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Frequency-truncated system norms

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Some applications require norm computation of frequency-truncated systems. A typical frequency-truncated system is one whose frequency response is rational in certain frequency bands and is zero in others. This note explains how to compute the $L^2$ norm of such systems.

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1. Introduction

The motivation for this note comes from a sampled-data problem depicted in Fig. 1. Given a sampling period $h$ and a system $G$, the game in this problem is to find a linear sampler $S$ and linear hold $H$ that minimize the $L^2$-norm of the error mapping $(I - HS)G$ from $w$ to $e$. The idea being that the smaller this norm is, the better the sampled-and-reconstructed $u = HSy$ resembles the analog $y$. The role of the system $G$ is mainly to emphasize certain frequency bands. Now samplers and holds, by their discrete nature, are not time invariant with respect to all time shifts. However, if $G$ is LTI then, in the standard sampled-data $L^2$-norm (Bamieh & Pearson, 1992) the $L^2$-optimal sampler-and-hold $HS$ is in fact LTI (Meinsma & Mirkin, 2010; Tsatsanis & Giannakis, 1995; Unser, 1993). In case the system $G$ has monotonically decreasing frequency response $|G(i\omega)|$ then the optimal sampler-and-hold is in fact nothing else than the ideal lowpass filter, $(HS)(i\omega) = \frac{1}{i\omega + \omega_u} \big|_{-\omega_u, \omega_u}$ with cut-off frequency equal to the Nyquist frequency $\omega_u := \pi / h$. (This ideal lowpass filter can indeed be implemented as a sampler-and-hold using Shannon’s formula.) As a result, the error mapping $(I - HS)G$ from $w$ to $e$ is just that of $G$ with the baseband $[-\omega_u, \omega_u]$ removed. Graphically,

For non-monotonic $|G(i\omega)|$ the optimal sampler-and-hold is more involved but still it is LTI and still it cancels certain frequency bands entirely and leaves other frequency bands untouched (Meinsma & Mirkin, 2010; Unser, 1993). In any event, monotonic or not, the computation of optimal error norms involves the computation of truncated system norms, either finite or semi-infinite,

$$\int_{-\omega_u}^{\omega_u} |G(i\omega)|^2 d\omega \quad \text{or} \quad \int_{\omega_u}^{\infty} |G(i\omega)|^2 d\omega.$$ 

In this note we analyze such integrals and, for rational $G(s)$, we survey some computational algorithms. This problem is not new and has for instance been dealt with in the context of model reduction (Juang & Gawronski, 1990). There are some subtleties that, to the best of our knowledge, have not been addressed earlier. This in particular pertains to the semi-infinite integrals and to the problem posed by imaginary poles of $G$.

Notation. The conjugate $G^\ast$ of a real transfer matrix is defined as $G^\ast(s) = [G(-s)]^T$. A constant square matrix $A$ is said to be stable if all its eigenvalues have strictly negative real part. The logarithm in...
this paper always refers to the principal logarithm. The principal logarithm is defined for square matrices without eigenvalues on the branch cut (the negative real axis, including zero). \( Q = \log(X) \) is then the unique matrix \( Q \) for which \( e^Q = X \) and whose spectrum lies in the open horizontal strip \( \{ z \in \mathbb{C} \mid -\pi < \text{Im}(z) < \pi \} \) of the complex plane (Higham, 2008, Thm. 1.31). The truncated \( L^2 \)-norm of a system \( G \) we denote with a subscript.

\[
\|G\|_{\text{infty}} = \sqrt{\frac{1}{\pi} \int_{-\infty}^{\infty} \text{tr} \ G^*(i\omega)G(i\omega) \, d\omega}, \quad \omega_a \geq 0. \tag{1}
\]

In Eq. (1) we allow MIMO systems \( G \).

### 2. Computation of frequency-truncated \( L^2 \)-norm

A complicating factor is that our \( G \) may have imaginary poles, as is often the case in the sampled-data problem, for instance, \( G(s) \) typically is rational with one pole at the origin. The truncated norm \( \|G\|_{\text{infty}} \) then exists only for \( \omega_a \) large enough and it also rules out splitting of the spectrum of \( G^* \) G into stable and antistable systems using, for instance, Lyapunov equations. We have to work instead with the full 2n-dimensional state representation of \( K \) := \( G^* \) \( G \). To this end, assume that \( G(s) \) is rational and strictly proper, and let \( G(s) = C(sI - A)^{-1}B \) be a realization of \( G \) and define \( \hat{A}, \hat{B} \) via the realization

\[
K := G^*G = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & 0 \end{bmatrix}.
\]

Notice that \( \hat{C} \hat{B} = 0 \) and that the imaginary poles of \( K \) are also imaginary poles of \( G \), not counting multiplicities.

