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Optimal Linear Stochastic Control for Systems with Multiplicative Noise

ARUNABHA BAGCHI AND TOM SCHILPEROORT

Abstract—**The stochastic control problem of a linear dynamical system with multiplicative noise and with incomplete and inaccurate observation, has been studied for quadratic performance criterion. A suboptimal solution, which is the best linear control based on the available observations, has been worked out when the observations are given only at discrete-time points.**

I. Introduction

We obtain the best linear control law for a stochastic control problem with linear dynamics, but with multiplicative noise, based on observations at discrete-time points. We convert the problem into successive control problems where the control depends on the current observation only. Mclane [1] solved this type of problem using the matrix maximum principle of Athans [2]. This gives explicit control laws for our problem, which involves successive solutions of nonlinear boundary value problems.

II. Problem Formulation

We consider a dynamical system described by

$$dX_t = A(t)X_t\,dt - B(t)U_t\,dt + D(t,X_t)\,dW_{1t} + E(t)\,dW_{2t} \tag{2.1}$$

$$dY_t = C(t)X_t\,dt + G(t,X_t)\,dW_{3t} + F(t)\,dW_{4t}, \qquad 0<t<T. \tag{2.2}$$

X_0, Y_0 are independent random vectors, X_t is an n-dimensional state, U_t is a p-dimensional control, Y_t is an m-dimensional observation, and

$$D(t,X_t) = \sum_{i=1}^{n} D_i(t)X_{it},\; G(t,X_t) = \sum_{i=1}^{n} G_i(t)X_{it}$$

with X_{it} being the ith component of X_t and $D_i(t)$ and $G_i(t)$ are appropriate dimensional matrices. W_{it}, $i=1,2,3,4$, are independent Brownian motions of dimensions d_i, independent of X_0 and Y_0. The matrices $A(t)$, $B(t)$, $C(t)$, $E(t)$, $F(t)$ have appropriate dimensions.

Let $t_0=0$ and $t_k=kT/N$, $k=1,\cdots,N$, and at these time points we observe the process Y_t. We can generalize to the case of nonequidistant t_k's. For $t\in[t_{j-1}, t_j)$, $j=1,\cdots,N$, let $Y^{j-1}=\mathrm{col}\,(Y_{t_{j-1}},\cdots,Y_0)$. Let $\Phi^j_k=\{\phi_j\colon [t_{j-1},t_j)XR^{mj}\to R^P$ such that $\phi_j(t,y)=\Sigma_{i=0}^{j-1}k_i(t)Y_{t_i}\}$. We denote by $\mathcal{U}_{lj}$ the class of control U_{jt} where

$$U_{jt}=\phi_j(t,Y^{j-1}), \qquad t\in\left[t_{j-1},t_j\right),\ \phi_j\in\Phi^j_k. \tag{2.3}$$

A control U_t, $t\in[0,T]$, is now *admissible* if

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A Bagchi is with the Department of Applied Mathematics, Twente University of Technology, Enschede, The Netherlands.

T. Schilperoort is with the Delft Hydraulics Laboratory, Laboratory "de Voorst," Emmeloord, The Netherlands.

i) $U_t=U_{jt}, \quad t\in\left[t_{j-1},t_j\right),\ j=1,\cdots,N$

ii) $U_{jt}\in\mathcal{U}_{lj}, \quad j=1,\cdots,N.$

Denote this class by $\mathcal{U}_{la}$. We want to determine a control U_t in this class that minimizes

$$J(U)=E\left\{\int_0^T (X_t'Q(t)X_t + U_t'R(t)U_t)\,dt + X_T'Q_fX_T\right\}$$

with "prime" denoting transpose, where the matrices $Q(t)>0$, $Q_f>0$, and $R(t)>0$ a.e. t.

III. Determination of the Optimal Linear Control

Let

$$Z_t=\begin{pmatrix}X_t\\ Y_t\end{pmatrix}, \qquad t\in[0,T]$$

and $\bar{C}=(0_{mn}I_{mm})$, where 0_{mn} is the $m\times n$ zero matrix and I_{mm} is the $m\times m$ identity matrix, so that $Y_t=\bar{C}Z_t$. For $j=1,\cdots,N$, define stochastic processes

$$Z_t^j=\begin{pmatrix}Z_t\\ Z^{j-1}\end{pmatrix}, \quad \text{with } Z^{j-1}=\mathrm{col}(Z_{t_{j-1}},\cdots,Z_0),\ t\in\left[t_{j-1},t_j\right). \tag{3.1}$$

