On the Consistent Formulation and Approximation of EHL Theory

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In this paper a quasi-variational formulation for the creeping flow of a lubricant between deformable surfaces is derived on the basis of a principle of virtual power. This physical principle states that the state of the lubricated system is such that any change of the deformation or of the flow increases the total power. The principle correctly produces the governing differential equations and boundary conditions. Performing the usual approximations that are based on the scaling of the problem directly into the power functional (and not into the differential expressions), an approximate variational formulation is obtained that contains only quantities for and at the deformed surface. In deriving the differential equations and boundary conditions from this approximate functional it is noted that the usual formulation is obtained provided the sliding speed of the surfaces vanishes at the cavitation boundary. When restricted to hydrodynamic lubrication, a true variational principle results; the functional correctly produces the Reynolds equation, but differs from the functional that is commonly derived solely from this equation. The quasi-variational formulation that is derived for EHL gives an alternative way to formulate finite element approximations. Furthermore it specifies the relevance of the contribution of centrifugal forces on the deformation.

1 Introduction

The formulation of the lubricated line contact between a deformable, rotating cylinder and a rigid surface forms the topic of this paper. The ingredients of an approximate differential formulation of this classical elastohydrodynamic lubrication (abbreviated EHL) problem are more than a century old (Reynolds 1886 and Herz 1881). Here an alternative formulation is presented that is based on a principle of virtual power: the state of the lubricated system is such that any change of the deformation or of the flow increases to total power (cf. also Oden and Reddy [1]). It is shown that this principle correctly produces the set of differential equations and boundary conditions.

One reason to choose the variational description of the EHL problem is the metaphysical desire to formulate laws of nature by optimization principles. A much more important and practical motive is that approximations can be better performed in the variational formulation. The reason for this is that when differential expressions are approximated the variational nature may be destroyed, while equations derived from an approximation of the variational principle have the presence of a variational structure in common with the original equations. This holds true for approximations that simplify the governing equations as well as for the approximations that describe a numerical scheme for solving the problem. This paper will not deal with the numerical exploitation of the principle. See [2]

where an efficient numerical method is derived from the variational formulation.

The rest of this paper is devoted to the formulation and approximation of the governing power functional, and to the derivation of the differential equations from the functionals. For that purpose, the deformation of the solid and the flow of the lubricant are described first. The fundamental concepts of deformation, velocity and stress are illuminated in Section 2. The equilibrium state is governed by a set of partial differential equations supplemented with boundary conditions. These equations are derived under the assumptions that the displacement and velocity gradients are so small that the constitutive relations can be linearized.

Following the description of the basic problem, the formulation based on the principle of virtual power is presented in Section 3. The separate variational formulations for the deformation of the solid and for the creeping flow of the lubricant are derived first. These separate variational structures have inspired many investigators. Recently, such quasi-variational formulations have been used as the basis of finite element calculations (see e.g., Wu [3], Rohde and McAllister [4], or Oh and Huebner [5]). Strozzi [6] has formulated the EHL problem in terms of an extended variational principle. Moreover, a formulation in terms of a nonlinear complementarity problem is given by Kostreva [7]. The principle of virtual power as formulated here leads to a unified quasi-variational formulation. As remarked above, this formulation is by its nature better suited than the differential formulation to perform further approximations in a consistent way. Therefore it is used in Section 4, as the starting point for further approximations.

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A side-step into hydrodynamic lubrication (HL) shows that the usual approximation that is based on the scaling of the problem, leads to a true variational formulation for the Reynolds equation. The variational approach of HL theory has proved to be useful in several aspects: to derive existence results (cf. Bayada and Chambat [8]), for numerical purposes (cf. Reddi [9]), and for the optimal construction of bearings (cf. McAllister and Rohde [10]). In the references quoted above the functional that is used in the variational formulation is derived by simply requiring that its Euler-Lagrange equation is the Reynolds equation (see e.g. Capriz and Cimatti [11]). Compared to this functional, the power functional that will be derived here contains an additional term. The presence of this term will imply that that approximation is only consistent if the sliding speed of the surfaces is almost zero, a requirement that is often not clearly stated as a necessary condition for the validity of the approximations. The additional term will probably also affect the optimal shape of bearings.

Returning to the EHL problem, the extremizing displacement and velocity are approximated and then substituted into the principle. The resulting quasi-variational formulation depends on the pressure and on the deformed surface only. As can be expected, it is shown that the deformation caused by the centrifugal force can not be neglected if the position of the cylinder is fixed.