**Theorem 1.** Suppose that \( G(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} \) with \( \hat{A}, \hat{B}, \hat{C} \) real matrices and that \( \hat{C} \hat{B} = 0 \). Then

\[
\int_{-\omega_a}^{\omega_a} K(i\omega) d\omega = i\hat{C} \log(o_\omega) + i\hat{A} \hat{B}
\]

provided that \( \omega_b > \omega_{\text{max}} := \max |\omega_k| \) where the maximum is taken over all imaginary eigenvalues \( i\omega_k \) of \( \hat{A} \).

This theorem and other results in the section are proved in the Appendix of this paper. The proof relies on elementary properties of the principal logarithm as documented in Higham (2008). The result appears intuitive because \( K(i\omega) \) equals

\[
K(i\omega) = -i\hat{C} \log(\omega I + i\hat{A})^{-1}\hat{B}
\]

and an anti-derivative, motivated by the scalar case, then indeed is

\[
-i\hat{C} \log(i\omega I - i\hat{A})\hat{B}.
\]

There are however some points in the proof that are easily overlooked. In particular, the following: from a systems theoretic perspective one might prefer not to extract the factor \( i \) in \( K(i\omega) \) and use instead \( K(i\omega) = \hat{C}(i\omega I - \hat{A})^{-1}\hat{B} \). This wrongly suggests that

\[
-i\hat{C} \log(i\omega I - i\hat{A})\hat{B}
\]

is a valid anti-derivative of \( K(i\omega) \) on \( (\omega\text{max}, \infty) \). It is generally wrong as \( \omega \) varies in \( (\omega_{\text{max}}, \infty) \) some eigenvalues of \( i\omega I - \hat{A} \) may cross the branch cut (the negative real axis) of the principal logarithm, and this makes the candidate anti-derivative (6) discontinuous (and wrong); see Fig. 2 (right).

Extracting \( i \) from the realization of \( K(i\omega) \) as done in (4) avoids this problem because now the matrix whose logarithm we take, \( i\omega I + i\hat{A} \), by construction has no eigenvalues on the branch cut when \( \omega \in (\omega_{\text{max}}, \infty) \); see Fig. 2 (middle). In addition, it has the advantage that the corresponding anti-derivative (5) is normalized to be zero at \( \omega = +\infty \), because \( \hat{C} \hat{B} = 0 \).

Given \( \omega > \omega_{\text{max}} \), the condition that \( \hat{C} \hat{B} = 0 \) is necessary and sufficient for the matrix \( \int_{-\omega_a}^{\omega_a} K(i\omega) d\omega \) to exist. This is equivalent to \( K(s) \) having relative degree 2 or more. The truncated system norm (1) now trivially follows as

\[
\|G\|_{\text{infty}} = \sqrt{\frac{1}{\pi} \text{tr} \ [\hat{C} \log(\omega I + i\hat{A})\hat{B}].}
\]

Owing to the symmetry properties of the realization (2), the term in (7) of which the square root is taken, is indeed real nonnegative. In the remaining subsections, we document some extensions and special cases of (3) and (7).

### 2.1. Finite integral for proper \( K(s) \)

If the relative degree of \( K(s) \) is less than 2, then the semi-infinite integral in (3) does not exist. A finite integral may still exist though. We formulate the result for proper \( K \).

**Proposition 2.** Let \( K(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D} \) be a realization with \( \hat{A}, \hat{B}, \hat{C}, \hat{D} \) matrices (possibly complex). Then

\[
\int_{-\omega_a}^{\omega_a} K(i\omega) d\omega = -i\hat{C} \log(\omega_I + i\hat{A}) - \log(\omega I + i\hat{A}) + \hat{D}(\omega_b - \omega_a)
\]

provided that \( \omega_a, \omega_b > \omega_{\text{max}} := \max |\omega_k| \) where the maximum is taken over all imaginary eigenvalues \( i\omega_k \) of \( \hat{A} \).

In this finite case the two logarithms can be combined into one, log(\( \Omega \)), with \( \Omega \) defined as

\[
\Omega = (\omega_b I + i\hat{A})(\omega_a I + i\hat{A})^{-1} = (i\omega_b I - \hat{A})(i\omega_a I - \hat{A})^{-1}
\]

Notice that the extraction of \( i \) from \( i\omega b I - \hat{A} \) – which was needed earlier to avoid eigenvalues on the branch cut – cancels in the formula for \( \Omega \). In fact this \( \Omega \) has no eigenvalues on the branch cut for all cases that one could possibly consider.

