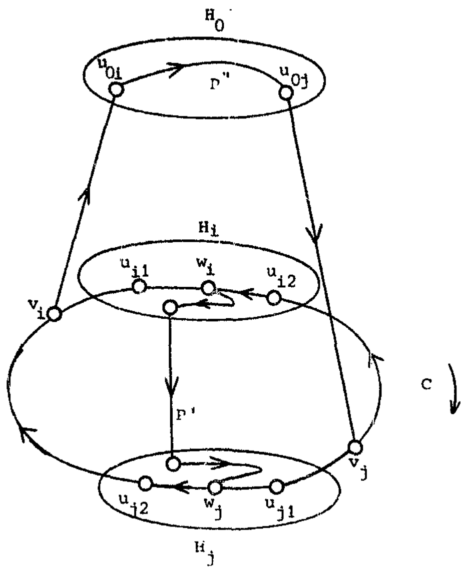

Fig. 1.

Thus we have shown that the connected subgraphs $H_0, H_1, \ldots, H_k$ of G of order t are mutually remote, so that $\alpha_t > k$. Since α_x is easily seen to be a nonincreasing function of x, it follows that $\alpha_\lambda \geqslant \alpha_t > k$. □

For $s \geqslant \lambda$, the graph $K_k \vee (k+1)K_s$ is non-D_λ-cyclic and satisfies $\alpha_\lambda = k+1$, showing that Theorem 2 is, in a sense, best possible.

Theorem 2 can be improved to a generalization of Theorem D. Referring to the proof of Theorem 2, it can be shown that

$$d(H_i) + d(H_j) \leqslant \nu + k - \lambda - k\lambda \qquad (0 \leqslant i < j \leqslant k).$$

Bondy [2] showed these inequalities to hold in case $\lambda = 1$. The proof of the general case is completely analogous and hence omitted. Summing the above inequalities eventually yields

Theorem 3. *Let k and λ be positive integers such that either $k \geqslant 2$ or $k = 1$ and $\lambda \leqslant 2$. If G is a k-connected graph, other than a tree, such that, for every $k+1$ mutually remote connected subgraphs $H_0, H_1, \ldots, H_k$ of order λ of G,*

$$\sum_{i=0}^{k} d(H_i) > \tfrac{1}{2}(k+1)(\nu + k - \lambda - k\lambda),$$

then G is D_λ-cyclic.

From Theorem 3 one easily deduces a generalization of Theorem B: a k-connected graph with $\delta_\lambda > \frac{1}{2}(\nu + k - \lambda - k\lambda)$ is D_λ-cyclic ($k \geq 2$ or $k = 1$ and $\lambda \leq 2$). However, we can do better.

Theorem 4. *Let k and λ be positive integers such that either $k \geq 2$ or $k = 1$ and $\lambda \leq 2$. If G is a k-connected graph, other than a tree, with*

$$\delta_\lambda > \begin{cases} (\nu - (k+1)\lambda + k^2)/(k+1) & \text{if } \lambda \geq k, \\ (\nu - \lambda)/(\lambda + 1) & \text{if } \lambda \leq k, \end{cases}$$

then G is D_λ-cyclic.

Proof. By contraposition. Assume that G is k-connected and non-D_λ-cyclic. Set $t + 1 = \min\{i \mid G \text{ is } D_i\text{-cyclic}\}$, so that $t \geq \lambda$. Let C be a D_{t+1}-cycle of G for which $\omega_t(G - V(C))$ is minimum. We may assume C to have length at least k. Let H_0 be a component of $G - V(C)$ of order t and let $v_1, \ldots, v_m$ be the vertices of C close to H_0, where $m = d(H_0)$. Choose to each v_i a subgraph H_i of G of order t as in the proof of Theorem 2 ($i = 1, \ldots, m$). The choice of C then implies, among other things, that the vertex sets $V(H_0)$, $V(H_1)$, ..., $V(H_m)$ and $\{v_1, \ldots, v_m\}$ are mutually disjoint. Thus

$$\nu \geq (d(H_0) + 1)t + d(H_0), \tag{1}$$

or, equivalently,

$$d(H_0) \leq (\nu - t)/(t + 1)$$

and consequently

$$\delta_\lambda \leq (\nu - t)/(t + 1) + t - \lambda. \tag{2}$$

Since G is k-connected, H_0 has degree at least k, so (1) implies that

$$\nu \geq (k + 1)t + k. \tag{3}$$

If $\lambda \geq k$, then also $t \geq k$. The inequality (3) is then equivalent to

$$\frac{\nu - t}{t + 1} + t - \lambda \leq \frac{\nu - (k+1)\lambda + k^2}{k + 1}. \tag{4}$$

Combination of (2) and (4) proves the first part of the theorem.