2 Basic Formulation

Consider the lubricated line contact between a deformable, rotating cylinder (rotating with angular velocity ω) and a rigid surface (see Fig. 1). The cylinder is so long that it is in a state of plane strain, i.e., the deformation is independent of the axial coordinate and has no components in the axial direction. Denoting the Lagrangian coordinate pair by $\xi = (\xi, \eta)$, the Eulerian coordinates will be denoted by $x = (x, y)$; the reference configuration of the (cross-section of the) cylinder is given by $\Omega_s = \{\xi \in \mathbb{R}^2 \mid -r < \xi < r \text{ and } -k_0(\xi) < \eta < k_0(\xi) = \sqrt{r^2 - \xi^2}\}$. The deformed cylinder is completely separated from the rigid surface $\Gamma = \{x \in \mathbb{R}^2 \mid y = y_0(x)\}$ by a thin film lubricant. The lubricant is assumed to be an incompressible, Newtonian fluid in a state of plane strain.

Basically, the deformation of an arbitrary body Ω_0 , $\xi \rightarrow x = \phi(\xi, t)$, is governed by three balance laws. Expressed in terms of the Cauchy stress T , the velocity $\underline{v}$ (defined as $\underline{v}(x = \phi(\xi, t)) = \partial \phi / \partial t(\xi, t)$) and the density ρ , the stationary equilibrium equations are (cf. Marsden and Hughes [12])

Fig. 1 The geometry

$$\text{div}(\rho \underline{v}) = 0, \quad \text{div} T = \rho (\underline{v} \cdot \nabla) \underline{v}, \quad T = T^T, \quad \text{in } \Omega = \phi(\Omega_0). \quad (1)$$

By specifying the Cauchy stress tensor, these three equations can describe (in the Eulerian setting) both the deformation of the cylinder and the flow of the lubricant.

The cylinder is assumed to be made of a homogeneous, elastic material. Hence, the constitutive equation gives the Cauchy stress T_s at each place $x = \phi(\xi, t)$ as a function of the deformation gradient $\nabla \phi$ only: $T_s = T(\nabla \phi)$. Since the motion of a particle is a result of both the elastic deformation and the rigid rotation of the cylinder, the map ϕ is taken as the composition of a rigid rotation $\xi \rightarrow X = R(t)\xi$ and an elastic deformation, which can be written in an Eulerian way like $X \rightarrow \underline{\psi}(X) = X + \underline{u}(X)$. Hence

$$\phi(\xi, t) = R\xi + \underline{u}(R\xi), \quad \text{with } R(t) = \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix}. \quad (2)$$

The velocity $\underline{v}_s$ of a particle in the cylinder is then easily seen to be:

$$\underline{v}_s(x = \underline{\psi}(X)) = \underline{v}_0(X) + (\underline{v}_0 \cdot \nabla) \underline{u}(X), \quad (3)$$

where $\underline{v}_0(X) = dR/dt \cdot R^T X = \omega(Y, -X)^T$ is the velocity of the undeformed, rotating cylinder, and the derivative operator $(\underline{v}_0 \cdot \nabla)$ can be interpreted as the material time-derivative in

Nomenclature

t = time coordinate, s
 ω = angular velocity of the cylinder, s^{-1}
 R = rotation matrix
 ξ = (ξ, η) = Lagrangian position of a particle, m
 $\underline{X}$ = (X, Y) = rotated Lagrangian position, m
 $\underline{x}$ = (x, y) = Eulerian position, m
 Ω = domain
 $\underline{n}$ = outward normal of Ω
 $\bar{\Gamma}$ = $\{\underline{x} \in \mathbb{R}^2 \mid y = y_0(x)\}$ = rigid surface
 k = shape of the cylinder, m
 r = radius of the cylinder, m
 ϕ = map of the deformation
 $\underline{\psi}$ = idem, in rotated frame
 T = Cauchy stress tensor, N/m^2
 S = Lagrangian stress tensor, N/m^2

P = idem, in rotated frame, N/m^2
 ρ = density, kg/m^3
 $\underline{v}$ = velocity, m/s
 V_k = x-component of the velocity at $y = k$, m/s
 V_0 = idem, at $y = y_0$
 $\underline{u}$ = displacement, m
 $\underline{u}_p$ = displacement caused by the centrifugal force, m
 $\underline{u}_h$ = displacement caused by the stress, m
 c = elasticity tensor, N/m^2
 λ_s = Lamé constant, N/m^2
 μ_s = idem
 μ = viscosity, Ns/m^2
 p = pressure, N/m^2 (cavitation pressure, $p_{cav} = 0$)
 q = reduced pressure = $\int \mu^{-1}(p) dp$, s^{-1}
 $\mathcal{P}$ = power, Nm/s