Define, for $j=1,\cdots,N$, matrices of dimensions $p\times mj$:

$$K^j(t)=\left(K^j_{j-1}(t),\cdots,K^j_0(t)\right). \tag{3.2}$$

Let $\bar{C}^j$ be a $j\times(j+1)$ block $m\times(n+m)$ matrices with $(\bar{C}^j)_{k,k+1}=\bar{C}$ and $(\bar{C}^j)_{kl}=0$ for $l\neq k+1$, $k=1,\cdots,j$; $l=1,\cdots,j+1$. A control $U_{jt}\in\mathcal{U}_{lj}$ can be expressed as $U_{jt}=K^j(t)\bar{C}^jZ^j_t$, so that, using this control, Z^j_t is the solution of

$$dZ^j_t=\left(A^j(t)-B^j(t)K^j(t)\bar{C}^j\right)Z^j_t\,dt+\sigma^j(t,Z^j_t)\,dW_t \tag{3.3}$$

where

$$W_t=\mathrm{col}(W_{1t},W_{2t},W_{3t},W_{4t}),$$

$$\sigma^j(t,Z^j_t)=\left(\sum_{i=1}^{n}D^j_i(t)(Z^j_t)_i\ E^j(t)\ \sum_{i=1}^{n}G^j_i(t)(Z^j_t)_i\ F^j(t)\right)$$

and the matrices $A^j(t)$, $B^j(t)$, $D^j_i(t)$, $E^j(t)$, $G^j_i(t)$, $F^j(t)$ have easily identifiable structure.

Define, for $t\in[t_{j-1},t_j)$, $j=1,\cdots,N$,

$$L^j_k(t,z^j)=(z^j)'\left[Q^j(t)+(\bar{C}^j)'K^j(t)'R(t)K^j(t)\bar{C}^j\right]z^j \tag{3.4}$$

where $Q^j(t)=$ block diag $(Q(t),0,0)$ so that we have

$$J=E\left\{\sum_{k=1}^{N}\int_{t_{k+1}}^{t_k}L^k_K(s,Z^k_s)\,ds+(Z^N_T)'Q^N_fZ^N_T\right\} \tag{3.5}$$

with $Q^j_f=$ block diag $(Q_f,0,0)$.

Let

$$J^j_K=E\left\{\sum_{k=j}^{N}\int_{t_{k-1}}^{t_k}L^k_K(s,Z^k_s)\,ds+(Z^N_T)'Q^N_fZ^N_T\right\} \tag{3.6}$$

and

$$W^k_K=\int_{t_{k-1}}^{t_k}L^k_K(s,Z^k_s)\,ds.$$

Bellman's principle of optimality yields the following procedure for determining the optimal control sequence $U^*_{1t},\cdots,U^*_{Nt}$.

1) Find $U_{Nt} \in \mathcal{U}_{1N}$ for which J_K^N is a minimum. Use J_K^{j*} to denote the minimum value of $J_K^j, j=1,\cdots,N$.

2) Find, successively, for $j=(N-1),\cdots,1$, $U_{jt} \in \mathcal{U}_{1j}$ for which

$$E\{W_K^j + J_K^{(j+1)*}\}$$

is a minimum.

To carry out these minimizations, consider functions $V_K^j(t,z^j)$, $j=1,\cdots,N$, which are solutions of the backward equations

$$\frac{\partial V_K^j(t,z^j)}{\partial t} + \mathcal{L}_K^j(t,z^j)V_K^j(t,z^j) + L_K^j(t,z^j) = 0$$

$$\cdot(t,z^j) \in [t_{j-1},t_j) \times R^{(n+m)(j+1)} \quad (3.7)$$

with the final conditions, for $j=N$,

$$V_K^N(T,z^N) = (z^N)' Q_f^N z^N \quad (3.8a)$$

and for $j=N-1,\cdots,1$,

$$V_K^j(t_j,z^j) = V_K^{j+1}(t_j,z^j,z_j) \quad (3.8b)$$

where the differential generator $\mathcal{L}_K^j(t,z^j)$, $z^j \in R^{(n+m)(j+1)}$ is defined by

$$\mathcal{L}_K^j(t,z^j) = (z^j)'\left[A^j(t) - B^j(t)K^j(t)\overline{C}^j\right]'\frac{\partial}{\partial z^j} + \frac{1}{2}\mathrm{tr}\left[\sigma^j(t,z^j)\sigma^j(t,z^j)'\frac{\partial^2}{\partial (z^j)^2}\right]. \quad (3.9)$$

