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Interconnection of J-lossless behaviours

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ABSTRACT

In this paper, motivated by the phenomenon of the interconnection of lossless electrical networks, a class of behaviours known as J-lossless behaviours is introduced, where J is a symmetric two-variable polynomial matrix. It is shown that for certain values of J, interconnection of J-lossless behaviours leads to an oscillatory behaviour. Physically this translates to the fact that interconnection of two multi-port lossless electrical networks results in an autonomous lossless electrical network. Finally, the problem of decomposition of an oscillatory behaviour with a given characteristic polynomial as an interconnection of two single-input–single-output behaviours, such that one has a lossless positive real transfer function and the other has a lossless negative real transfer function is also considered. This problem can be viewed as an inverse problem to the one of interconnection of J-lossless behaviours.

1. Introduction

Consider the interconnection of two lossless one-port electrical networks as depicted in Fig. 1. It can be inferred that this interconnection will result in an autonomous lossless electrical network as it will not have any dissipative component. In this paper, this result is proved mathematically. The main result of this paper is a generalization of this result for the case of interconnection of multi-port lossless electrical networks which is proved in the framework of behavioural systems theory. It is assumed that the reader is familiar with the behavioural framework and with the calculus of quadratic differential forms, and the interested readers are referred to respectively [1] and [2] for a thorough exposition of the concepts and mathematical techniques.

In [3], losslessness has been studied in the context of nonlinear network theory. Here, a state representation of a multi-port electrical network is defined as lossless if the energy required to travel between any two points in the state space is independent of the path taken. In [3], it has been also shown that under certain conditions, the interconnection of multi-port networks with lossless state representations has a lossless state representation. In contrast, the technique used in this paper to prove the oscillatory nature of the interconnection of lossless networks is independent of the representation of its behaviour.

In this paper, for a given nonzero finite-dimensional symmetric two-variable polynomial square matrix J, a class of behaviours known as J-lossless behaviours is defined. The main result of this paper is Theorem 16 where it is proved that for certain J's that are associated with lossless electrical networks, interconnection of two J-lossless behaviours results in an oscillatory behaviour which is a behaviour whose trajectories are linear combinations of vector sinusoidal functions. Physical examples of oscillatory systems are mechanical systems consisting of frictionless springs and masses having as external variables the displacements or the velocities of the masses from the equilibrium positions; and electrical systems consisting of the interconnection of inductors and capacitors, having as external variables the voltages across the capacitors or the currents in the inductance components.

In this paper, it is shown that the main result (Theorem 16) regarding interconnection of J-lossless behaviours is related to a well-known result in state-space theory known as the *Kalman–Yakubovich–Popov* (KYP) *Lemma*, which is also called the *positive real lemma*. This lemma holds for systems with transfer function matrices that are positive real (See Appendix for a definition). In [4], it has been proved that G is a hybrid transfer function matrix (see Appendix D) of a multi-port electrical network consisting of a finite number of resistors, capacitors, inductors, transformers and gyrators if and only if it is positive real. This result was earlier proved for the case of one-port electrical networks by Brune [5]. Thus KYP lemma holds for multi-port electrical networks consisting of a finite number of resistors, capacitors, inductors, transformers and gyrators. In this paper, it is shown that the external behaviours of certain systems with lossless positive real transfer functions (see Appendix for a definition) are J-lossless for a certain J. Thus the physical interpretation of the main result of this paper is that the interconnection of a certain type of lossless electrical networks leads to oscillatory systems.

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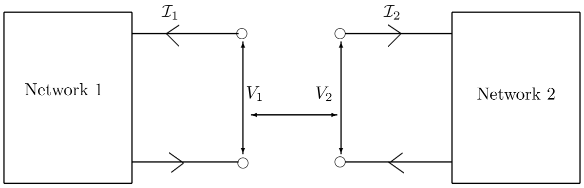

Fig. 1. Interconnection of one-port electrical networks.