$\mathcal{E}$ = energy, Nm
 a = inlet boundary, m
 b = outlet boundary, m
 h = film thickness, m
 β = constant, $\rho_0 \omega^2 r^2 / (8(\lambda_s + \mu_s))$, $kgm/(Ns^2)$
 γ = constant, $(\lambda_s + 2\mu_s) / (2\pi\mu_s(3\lambda_s + 2\mu_s))$, m^2/N

Subscripts

0 = undeformed
 s = solid
 l = lubricant
 1 = x-direction
 2 = y-direction
 x = differentiation with respect to x
 y = idem, with respect to y

the rotating X -frame. Since the deformed configuration is not known in advance, it is not convenient to use the Eulerian description (1) for the deformation of the cylinder. For this reason the stress tensors P and S are introduced. The tensor P measures the force per unit area in the rotating, but elastically undeformed, configuration, and the Lagrangian stress S measures the force per unit area in the reference configuration. That is

$$S n_0 ds_0(\xi) = P n_0 ds_0(X = R\xi) = T_s n ds(X = \phi(\xi, t)), \quad (4)$$

where ds_0 is an undeformed surface element with unit normal n_0 . It is to be noticed that the stress P is related to the (Lagrangian) Piola Kirchhoff stress S like $P(R\xi) = S(\xi)R^T$. Furthermore, an argument similar to the one used to establish the Lagrangian equivalent of (1) (cf., Gurtin [13]), shows that the equilibrium expressed in the X -frame reads

$$\text{div} P = \rho_0(-\omega^2 X + (v_0 \cdot \nabla)^2 u) \quad \text{and} \quad (P \nabla \psi)^T = P \nabla \psi \quad \text{for} \quad X \in \Omega_s. \quad (5)$$

Here ρ_0 denotes the density in the undeformed, rotating configuration. The term $\omega^2 \rho_0 X$ represents the centrifugal force caused by the rotation R and $\rho_0(v_0 \cdot \nabla)^2 u$ is the acceleration force caused by the displacement u . Moreover, for an isotropic elastic material the constitutive equation in Eulerian terms can be transformed into a constitutive relation in terms of S , and thus also in terms of P : $P = P(\nabla \psi)$. The linearization of this relation (i.e., neglecting terms of the order ∇u and higher) yields (cf. Marsden and Hughes [12])

$$P = c \nabla u = \lambda_s \text{div} u I + \mu_s (\nabla u^T + \nabla u), \quad (6)$$

where λ_s and μ_s are the two Lamé constants. Concluding: the deformation of the cylinder is governed by the equations (5) and (6) for small ∇u .

Concerning the lubricant, in that part, denoted by Ω_1 , of the gap between the deformed cylinder and the rigid surface Γ that is completely filled with lubricant the Cauchy stress T_1 is assumed to be a linear function of the velocity gradient (cf. Malvern [14]):

$$T_1 = -pI + \mu(\nabla v + \nabla v^T), \quad (7)$$

where the constant μ denotes the viscosity of the lubricant. In the Eulerian picture, the flow of the lubricant is governed by the equations (1) and (7) with $\Omega = \Omega_1$ and ρ constant.

The unknown domain Ω_1 is described by the condition that the pressure p exceeds the cavitation pressure p_{cav} (which may be normalized by setting $p_{\text{cav}} = 0$). Thus $\Omega_1 = \{x \in \mathbb{R}^2 \mid k(x) < y < y_0(x) \text{ and } p > p_{\text{cav}}\}$. The function k , which describes the unknown shape of the lower half of the deformed cylinder, can be expressed in terms of the undeformed shape k_0 and the displacement on the boundary $y = k_0(x)$. A first order Taylor expansion, neglecting terms of the order $|\nabla u|^2$, leads to $k(x) = k_0(x) + u(x, k_0) \cdot (-k_0)_x, 1)^T$.