**Theorem 3.** Take \( \omega_b, \omega_a \in \mathbb{R} \). Let \( K(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D} \) be a realization with \( \hat{A}, \hat{B}, \hat{C}, \hat{D} \) matrices and suppose that \( \hat{A} \) has no imaginary eigenvalue \( i\omega \) with \( \omega \in (\omega_a, \omega_b) \). Then log(\( \Omega \)) exists for the \( \Omega \) defined above, and we have

\[
\int_{-\omega_a}^{\omega_a} K(i\omega) d\omega = -i\hat{C} \log(\Omega) + \hat{D}(\omega_b - \omega_a).
\]
2.2. Stable A matrix

If the A matrix of G is stable then the computational effort can be further reduced and connections with Lyapunov and the classic L2-norm can be established. It is a classic result that the L2-norm of a stable finite dimensional system $G(s) = C(sI - A)^{-1}B$ can be computed via the solution of a linear equation. Specifically, if $A$ is stable then

$$\|G\|_2^2 := \frac{1}{\pi} \int_{-\infty}^{\infty} G^*(i\omega) G(i\omega) d\omega$$

of $G$ can be computed via the solution of a linear equation. Specifically, if $A$ is stable then

$$\|G\|_2^2 := \frac{2}{\pi} \text{tr} \left( B^*P \right)$$

where $P$ is the unique solution of the Lyapunov equation

$$A^TP + PA = -C^T C,$$ (10)

(see e.g. Zhou and Doyle (1998, Lemma 2.1)). Now for given $\omega_k \geq 0$, the squared truncated norm (7) entails computation of a logarithm of a 2n × 2n Hamiltonian matrix. Given the stability of $A$ one can reduce the computational burden somewhat.

Theorem 4. Suppose $G$ is stable and strictly proper and let $G(s) = C(sI - A)^{-1}B$ be a realization with $A$, $B$, $C$ real matrices and $A$ stable. Then

$$\|G\|_{\infty}^2 = \frac{2}{\pi} \text{tr} \left( B^*P \right),$$

$$\|G\|_{\infty}^2 = \frac{2}{\pi} \text{tr} \left( B^*P \right) \log(\omega I - A)B)$$

where $P$ is the unique solution of (10).

Stability of $A$ in Theorem 4 is exploited in two different ways, because then (a) a solution of the Lyapunov equation (10) is guaranteed to exist, and (b) the eigenvalues of the matrices $\omega I + A$ and $i\omega - A$, which we take are not on the branch cut (the negative real axis, including zero). This holds irrespective of the choice of $\omega_k \in \mathbb{R}$.

For $\omega_k = 0$ one recovers (9). Indeed for $\omega_k = 0$, Eq. (11) reduces to

$$\|G\|_{\infty}^2 = \frac{2}{\pi} \text{tr} \left( B^*P \right) \log(\omega I + A)B) = \frac{2}{\pi} \text{tr} \left( B^*P \right) \log(\omega I - A)B) = (\omega I - A)^{-1}B.$$ (11)

Here we used Lemma 6 of the Appendix, which states that $\text{Im}(\log(\omega I)) = -\frac{\omega}{2} I$ for every stable $A \in \mathbb{R}^{n \times n}$.

2.2.1. Stable A matrix, finite interval of integration

If $G(s)$ is stable and proper but possibly not strictly proper, $G(s) = C(sI - A)^{-1}B + D$ with $A$ stable, then the semi-infinite (1) may not be well defined, but the finite integral

$$\frac{1}{\pi} \int_{-a}^{a} G^*(i\omega) G(i\omega) d\omega$$

exists whenever $-\infty < \omega_a < \omega_b < \infty$. Then (13) equals

$$\frac{2}{\pi} \text{tr} \left[ R(\log(\omega I + A) - \log(\omega I + A))B) + \frac{1}{\pi} \text{tr}[D^T D(\omega_a - \omega_b)] \right].$$

On the other hand, the two logarithms can be combined into one, $\log(2)$, with $\Omega := (\omega I + A)(\omega I + A)^{-1} = (i\omega I - A)$, which implies that (13) can also be computed as

$$\frac{2}{\pi} \text{tr} \left[ R(\log(\Omega)B) + \frac{1}{\pi} \text{tr}[D^T D(\omega_a - \omega_b)].$$

3. Concluding remarks

The principal logarithm is available in MATLAB and it appears to be numerically reliable (Davies & Higham, 2003) (this is in stark contrast with the matrix exponential Moler & van Loan, 1978).

Appendix. The principal logarithm and proofs

This Appendix collects basic properties of the principal logarithms (Higham, 2008, Thm. 1.31) and proofs of the results of Section 2.