The structure of this paper is as follows. In Section 2, some basic concepts from behavioural systems theory and some properties of quadratic differential forms and properties of autonomous, oscillatory and lossless systems are discussed. In Section 3, a special class of behaviours known as J-lossless behaviours is introduced followed by a discussion of properties of their image representation and another property of such behaviours. Section 4 presents results pertaining to interconnection of J-lossless behaviours and those pertaining to decomposition of an oscillatory behaviour with a given characteristic polynomial as an interconnection of two SISO behaviours, such that one has a lossless positive real transfer function and the other has a lossless negative real transfer function. Section 5 presents conclusions based on the results in Sections 3 and 4.

The notation used in this paper is standard: the space of $\mathtt{n}$-dimensional real, respectively complex vectors is denoted by $\mathbb{R}^{\mathtt{n}}$, respectively $\mathbb{C}^{\mathtt{n}}$, the space of $\mathtt{m}\times\mathtt{n}$ real matrices by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$, and the space of $\mathtt{m}\times\mathtt{m}$ symmetric real matrices, by $\mathbb{R}_s^{\mathtt{m}\times\mathtt{m}}$. Whenever one of the two dimensions is not specified, a bullet $\bullet$ is used; so that for example, $\mathbb{R}^{\bullet\times\mathtt{n}}$ denotes the set of real matrices with $\mathtt{n}$ columns and an unspecified number of rows. In order to enhance readability, when dealing with a vector space $\mathbb{R}^{\bullet}$ whose elements are denoted with w (or x), the notation $\mathbb{R}^{\mathtt{w}}$ (or $\mathbb{R}^{\mathtt{x}}$) (note the typewriter font type!) is used and when dealing with a vector space $\mathbb{R}^{\bullet}$ whose elements are denoted with ℓ, the notation $\mathbb{R}^{l}$ is used; similar considerations hold for matrices representing linear operators on such spaces. The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$; the set of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta,\eta]$. The space of all $\mathtt{w}\times l$ polynomial matrices in the indeterminate ξ is denoted by $\mathbb{R}^{\mathtt{w}\times l}[\xi]$, and that consisting of all $\mathtt{w}\times l$ polynomial matrices in the indeterminates ζ and η by $\mathbb{R}^{\mathtt{w}\times l}[\zeta,\eta]$. The space of real rational functions in the indeterminate ξ is denoted by $\mathbb{R}(\xi)$ (note the difference in bracket type as compared to the ring of polynomials) and the space of all matrices of size $\mathtt{w}\times l$, whose entries are real rational functions of the indeterminate ξ are denoted by $\mathbb{R}^{\mathtt{w}\times l}(\xi)$. The set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$ is denoted by $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. $\mathbb{R}^{+}$ denotes the set of positive real numbers. $\mathbb{N}$ denotes the set of positive integers. I_l stands for identity matrix of size l. $0_{\mathtt{w}\times l}$ denotes a matrix of size $\mathtt{w}\times l$ consisting of zeroes and 0_n denotes a vector of dimension n consisting of zeroes. $\mathrm{col}(L_1,L_2)$ denotes the matrix obtained by stacking the matrix L_1 over L_2, which has the same number of columns as L_1 and $\mathrm{row}(R_1,R_2)$ denotes the matrix obtained by stacking the matrix R_2 to the right of R_1, which has the same number of rows as R_2. $\mathrm{diag}(a_1,\ldots,a_n)$ denotes a block diagonal matrix with entries $a_1,\ldots,a_n$ along the diagonal in the given order if $a_1,\ldots,a_n$ are real square matrices. The class of linear differential behaviours with infinitely differentiable manifest variable w is denoted by $\mathcal{L}^{\mathtt{w}}$. $\det(A)$ denotes the determinant of a square matrix A. $\deg(r)$ denotes the degree of a polynomial r. $\dim(V)$ denotes the dimension of a vector V. $\mathrm{Im}(M)$ denotes the image of a linear map M.

2. Preliminaries

In this section, the basic definitions and concepts of [1,2,6–8] that are necessary to understand the results given in this paper are illustrated.