If $\lambda \leq k$, then from (3) it follows that

$$\nu \geq (\lambda + 1)t + \lambda. \tag{5}$$

Since $t \geq \lambda$, the inequality (5) is satisfied if and only if

$$\frac{\nu - t}{t + 1} + t - \lambda \leq \frac{\nu - \lambda}{\lambda + 1}. \tag{6}$$

The proof is now completed by combining (2) and (6). □

For $\lambda \geq k$, the collection $\{K_k \vee (k+1)K_t \mid t \geq \lambda\}$ consists of infinitely many k-connected non-D_λ-cyclic graphs with $\delta_\lambda = (\nu - (k+1)\lambda + k^2)/(k+1)$. If $\lambda \leq k$, then $\{K_t \vee (t+1)K_\lambda \mid t \geq k\}$ is an infinite collection of k-connected non-D_λ-cyclic graphs with $\delta_\lambda = (\nu - \lambda)/(\lambda+1)$. Thus Theorem 4 is, in a sense, best possible.

In view of Theorem 4, Theorem 3 might be improved to

Conjecture 1. Let k and λ be positive integers satisfying either $k \geq 2$ or $k = 1$ and $\lambda \leq 2$. If G is a k-connected graph, other than a tree, such that, for every $k+1$ mutually remote connected subgraphs $H_0, H_1, \ldots H_k$ of order λ of G,

$$\sum_{i=0}^{k} d(H_i) > \begin{cases} \nu - (k+1)\lambda + k^2 & \text{if } \lambda \geq k \\ (k+1)(\nu - \lambda)/(\lambda+1) & \text{if } \lambda \leq k, \end{cases}$$

then G is D_λ-cyclic.

If H is a subgraph of order k of a graph G and v is a vertex of H, then $d(H) \geq d(v) - k + 1$. From this observation one easily deduces that the truth of Conjecture 1 (for $\lambda = k$) would imply the truth of the following, which is a weaker version of a conjecture due to Bondy.

Conjecture A (cf. [2, Conjecture 1]). Let G be a k-connected graph such that the degree-sum of every $k+1$ independent vertices is at least $\nu + k(k-1)$, where $\nu \geq 3$. Then there exists a cycle C of G such that $G - V(C)$ contains no path of length $k-1$.

In fact, Bondy conjectured that, under the condition of Conjecture A, every longest cycle C of G has the property that $G - V(C)$ contains no path of length $k-1$.

So far, the truth of Conjecture 1 has been established in the following cases:

(a) $\lambda = 1$ and $k \geq 1$ (Theorem D),

(b) $\lambda = 2$ and $k = 1$ [6, Theorem 2],

(c) $\lambda = 2$ and $k = 2$ [6, Corollary 3.2].

Without giving it we mention that the proof of [6, Corollary 3.2] is easily extended to a proof of Conjecture 1 for $k = 2$ and $\lambda > 2$.

Theorem 5. *Let G be a 2-connected graph such that the degree-sum of every three mutually remote connected subgraphs of order $\lambda \geq 2$ is at least $\nu - 3\lambda + 5$. Then G is D_λ-cyclic.*

By Theorem 4, a 2-connected graph G has a D_λ-cycle ($\lambda \geq 2$) if $\delta_\lambda \geq \frac{1}{3}(\nu - 3\lambda + 5)$. Under the assumption that $G \notin \mathcal{K}_\lambda$ the existence of a D_λ-cycle can be proved if the weaker inequality $\delta_\lambda \geq \frac{1}{3}(\nu - 3\lambda + 3)$ is satisfied (the proof is a slight extension of the proof of Theorem 4; instead of inequality (1) one demonstrates the inequality $\nu \geq (m+1)t + m + 2$, where $m = d(H_0)$, using the fact that deletion of the m vertices of C close to H_0 does not create a graph with more than m components of order at least t). Thus, in particular, every 2-connected graph G satisfying $G \notin \mathcal{K}_2$ and $\delta_2 \geq \frac{1}{3}\nu - 1$ is D_2-cyclic, providing an extension of the following consequence of a result of Bigalke and Jung [1, Satz 1]: a graph G with $G \notin \mathcal{K}_1$ and $\delta \geq \frac{1}{3}\nu$ has a D_2-cycle. The latter result, in turn, easily implies the following, due to Nash-Williams [5, Lemma 4]: if G is a 2-connected graph and $\delta \geq \max(\alpha, \frac{1}{3}(\nu + 2))$, then G is hamiltonian.

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