Summarizing so far, the equations (5) and (6) that govern the deformation of the cylinder should be solved together with the equations (1) and (7) that describe the flow of the lubricant in the domain Ω_1 . To complete the formulation, boundary conditions have still to be specified. On the rigid surface Γ the no-slip condition implies that the velocity is prescribed. On the upper boundary $Y = -k_0(X)$ the elastic deformation will be neglected. On the deformed boundary $y = k(x)$ two conditions are required (two since the boundary itself is unknown in advance). The first condition follows from the physical requirement that the forces on this boundary are in equilibrium. Hence on that part of the boundary $y = k(x)$ where $p > p_{\text{cav}}$ it must hold that $T_1 n ds = T_s n ds$, and on the remainder $T_s n ds = 0$. Since the deformation of the cylinder is described in terms of the stress tensor P , using equation (4) and the linearization (6) this boundary condition becomes

$$c \nabla u n_0(X, k_0(X))$$

$$= \begin{cases} (-pI + \mu(\nabla u + \nabla u^T)) n(x, k(x)) & \text{if } p(x, k) > p_{\text{cav}} \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

Here $x = X + u_1(X, k_0(X))$, n_0 is the unit outward normal of Ω_s , and n is the unit outward normal of the deformed cylinder.

Secondly, the no-slip condition is imposed on that part of $y = k(x)$ where $p > p_{\text{cav}}$. This means that an elastic particle and a fluid particle that are in contact move with the same velocity:

$$v(x = \psi(X)) = v_0(X) + (v_0 \cdot \nabla) u(X). \quad (9)$$

(A similar expression can be found in the thesis of Kalker [15] p. 66.) Since the cavitation boundary is also treated as a free boundary, an additional condition has to be introduced. Following Coyne and Elrod [16], we impose that the cavitation boundary is free of forces: $T_1 n = 0$. In this way a complete description of the EHL problem is obtained.

3 Quasi-Variational Formulation

In this section the basic variational principles for the two constituents (the cylinder and the lubricant) are combined into one variational formulation for the complete EHL problem. The flow of the lubricant can be described by the principle of virtual power. Arbitrary variations δv and δp , with $\delta v = 0$ on $\partial\Omega_1$ in the expression

$$G_1(v, p; \delta v, \delta p) = \iint_{\Omega_1} (\delta v \cdot \rho_1 (v \cdot \nabla) v + T_1 : \nabla \delta v - \delta p \text{div} v) d\Omega_1 = 0 \quad (10)$$

provide the correct equation (1). It is to be noticed that a variation δp is introduced, which takes account for the constraint $\text{div} v = 0$, and moreover that the boundary conditions are, at this point, not included in (10). As it stands, equation (10) is not of the form of the vanishing of the first variation of a certain functional, due to the presence of the inertia term. However, in the usual restriction to creeping flow (see e.g., Schlichting [17]), i.e., assuming that the reduced Reynolds number is very small compared to unity, the inertia term is omitted. This leads to the power functional

$$\mathcal{P}_1(p, v) = \iint_{\Omega_1} \left(-p \text{div} v + \frac{\mu}{2} \nabla v : (\nabla v + \nabla v^T) \right) d\Omega_1. \quad (11)$$

$\mathcal{P}_1$ is the power of the flow, and the pair (p, v) is an approximate solution if and only if (p, v) is a critical points of $\mathcal{P}_1$. The deformation of the cylinder can be derived from

$$G_s(u; \delta u) = \iint_{\Omega_s} \left(\delta u \cdot \rho_0 [(v_0 \cdot \nabla)^2 u - \omega^2 X] + P : \nabla \delta u \right) d\Omega_s = 0 \quad (12)$$

for all variations δu subject to $\delta u = 0$ on $\partial\Omega_s$. And G_s is the first variation of the functional

$$\mathcal{E}_s(u) = \frac{1}{2} \iint_{\Omega_s} \left(\nabla u : c \nabla u - 2\rho_0 \omega^2 X \cdot u + \rho_0 (v_0 \cdot \nabla) u \cdot (v_0 \cdot \nabla) u \right) d\Omega_s, \quad (13)$$

which can be interpreted as the potential elastic energy. Equation (12) expresses the fact that u is a solution if and only if the energy corresponding to the state u is minimal.

Due to the different physical interpretations, the functionals $\mathcal{E}_s$ and $\mathcal{P}_1$ can not simply be added to obtain a unified func-

tional for the EHL problem. However, the virtual elastic energy G_s can be transformed in a virtual elastic power. Indeed, the velocity in Ω_s is defined by $G_s(\underline{u}; \delta \underline{v}(x) = (\underline{v}_0 \cdot \nabla) \delta \underline{u}(X))$. Note that the variation of equation (3) is restricted, since variations of the argument x of $\underline{v}$ with respect to the displacement are not considered. Now, on the basis of the principle of virtual power, the quasi-variational structures of both the flow and the deformation can be combined to a unified quasi-variational formulation:

$$G(\underline{u}, \underline{v}, p; \delta \underline{u}, \delta \underline{v}, \delta p) = G_1(\underline{v}, p; \delta \underline{v}, \delta p) + G_s(\underline{u}; (\underline{v}_0 \cdot \nabla) \delta \underline{u}) = 0 \quad (14)$$

for all variations $\delta \underline{u}$, $\delta \underline{v}$, and δp satisfying

$$\delta \underline{v}(x, k) = (\underline{v}_0 \cdot \nabla) \delta \underline{u}(X, k_0) \quad \text{if } p(x, k) > p_{cav} \quad (15)$$

$$\delta \underline{v} = 0 \text{ on } y = y_0(x) \text{ and } \delta \underline{u} = 0 \text{ on } Y = -k_0(X).$$