2.1. Behaviours

A *behaviour* $\mathfrak{B}$ is a subspace of $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ consisting of all solutions w of a given system of linear constant-coefficient differential equations of the form

$$R_0 w + R_1\frac{\mathrm{d}w}{\mathrm{d}t} + \cdots + R_L\frac{\mathrm{d}^L w}{\mathrm{d}t^L} = 0 \tag{1}$$

where $R_i\in\mathbb{R}^{\mathtt{g}\times\mathtt{w}}$ for $i=0,1,\ldots,L$. Define the polynomial matrix $R\in\mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$ as

$$R(\xi) := R_0 + R_1\xi + \cdots + R_L\xi^L.$$

Using the above equation, Eq. (1) can also be written as

$$R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0. \tag{2}$$

The behaviour $\mathfrak{B}$ can be defined as

$$\mathfrak{B} := \left\{ w\in\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0 \right\}.$$

Thus, considering $R(\frac{\mathrm{d}}{\mathrm{d}t})$ as an operator from $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ to $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{g}})$, $\mathfrak{B}=\ker\left(R(\frac{\mathrm{d}}{\mathrm{d}t})\right)$. Linearity of the differential operator $R(\frac{\mathrm{d}}{\mathrm{d}t})$ results in linearity of the behaviour $\mathfrak{B}$. $\mathfrak{B}$ is shift-invariant as the coefficients of the polynomial matrix R are constant. The system of linear constant coefficient differential equation (2) is called a *kernel representation* of the behaviour $\mathfrak{B}$. It is called *minimal* if every other kernel representation of $\mathfrak{B}$ has at least $\mathtt{g}$ rows. The set of behaviours with infinitely often differentiable manifest variable w is denoted by $\mathcal{L}^{\mathtt{w}}$ (the superscript $\mathtt{w}$ in $\mathcal{L}^{\mathtt{w}}$ refers to the dimension of $w\in\mathfrak{B}$).

When modelling a system, two types of variables namely *manifest variables* (denoted by w) and *latent variables* (denoted by ℓ) are encountered. Manifest variables are the variables whose evolution with time is of interest, while latent variables are the other variables that come up during the process of modelling. If the system under consideration is a linear differential system, then the trajectories belonging to the system can be described by a set of linear constant coefficient ordinary differential equations

$$R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell \tag{3}$$

where $R\in\mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$ and $M\in\mathbb{R}^{\mathtt{g}\times l}[\xi]$. The above equation describes the *full behaviour*

$$\mathfrak{B}_{\mathrm{f}} := \{(w,\ell)\in\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}+l}) \mid (3)\text{ holds}\}$$

and the projection of $\mathfrak{B}_{\mathrm{f}}$ on the w variable, i.e.

$$\mathfrak{B} := \{w \mid \exists\ell \text{ such that (3) holds}\}$$

is called the *manifest behaviour* associated with (3). If $R(\xi)=I_{\mathtt{w}}$ in Eq. (3),

$$w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell. \tag{4}$$

The above equation is called an image representation of $\mathfrak{B}$. It can be showed that an image representation exists for $\mathfrak{B}$ iff $\mathfrak{B}$ is *controllable* in the behavioural sense (see chapter 5 of [1]). The image representation (4) is called *observable* if ℓ is observable from w, i.e if $[w=M(\frac{\mathrm{d}}{\mathrm{d}t})\ell=0]\implies[\ell=0]$ (see chapter 5 of [1] for a description of observability). Using Theorem 5.3.3, p. 174 of [1], it can be proved that this is the case if and only if the matrix $M(\lambda)$ has full column rank for all $\lambda\in\mathbb{C}$. It can also be proved that if $\mathfrak{B}$ is controllable, then there exists an observable image representation of $\mathfrak{B}$.