The system of equations (3.7), (3.8a), (3.8b) has a unique solution which follows from standard existence results in partial differential equations with slight modification [3]. The Itô differentiation rule [4] gives

$$J_K^j = E\left\{V_K^j\left(t_{j-1}, Z_{t_{j-1}}^j\right)\right\}, \qquad j=1,\cdots,N. \quad (3.10)$$

We propose a solution of (3.7) in the form

$$V_K^j(t,z^j) = (z^j)'P_K^j(t)z^j + p_K^j(t) \quad (3.11)$$

with $P_K^j(t) \in H_{(n+m)(j+1)}$, $j=1,\cdots,N$, where H_n stands for the class of all $n \times n$ symmetric matrices.

Substituting (3.11) in (3.7), we get for $j=1,\cdots,N$,

$$\dot{P}_K^j + P_K^j(A^j - B^jK^j\overline{C}^j) + (A^j - B^jK^j\overline{C}^j)'P_K^j + Q^j + \Delta^j(t,P_K^j) + \Gamma^j(t,P_K^j) + (\overline{C}^j)'(K^j)'RK^j\overline{C}^j = 0 \quad (3.12)$$

with the maps Δ^j and Γ^j defined by

$$\Delta^j, \Gamma^j : [t_{j-1},t_j) \times H_{(n+m)(j+1)} \rightarrow H_{(n+m)(j+1)}$$

$$[\Delta^j(t,M)]_{ik} = \begin{cases} \mathrm{tr}[D_i^j(t)'MD_k^j(t)]; & i,k=1,\cdots,n \\ 0 & \text{otherwise} \end{cases} \quad (3.13)$$

$$[\Gamma^j(t,M)]_{ik} = \begin{cases} \mathrm{tr}[G_i^j(t)'MG_k^j(t)]; & i,k=1,\cdots,n \\ 0 & \text{otherwise.} \end{cases}$$

The final conditions are

$$P_K^N(t) = Q_f^N \quad (3.14)$$

and for $j=1,\cdots,N-1$,

$$(z_j,\cdots,z_0)P_K^j(t_j)\begin{bmatrix} z_j \\ \vdots \\ z_0 \end{bmatrix} = (z_j z_j \cdots z_0)P_K^{j+1}(t_j)\begin{bmatrix} z_j \\ z_j \\ \vdots \\ z_0 \end{bmatrix}. \quad (3.15a)$$

The last condition, after simple calculation, yields

$$\left[P_K^j(t_j)\right]_{11} = \sum_{k=1}^{2}\sum_{l=1}^{2}\left[P_K^{j+1}(t_j)\right]_{kl} \quad (3.15b)$$

$$\left[P_K^j(t_j)\right]_{l1} = \left[P_K^j(t_j)\right]_{1l} = \sum_{k=1}^{2}\left[P_K^{j+1}(t_j)\right]_{k,l+1}$$

$$l=2,\cdots,(n+m)(j+1) \quad (3.15c)$$

$$\left[P_K^j(t_j)\right]_{kl} = \left[P_K^j(t_j)\right]_{lk} = \left[P_K^{j+1}(t_j)\right]_{k+1,l+1}$$

$$k,l=2,\cdots,(n+m)(j+1). \quad (3.16)$$

For $p_K^j(t)$, we have

$$\dot{p}_K^j(t) = -\left[\mathrm{tr}\{E^j(E^j)'P_K^j(t)\} + \mathrm{tr}\{F^j(F^j)'P_K^j(t)\}\right].$$

$$p_K^j(t_j) = 0. \quad (3.17)$$

Finally, we have

$$E(W_K^j + J_K^{j+1}) = \mathrm{tr}\left[P_K^j(t_{j-1})E\left\{Z_{t_{j-1}}^j\left(Z_{t_{j-1}}^j\right)'\right]+p_K^j(t_{j-1}). \quad (3.18)$$

The determination of the optimal sequence of controls $U_{1t}^*,\cdots,U_{Nt}^*$ can be accomplished in two stages.

1) First, find $K^N(t)$ for which J_K^N given by (3.17) is a minimum, where $P_K^N(t)$ and $p_K^N(t)$ are solutions of (3.12), (3.14), and (3.16). Denote the optimal $P_K^N(t)$ and $p_K^N(t)$ by $P_K^{N*}(t)$ and $p_K^{N*}(t)$.