It should be noted that the no-slip condition (9) restricts the space of admissible variations, while the force equilibrium condition (8) is built into the weak form (14).

In accordance with the physical interpretations of (10) and (12), the principle (14)–(15) states that the displacement $\underline{u}$ in Ω_s and the velocity $\underline{v}$ in Ω_1 , related by the no-slip condition (9), and the pressure p , are such that any changes $\underline{u} + \delta \underline{u}$, $\underline{v} + \delta \underline{v}$ and $p + \delta p$ subject to (15), lead to an increase of power. In addition, it should be emphasized that, due to the fact that the time-derivative and the δ operators do not commute, equation (14) is not of the form of the vanishing of the first variation of the total power. This point is discussed in detail in a forthcoming paper, where a further explanation is presented that is based on a time-discretized variational formulation. Up till now, it is assumed that the angular velocity ω is given. However, if the cylinder is not externally forced to rotate, the angular velocity ω has to be determined. This motivates that variations with respect to ω should be considered. Arbitrary variations with respect to ω in (11), subject to the no-slip condition (9), lead approximately to $\int_a^b T_1 \underline{n}(x, k) \cdot (k_0, -x)^T dx = 0$. This relation expresses that the total moment performed by the lubricant on the cylinder is zero. Note that the contribution of variations of the elastic power with respect to ω is of second order in terms of $\nabla \underline{u}$, and thus can be neglected.

4 Approximate Surface Formulation

The formulation in the foregoing section makes the variational structure visible that lies beyond the differential formulation of Section 2. To derive any (further) approximation in a consistent way it is required to retain this special structure and therefore to perform such an approximation in the variational formulation.

The further approximation is based on the usual assumption that the scaling of the problem is typically such that $\epsilon \ll 1$, where ϵ is defined as the quotient of the characteristic lengths of Ω_1 in the y - and in the x -direction. This will enable us to formulate the quasi-variational principle (14)–(15) exclusively in terms of the displacement, velocity and pressure at the surface $y = k(x)$. For convenience, the variational surface formulation of hydrodynamic lubrication theory is derived first.

4.1 Hydrodynamic Lubrication. In hydrodynamic lubrication (HL) theory the deformation of the cylinder is prescribed and fixed. So, one can focus on the flow of the lubricant between the two surfaces $y = k(x) = k_0(x)$ and $y = y_0(x)$. The velocity on these two surfaces is prescribed by the no-slip condition, i.e., the horizontal (= x -direction) components are given by $V_k = \omega k_0$ and V_0 , respectively. In the previous section the principle of virtual power (neglecting the inertia term) leads to the functional $\mathcal{P}_1$ (11). A further approx-

imation, based on the assumption that $\epsilon \ll 1$ and the introduction of the film thickness, $h(x) = y_0(x) - k(x)$, leads to

$$\mathcal{P}_1(p) = -\frac{1}{2} \int_a^b \left(\frac{h^3}{12\mu} p_x^2 - (V_k + V_0) h p_x - \mu \frac{(V_k - V_0)^2}{h} \right) dx. \quad (16)$$

This functional will be derived in several steps below, but the basic idea is the following. Starting from the variational principle for a creeping flow, equation (11), the extremizing velocity can be calculated, symbolically $\underline{v} = \Delta^{-1}(\nabla p/\mu)$, and substituted in $\mathcal{P}_1$. Finally, this functional of p is approximated up to first order in ϵ , by approximating Δ^{-1} . This idea will now be made explicit in four steps.

Step I. By introducing scaled, dimensionless variables, it can be shown that the functional (11) approximates in lowest order to

$$\mathcal{P}_1(p, \underline{v}) = \iint_{\Omega_1} \left(\frac{\mu}{2} \left(\frac{\partial v_1}{\partial y} \right)^2 - p \operatorname{div} \underline{v} \right) d\Omega_1, \quad (17)$$

where terms of order ϵ are omitted.