Lemma 5. Let $\omega_n \in \mathbb{R}$. Let $A \in \mathbb{C}^{n \times n}$ and suppose that $\omega I + A$ is invertible for every $\omega \in [\omega_a, \infty)$. Then

1. $\log(\omega I + A)$ is analytic in $\omega \in [\omega_a, \infty)$ and $rac{d}{d\omega} \log(\omega I + A) = (\omega I + A)^{-1}$ for all $\omega \in [\omega_a, \infty)$
2. $\lim_{\omega \to \infty} \log(\omega I + A) \to \lim(\omega I + A) = 0$
3. $\lim_{\omega \to \infty} \text{Im}(\log(\omega I + A)) = 0$.

Proof. We write the Jordan canonical form of a matrix $-iA \in \mathbb{C}^{n \times n}$ as $-iA = \text{Zdiag}(I_1, I_2, \ldots, I_p)Z^{-1}$ where $Z \in \mathbb{C}^{n \times n}$ and $I_k$ is the kth Jordan block with eigenvalue $\lambda_k$.

1. Using Higham (2008, Dn. 1.2) and Higham (2008, Eq. (1.34)), we have $\log(\omega I + A) = Z\text{diag}(F_1(\omega), \ldots, F_p(\omega))Z^{-1}$, where $F_k(\omega)$ is given by

$$\log(\omega - \lambda_k) \log(\omega - \lambda_k)^{-1} \cdots \log(\omega - \lambda_k)^{-m_k}.$$ Clearly $\log(\omega - \lambda_k)$ and $\log(\omega - \lambda_k)^{-1}$ for every $j \in [1, 2, \ldots]$ is analytic for $\omega > \omega_a$. Therefore, $\frac{d}{d\omega} \log(\omega I + A) = \text{Zdiag}(F_1(\omega), \ldots, F_p(\omega))Z^{-1}$ where $F_k(\omega)$ is given by

$$\begin{pmatrix} (\omega - \lambda_k)^{-1} & (\omega - \lambda_k)^{-2} & \cdots & (\omega - \lambda_k)^{-m_k} \\ (\omega - \lambda_k)^{-1} & \cdots & (\omega - \lambda_k)^{-2} \\ \vdots & \ddots & \ddots & \vdots \\ (\omega - \lambda_k)^{-1} & \cdots & (\omega - \lambda_k)^{-2} & (\omega - \lambda_k)^{-1} \end{pmatrix}.$$ The result now follows because $F_k(\omega) = \lim_{\omega \to \infty} \log(\omega I - A) = I_{m_k}$, where $I_{m_k}$ is identity matrix of size $m_k$. (2) Since $\lim_{\omega \to \infty} \log(\omega I - A) \to \lim(\omega I - A) = 0$ for every $j \in [1, 2, \ldots]$, we have that $\lim_{\omega \to \infty} \text{Im}(F_k(\omega)) = 0$, which immediately yields the property. (3) As $\text{Im}(\log(\omega I)) = 0$, we have that $\lim_{\omega \to \infty} \text{Im}(F_k(\omega)) = 0$. Therefore, $\lim_{\omega \to \infty} \text{Im}(\log(\omega I - A) = 0$.

Lemma 6. If $A \in \mathbb{C}^{n \times n}$ is stable, then $\text{Im}(\log(\omega I)) = \text{Im}(\log(\omega I - A)) = \text{Im}(\log(\omega I - A)) = -\frac{\omega}{2} I$. If $A \in \mathbb{R}^{n \times n}$ is stable, then $\text{Im}(\log(\omega I)) = -\frac{\omega}{2} I$.

Proof. Let $\omega \in \mathbb{C}$ and $\Re(\omega) < 0$. Then $-\omega$ has a positive real part and $-\omega I$ a negative imaginary part. Considering that the branch cut of the principal logarithm is the negative real axis, we get $\log(-\omega I) = \log(-\omega) - i\frac{\omega}{2}$. Since $A$ is a stable matrix, $\log(-\omega I)$ exists. Therefore, using (Higham, 2008, Thm. 1.15a), $\text{Im}(\log(\omega I)) = \text{Im}(\log(-\omega I) - i\frac{\omega}{2}) = \text{Im}(\log(-\omega)) - i\frac{\omega}{2}$, $I$. If $A \in \mathbb{R}^{n \times n}$, then $\text{Im}(\log(-\omega I)) = 0$ (Higham, 2008, Thm. 1.16).□