2) Suppose that optimal $K^{n*}(t)$, $n=j+1,\cdots,N$, has been determined. Determine $K^j(t)$ for which $E(W_K^j + J_K^{(j+1)*})$ given by (3.17) is a minimum, where $P_K^j(t)$ and $p_K^j(t)$ are solutions of (3.12), (3.15a)–(3.15c), and (3.16), where in the right-hand sides of (3.15a)–(3.15c) we use $P_K^{(j+1)*}(t_j)$, the optimal values of $P_K^{(j+1)}(t_j)$. This is possible because $E\{Z_{t_{j-1}}^j(Z_{t_{j-1}}^j)'\}$ is independent of the choice of $K^j(t)$ in $[t_{j-1},t_j)$.

The minimizations may be carried out successively. We briefly outline the minimization procedure for 1). The rest can be performed similarly. We may write

$$J_K^N = \mathrm{tr}\left[P_K^N(t_{j-1})E\left\{Z_{t_{N-1}}^N\left(Z_{t_{N-1}}^N\right)'\right\} + \int_T^{t_{N-1}}\dot{p}_K^N(s)\,ds.\right.$$

Take $P_K^N(t)$, $t \in [t_{N-1},T)$ as the "state matrix" and $K^N(t)$ as the "control matrix" and define the *Hamiltonian*

$$H(P_K^N(t), S_K^N(t), t, K^N(t)) \triangleq \dot{p}_K^N(t) + \mathrm{tr}\left(\dot{P}_K^N(t)S_K^N(t)\right)$$

with $S_K^N(t)$ denoting the "costate matrix." The matrix maximum principle gives the following result (see [2] for details).

Assuming that the matrix $\overline{C}^N S_K^N(\overline{C}^N)'$ is invertible for $t \in [t_{N-1},T]$ there is a unique $K^{N*}(t)$ that minimizes J_K^N, given by

$$K^{N*}(t) = R(t)^{-1}B^N(t)'P_K^{N*}(t)(\overline{C}^N)(\overline{C}^N)'\left(\overline{C}^N S_K^{N*}(\overline{C}^N)'\right)^{-1}$$

where $S_K^{N*}(t)$ is the unique solution of

$$\dot{S}_K^{N*} = S_K^{N*}\{A^N - B^NK^{N*}\overline{C}^N\} + \{A^N - B^NK^{N*}\overline{C}^N\}S_K^{N*} + M(t,S_K^{N*}) + N(t,S_K^{N*}) + E^N(E^N)' + F^N(F^N)'$$

$$S_K^{N*}(t_{N-1}) = E\left\{Z_{t_{N-1}}^{N*}(Z_{t_{N-1}}^{N*})'\right\}$$

where M: $[0,T] \times H_{n+m} \rightarrow H_{n+m}$ with

$$M(t,S) = \sum_{i,j=1}^{n} S_{ij}D_i.$$

We see that the optimal controls $K^{j*}(t)$ need solutions of nonlinear two-point boundary value problems, $j=N,\cdots,1$.

IV. Conclusion

We indicated a solution technique for obtaining the optimal linear stochastic control problem for dynamical systems with multiplicative noise and with quadratic criterion, where observations are available only at discrete-time points. The details may be found in [5].

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Suboptimal Control Using Pade Approximation Techniques

JAYANTA PAL

Abstract—**A method is given for the design of suboptimal controllers for single-input single-output systems using partial state feedback. This is based on the Pade approximation technique for model order reduction.**

INTRODUCTION

One of the drawbacks of optimal control theory is that it requires feedback from all the state variables that are defined to describe the dynamics of the plant. Unfortunately, the whole state vector is seldom available for measurement. One alternative is to reconstruct the missing states by using a Kalman filter or an observer. This introduces high-order dynamics in the control function and leads to a complicated and costly controller. This has motivated the design of incomplete state feedback suboptimal control laws using only the measurable states [1], [2].

In this paper a method for suboptimal controller design using measurable states for feedback is proposed. The suboptimal controller is derived from the "optimal" one by introducing constraints in the control structure. The Pade approximation technique for model order reduction [3] is used for arriving at the controller parameters.