Step II. Taking variations of (17) with respect to v_2 it follows that $p_y = 0$ in Ω_1 . Hence isobars are straight lines, and in particular, the part of the boundary of Ω_1 at which $p = 0$ is straight. Consequently, Ω_1 is given by $\{(x, y) \in \mathbb{R}^2 \mid k(x) < y < y_0(x) \text{ and } a < x < b\}$ for some, yet unknown, x -bounds a and b . It may be emphasized that $p_y = 0$ is not a basic assumption, but appears as a consequence of the assumption $\epsilon \ll 1$.

Step III. Variations of (17) with respect to v_1 lead to $p_x = \mu(v_1)_{yy}$. The solution of this equation, supplemented by the boundary conditions $v_1(x, k) = V_k(x)$ and $v_1(x, y_0) = V_0(x)$, is given by

$$v_1(x) = \frac{1}{2\mu} (y - y_0)(y - k) p_x + \frac{(y - k)}{h} V_0 + \frac{(y_0 - y)}{h} V_k. \quad (18)$$

Step IV. Exploiting the results so far, a part of the integration in (17) can be performed explicitly and (16) results in a straightforward way.

Having obtained the approximate functional $\mathcal{P}_1(p)$, equation (16), let us now concisely investigate the consequences of the corresponding variational principle. The Euler-Lagrange equation resulting from arbitrary variations of $\mathcal{P}_1$ with respect to p (subject to $p(a) = p(b) = 0$ and $p(x) > 0$ for $a < x < b$) is

$$\frac{1}{12\mu} (h^3 p_x)_x = \frac{1}{2} ((V_k + V_0) h)_x. \quad (19)$$

This equation governs the conservation of mass in the required approximation, and is nothing but the well-known Reynolds equation. In this respect, it is to be noticed that $\mathcal{P}_1$ differs from the functional that is commonly obtained if the only objective is to produce the Reynolds equation as the Euler-Lagrange equation of the functional (see e.g. Capriz and Cimatti [11]). Compared to (16) such a functional lacks the term depending on the velocity difference $V_k - V_0$. Since (16) is derived in a consistent way from a basic physical principle, it can be expected that this term should be incorporated.

Furthermore, taking $V_k + V_0 > 0$, and assuming (as usual) that the inlet boundary $x = a$ is fixed, the outlet boundary $x = b$ can be treated as a free boundary. Then arbitrary variations of (17) with respect to b give $\int_k^{y_0} (-p \operatorname{div} \underline{v} + \mu((v_1)_y)^2/2) dy = 0$ at $x = b$. This states that, in the approximation under consideration, the tangential stress at $x = b$ must vanish. Such a condition has been proposed before by Coyne and Elrod [15], and can physically be understood as the assumption that the vapor or gas in the cavity ($x > b$) has negligible viscosity. Variations with respect to b in (16) (note $p(b) = 0$!), yield $(p_x)^2 + 12(\mu(V_k - V_0)/h^2)^2 = 0$ at $x = b$. This relation is only solvable if both surfaces move approximately with the same velocity, and leads to the Swift-Stieber cavitation condition $p_x = 0$ at $x = b$.

It is to be noticed that the difference $V_k - V_0$, measuring the sliding of the surfaces, has to be zero at $x=b$ to insure that the approximations under consideration are consistent. Indeed, the viscous force in the lubricant is approximated by $\mu(v_1)_y$. Together with the approximation that the cavitation boundary is both straight and free of forces it follows that

$$\int_k^{y_0} \mu(v_1)_y dy = \mu(V_0 - V_k) = 0 \quad \text{at } x=b \quad (20)$$

This condition has not been stated explicitly before. The violation of this condition is equivalent to the introduction of some unknown force acting on the cavitation boundary.

In most applications the viscosity is pressure dependent. In that case the functional (16) needs to be modified in order to produce the Reynolds equation (19). This is achieved by considering $q = \int \mu^{-1} dp$, instead of p , as the basic variable in the variational principle

$$\tilde{\mathcal{P}}_1(q) = -\frac{1}{2} \int_a^b \left(\frac{h^3}{12} (q_x)^2 - (V_k + V_0) h q_x - \frac{(V_k - V_0)^2}{h} \right) dx. \quad (21)$$

Rewriting $\tilde{\mathcal{P}}_1$ in terms of p shows that the integrand of (21) contains an additional multiplicative factor μ^{-1} compared to (16). Therefore the functional $\tilde{\mathcal{P}}_1$ has a physical meaning different from the original functional $\mathcal{P}_1$. A dimension analysis shows that $\tilde{\mathcal{P}}_1$ can be interpreted as the work per unit mass performed by the lubricated system. Hence (21) leads to a principle of virtual work:

4.2 Elastohydrodynamic Lubrication. Here, the principle of virtual power (14)–(15) will be rewritten in terms of quantities on the surface $y=k(x)$ solely. This will be achieved by substituting an approximate, extremizing displacement into (14), and by simplifying the flow field as in the previous section. Hence, the quasi-variational formulation (14) of the elastohydrodynamic lubrication problem can be approximated by

$$\delta_{p,u} \mathcal{P}_1(p, u) + G_s(u; (v_0 \cdot \nabla) \delta u) = 0, \quad (22)$$

where $\mathcal{P}_1$ is given by equation (17). An analysis similar to the one used to omit the inertia force in equation (10) shows that if one additionally assumes that $\rho_0 \omega^2 r^2 \ll \mu_s$, the term $\rho_0 (v_0 \cdot \nabla)^2 u$ can be neglected in (12). Next, the displacement is split into a particular and a homogeneous part: $u = u_h + u_p$, with $\text{div } c \nabla u_p = -\rho_0 \omega^2 X$ in Ω_s and $c \nabla u_p n_0 = 0$ on $\partial\Omega_s$. The solution of this problem which satisfies $u_p(0) = 0$, is

$$u_p(X) = \frac{\rho_0 \omega^2}{8(\lambda_s + 2\mu_s)} \left(\frac{3\mu_s + 2\lambda_s}{\lambda_s + \mu_s} r^2 - (X^2 + Y^2) \right) X \quad (23)$$

Substituting this particular solution into equation (22) gives

$$\iint_{\Omega_1} \delta \left(-p \text{div } v + \frac{\mu}{2} \left(\frac{\partial v_1}{\partial y} \right)^2 \right) dy dx + \iint_{\Omega_s} c \nabla u_h : \nabla \delta (v_0 \cdot \nabla) u_h dY dX = 0. \quad (24)$$

Here the space of admissible variations is still restricted by the equations (15). Since the stress tensor P is linearized, or in other words, since it is assumed that $|\nabla u| \ll 1$, the first component of the no-slip condition (9) may also be written as $v_1(x, k) = \omega k_0(x) = V_k$, which implies that $\delta v_1(x, k) = 0$. Now, the equations resulting from the variations δu_h and δv_2 in (24) are

$$\text{div } c \nabla u_h = 0 \quad \text{in } \Omega_s \quad (25)$$

with $u_h = 0$ on $Y = -k_0(X)$ and on $y = h_0(X)$

$$(c \nabla u_h n_0)_1 = 0 \quad \text{and} \quad (c \nabla u_h n_0)_2 = \begin{cases} (pn)_2 & \text{if } p > p_{\text{cav}} \\ 0 & \text{otherwise} \end{cases}$$

and $p_y = 0$ in Ω_1 . Denote the solution of problem (25) by $u_h(p)$. Substituting this solution into equation (24), and repeating the analysis of Section 4.1 leads to the following quasi-variational surface formulation

$$\int_a^b \left(\frac{h^3(p)}{6\mu} p_x \delta p_x - (V_k + V_0) h(p) \delta p_x \right) dx = 0 \quad (26)$$

with $h(p) = y_0(x) - k(x) = y_0(x) - k_0(x) - (u_h(p) + u_p)_x(x, k_0) \cdot (-k_0)_x, 1)^T$. Usually, the boundary value problem (25) is not solved exactly, and the solution is replaced by a half-plane approximation (Love [18]):

$$u_h(p) \cdot (-k_0)_x, 1)^T \approx -\gamma \int_a^b p(s) \log |x-s| ds =: -\gamma L(p) \quad (27)$$

where the constant γ is given by $\gamma = (\lambda_s + 2\mu_s) / (2\pi\mu_s(3\lambda_s + 2\mu_s))$, and where the linear integral operator L is introduced. Using the equations (23) and (27), it can be shown that the film thickness $h(p)$ is approximately given by

$$h(p) = y_0(x) - k_0(x) - \frac{\beta r^2}{k_0(x)} + \gamma L(p) \quad (28)$$

with $\beta = \rho_0 \omega^2 r^2 / (8\lambda_s + \mu_s)$.