Proof of Theorem 1. For $\omega \in (\omega_{\max}, \infty)$ the matrix $\omega I + A$ has no eigenvalues on the branch cut because $\omega > \omega_{\max}$. So the principal logarithm $\log(\omega I + A)$ exists for all such $\omega$ (Higham, 2008, Thm. 1.31) and it is an anti-derivative of $(\omega I + A)^{-1}$ (Lemma 5, Item 1). Using Lemma 5 (Item 2) and the fact that $\tilde{C} \tilde{B} = 0$ we now obtain $\sum_{k=0}^{\infty} K(\omega) d\omega = -i \sum_{k=0}^{\infty} (\tilde{C}(\omega I + A)^{-1} \tilde{B} d\omega = i(\tilde{C}(\omega I + A)^{-1} \tilde{B} d\omega = i(\tilde{C} \log(\omega I + A)B) - i \lim_{\omega \to \infty} \log(\omega) (\tilde{C} \tilde{B}) = i(\tilde{C} \log(\omega I + A)B)$. □
The proof of Proposition 2 is similar to that of Theorem 1. As a preparation for the proof of Theorem 3 we use Richter’s theorem (Higham, 2008, Thm. 11.1) which implies that \( \frac{d}{d \omega} \log(\omega Q + I) = Q(\omega Q + I)^{-1} \) whenever \( Q \in \mathbb{C}^{n \times n} \) and \( \omega \in \mathbb{R} \) is such that \( \omega Q + I \) has no eigenvalues on the branch cut.

**Lemma 7.** Let \( A \in \mathbb{C}^{n \times n} \). If \( \omega I - A \) is nonsingular for all \( \omega \in [\omega_a, \omega_b] \), then \( \Omega_{\omega_a} := (\omega I - A)(\omega I - A)^{-1} \) has no eigenvalues on the branch cut and
\[
\int_{\omega_a}^{\omega_b} (\omega I - A)^{-1} d\omega = -i \log(\Omega_{\omega_b}). \tag{15}
\]

**Proof.** It is easy to show that \( \Omega_{\omega_b} \) has eigenvalues on the branch cut iff \( A \) has eigenvalues of the form \( i\omega \) with \( \omega \in [\omega_a, \omega_b] \). Note that for \( \omega_b = \omega_a \) equality (15) is trivially correct. So we need only establish that the derivative of the left- and right-hand sides of (15) with respect to \( \omega_b \) is the same. Since \( \Omega_{\omega_b} = (\omega_b - \omega_a)Q + I \) for \( Q = i(\omega I - A)^{-1} \), we have by Richter’s theorem that \( \frac{d}{d \omega} \log(\Omega_{\omega_b}) = Q((\omega_b - \omega_a)Q + I)^{-1} = i(\omega I - A)^{-1} \). So the left- and right-hand sides of (15) have the same derivative. \( \square \)

Lemma 7 can also be proved with Jordan canonical forms. This lemma immediately proves Theorem 3. The validity of Eq. (14) follows similarly.

**Proof of Theorem 4.** With \( P \) the solution of (10) we can split \( G^{-1} G = H + H^\perp \) with \( H(s) = B^T P(sI - \tilde{A})^{-1} B \); see e.g. Zhou and Doyle (1998, Proof of Lemma 12.8). Now the anti-derivative of \( H(i\omega) \) with respect to \( \omega \) (see Lemma 5, Item 1) is, with slight abuse of notation, \( \int H(i\omega) = B^T P \int (i\omega I - A)^{-1} B = -iB^T P \log(\omega I + iA)B \) (up to a constant). Since \( H^\perp(i\omega) \) is the complex conjugate transpose of \( H(i\omega) \) we thus have (up to a constant) \( \text{tr} \int G^\perp(i\omega)G(i\omega) = 2\text{Re} \text{tr} \int H(i\omega) = 2\text{Re} \text{tr} (-iB^T P \log(\omega I + iA)B) = 2\text{Im} \text{tr} (B^T P \log(\omega I + iA)B) \).

Therefore, using Lemma 5,
\[
\pi \|G\|^2_{\omega_B} = 2\text{Im} \text{tr} (B^T P \log(\omega I + iA)B) \bigg|_{\omega_B} = 2\text{Im} \text{tr} (-iB^T P \log(\omega I + iA)B). \tag{16}
\]

Note that \( -i(\omega I - A) \) is stable for every \( \omega_B \in \mathbb{R} \), therefore, using Lemma 6, Eq. (16) can also be written as \( \pi \|G\|^2_{\omega_B} = 2\text{tr} \left( \frac{1}{2} B^T PB \right) - 2\text{Im} \text{tr} (B^T P \log(\omega I - A)B). \) \( \square \)

**References**