THE DESIGN METHOD

The basic optimal control problem may be stated as follows. Consider the nth order single-input single-output linear dynamic system described by

$$\begin{aligned}\dot{x}&=Ax(t)+bu(t)\\ y&=c^Tx(t)\end{aligned} \tag{1}$$

with the quadratic cost function

$$J=\int_0^\infty\{x^T(t)Qx(t)+ru^2(t)\}\,dt \tag{2}$$

where Q is an $(n\times n)$ positive semidefinite matrix and r is a positive weight. A, b, and c are matrices of appropriate dimensions. It is well known that the optimal feedback control law is a linear combination of the state variables

$$u(t)=-r^{-1}b^TPx=-k^Tx \tag{3}$$

where P is a symmetric positive definite matrix whose elements may be found by solving the matrix Riccati equation

$$A^TP+PA-Pbr^{-1}b^TP+Q=0. \tag{4}$$

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The author is with the Department of Electrical Engineering, University of Roorkee, Roorkee, India.

The closed-loop transfer function with the optimal controller of (3) is

$$\begin{aligned}T^*(s)&=c^T[sI-A+bk^T]^{-1}b\\ &=\frac{b_0+b_1s+b_2s^2+\cdots+b_{m-1}s^{m-1}+b_ms^m}{a_0+a_1s+a_2s^2+\cdots+a_{n-1}s^{n-1}+a_ns^n} &(5)\\ &=d_0+d_1s+d_2s^2+\cdots &(6)\end{aligned}$$

where (6) is the power series expansion of (5) about $s=0$. Restricting the admissible control law to be linear and utilizing only the available states for feedback, the suboptimal controller may be specified as

$$\tilde{u}=-\tilde{k}^Tx \tag{7}$$

where $\tilde{k}_j=0$ if $x_j(t)$ is not available for feedback. Assuming that such a suboptimal controller exists, i.e., system (1) may be stabilized by the control law (7), the overall transfer function becomes

$$\begin{aligned}\tilde{T}(s)&=c^T[sI-A+b\tilde{k}^T]^{-1}b\\ &=\frac{b_0+b_1s+b_2s^2+\cdots+b_{m-1}s^{m-1}+b_ms^m}{f_0+f_1s+f_2s^2+\cdots+f_{n-1}s^{n-1}+f_ns^n}.\end{aligned} \tag{8}$$

For the choice of control laws in (3) and (7), the numerator polynomials in (5) and (8) will be the same. f_j $(j=0,1,\cdots,n)$ will contain the unknown feedback parameters in $\tilde{k}^T$. The incomplete state feedback problem is concerned with finding the elements of $\tilde{k}^T$ on some basis. For the suboptimal system response to be favorably comparable with that of the optimal one, the function in (8) should approximate $T^*(s)$ in (5) in some sense. The design technique is to use the Pade approximation method to find the unknown controller parameters in $\tilde{k}^T$.

For $\tilde{T}(s)$ to approximate $T^*(s)$ in the Pade sense, we have [3]

$$\begin{aligned}b_0&=f_0d_0\\ b_1&=f_0d_1+f_1d_0\\ b_2&=f_0d_2+f_1d_1+f_2d_0\\ &\vdots\quad\cdots\cdots\cdots\cdots\\ b_m&=f_0d_m+f_1d_{m-1}+f_2d_{m-2}+\cdots+f_md_0\\ 0&=f_0d_{m+1}+f_1d_m+f_2d_{m-1}+\cdots+f_{m+1}d_0\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ 0&=f_0d_{m+n}+f_1d_{m+n-1}+\cdots+f_{n-1}d_{m+1}+f_nd_m.\end{aligned} \tag{9}$$

Assuming that v state variables $(v<n)$ are available for feedback, the v unknown elements of $\tilde{k}^T$ can be explicitly determined by solving the first v linear equations in (9). The method is illustrated by the following example.

EXAMPLE

The voltage regulator example [4] is given by (1), where

$$A=\begin{bmatrix}-0.2&0.5&0.0&0.0&0.0\\ 0.0&-0.5&1.6&0.0&0.0\\ 0.0&0.0&-14.29&85.715&0.0\\ 0.0&0.0&0.0&-25.0&75.0\\ 0.0&0.0&0.0&0.0&-10.0\end{bmatrix}$$

$$b=[0.0\quad 0.0\quad 0.0\quad 0.0\quad 30.0]^T.$$

Choosing $Q=\mathrm{diag}\{1,0,0,0,0\}$; and $r=1$, in (2), on solving (4) we get

$$k^T=-[0.9245\quad 0.1711\quad 0.0161\quad 0.0492\quad 0.2643].$$

Using the above k^T, we have $T^*(s)=432.0/a(s)$, where

$$a(s)=432.05518+169.89431s+33.364525s^2+3.3945708s^3+0.1621612s^4+0.0028s^5.$$