From equation (25) it follows that the x -component of the global force acting on the cylinder is approximately zero, while the y -component is given by $w \approx \int_a^b p dx$. Up till now, it is assumed that the position of the cylinder is fixed, i.e., that y_0 is given in advance. The contribution of the displacement caused by the centrifugal force to the film thickness can be neglected if $\beta r \ll y_0 - r + \gamma L(p)$. This condition is not always satisfied in practical situations (For instance, in Wu [3] u_p is of the same order as the minimal film thickness when $\rho_0 = 8000 \text{ kg/m}^3$). By contrast, if the position of the cylinder is not fixed in advance, but if, instead, the load w is given, the level y_0 has to be determined. Then the effect of the centrifugal force can be included in this unknown level because the displacement on the boundary $y = k_0(x)$ caused by the centrifugal force is approximately constant. Indeed

$$h(x) \approx y_0(x) - k_0(x) - \beta r + \beta \frac{x^2}{2r} + \gamma L(p), \quad (29)$$

and the term $\beta x^2/2r$ is in practice usually much smaller than $h(x)$. The weak form (26) has been used before as the starting point of finite element calculations (see e.g., Wu [3]). As noted above, the particular solution u_p was missing there.

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References

- Oden, J. T., and Reddy, J. N., *Variational Methods in Theoretical Mechanics*, Springer-Verlag, Berlin, 1976.
- Verstappen, R., and Van Groesen, E., "The Variational Structure and Numerical Solution of an Elastohydrodynamic Lubrication Problem," to be submitted to *Journal of Computational Physics*.
- Wu, S. R., "A Penalty Formulation and Numerical Approximation of the Reynolds-Hertz Problem of Elastohydrodynamic Lubrication," *International Journal Engineering Science*, Vol. 24, 1986, pp. 1001–1013.
- Rohde, S. M., and McAllister, G. T., "A Variational Formulation for a Class of Free Boundary Problems Arising in Hydrodynamic Lubrication," *International Journal Engineering Science*, Vol. 13, 1975, pp. 841–850.
- Oh, K. P., and Huebner, K. H., "Solution of the Elastohydrodynamic Finite Journal Bearing Problem," *ASME JOURNAL OF LUBRICATION TECHNOLOGY*, Vol. 95, 1973, pp. 342–352.
- Strozzi, A., "The Elastohydrodynamic Problem Expressed in Terms of Extended Variational Formulation," *ASME JOURNAL OF TRIBOLOGY*, Vol. 108, 1986, pp. 557–564.

7 Kostreva, M. M., "Elasto-hydrodynamic Lubrication: A Non-Linear Complementarity Problem," *Int. J. for Numerical Methods in Fluids*, Vol. 4, 1984, pp. 377-397.

8 Bayada, G., and Chambat, M., "Existence and Uniqueness for a Lubrication Problem with Nonregular Conditions on the Free Boundary," *Unione Matematica Italiana*, Bollettino B. Series 6, Vol. 3B, 1984, pp. 543-557.

9 Reddi, M. M., "Finite-Element Solution of the Incompressible Lubrication Problem," *ASME JOURNAL OF LUBRICATION TECHNOLOGY*, Vol. 91, 1969, pp. 524-533.

10 McAllister, G. T., and Rohde, S. M., "Optimum Design of One-Dimensional Journal Bearings," *Journal of Optimization Theory and Applications*, Vol. 41, 1983, pp. 599-617.

11 Capriz, G., and Cimatti, G., "Free Boundary Problems in the Theory of Hydrodynamic Lubrication," in *Free Boundary Problems: Theory and Applications*, Vol. 2, Eds. A. Fasano and M. Primicerio, Research Notes in Mathematics 79, Pitman, Boston, 1983, pp. 613-635.

12 Marsden, J. E., and Hughes, J. R., *Mathematical Foundations of Elasticity*, Prentice-Hall Inc., Englewood Cliffs, 1983.

13 Gurtin, M. E., *Topics in Finite Elasticity*, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 1981.

14 Malvern, L., *Introduction to the Mechanics of a Continuous Medium*, Prentice-Hall Inc., New York, 1969.

15 Kalker, J. J., "On the Rolling Contact of Two Elastic Bodies in the Presence of Dry Friction," Thesis Delft University of Technology, Delft, The Netherlands, 1967.

16 Coyne, J. C., and Elrod, Jr., H. G., "Conditions for the Rupture of a Lubricating Film. Part I: Theoretical Model," *ASME JOURNAL OF LUBRICATION TECHNOLOGY*, Vol. 92, 1970, pp. 451-457.

17 Schlichting, H., *Boundary Layer Theory*, 6th Edn., McGraw-Hill, New York, 1968.

18 Love, A. E. H., *A Treatise on the Mathematical Theory of Elasticity*, 4th Edn., University Press, Cambridge, 1952.