In this paper the concepts of state and of state representation are also used (see [6] for a thorough discussion). A latent variable ℓ is a *state variable* for $\mathfrak{B}$ if $\mathfrak{B}$ admits a representation (3) of first order in ℓ and zeroth order in w : $E\frac{d\ell}{dt} + F\ell + Gw = 0$. Such a representation is called a *state-space representation* of $\mathfrak{B}$. The state variable ℓ of $\mathfrak{B}$ is often denoted by x. By combining the notion of state with that of inputs and outputs one can arrive at the *input/state/output (i/s/o) representation*

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx + Du \tag{5}$$
$$w = \text{col}(y, u).$$

The state representation (5) is said to be *minimal* if any other state representation of $\mathfrak{B}$ with state variable x_1 is such that $\dim(x_1) \geq \dim(x)$. If the state representation (5) is minimal, then it can be shown that x is observable from w, i.e $[w = 0] \implies [x = 0]$. In such a case, there exists $X \in \mathbb{R}^{\mathtt{x}\times\mathtt{w}}[\xi]$, with $\mathtt{x} = \dim(x)$, such that $x = X(\frac{d}{dt})w$.

2.2. Quadratic differential forms

Consider the set of bilinear functionals acting on an infinitely differentiable trajectory w of the form

$$Q_\Phi(w) = \sum_{h,k=0}^{N} \left(\frac{d^h w}{dt^h}\right)^\top \Phi_{h,k} \left(\frac{d^k w}{dt^k}\right) \tag{6}$$

where $\Phi_{h,k}$ are $\mathtt{w} \times \mathtt{w}$-dimensional real matrices, and N is a nonnegative integer. Such a functional is called a *quadratic differential form* (QDF). With the QDF given by Eq. (6), is associated a two-variable polynomial matrix $\Phi(\zeta, \eta)$, which is given by

$$\Phi(\zeta, \eta) = \sum_{h,k=0}^{N} \Phi_{h,k}\zeta^h\eta^k.$$

A QDF Q_Φ is called *symmetric* if $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^\top$. The notion of the *derivative* of a QDF is defined below.

Definition 1. A QDF Q_Ψ is called the *derivative of a QDF* Q_Φ with $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$, denoted by $Q_\Psi = \frac{d}{dt}Q_\Phi$ if $\frac{d}{dt}Q_\Phi(w) = Q_\Psi(w)$ for all $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

In terms of the two-variable polynomial matrices associated with Q_Ψ and Q_Φ, the relationship in Definition 1 can be expressed as follows: Q_Ψ is the derivative of Q_Φ if and only if for the corresponding two-variable polynomial matrices, there holds $(\zeta + \eta)\Phi(\zeta, \eta) = \Psi(\zeta, \eta)$ (see [2], p. 1710).

Defined below are the notions of nonnegativity and positivity of QDFs.

Definition 2. Let $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. Q_Φ is said to be *nonnegative*, denoted by $Q_\Phi \geq 0$ if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$; and *positive*, denoted by $Q_\Phi > 0$, if $Q_\Phi \geq 0$, and $[Q_\Phi(w) = 0] \implies [w = 0]$.

The notion of a QDF being nonnegative or positive along a particular behaviour is defined below.

Definition 3. Let $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. Q_Φ is said to be *nonnegative along* $\mathfrak{B}$, denoted by $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$ if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{B}$, and *positive along* $\mathfrak{B}$, denoted by $Q_\Phi \overset{\mathfrak{B}}{>} 0$, if $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$ and $[Q_\Phi(w) = 0] \implies [w = 0]$.

Given below are algebraic conditions on the two-variable polynomial matrix corresponding to a QDF under which it is nonnegative and positive along a given behaviour.

Proposition 4. *Let $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ and let $\mathfrak{B} = \ker\left(R(\frac{d}{dt})\right)$. Then*

1. *$Q_\Phi \overset{\mathfrak{B}}{\geq} 0$ iff there exists $F \in \mathbb{R}^{\bullet\times\mathtt{w}}[\zeta, \eta]$ and $D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$, such that $\Phi(\zeta, \eta) = D(\zeta)^\top D(\eta) + F(\eta, \zeta)^\top R(\eta) + R(\zeta)^\top F(\zeta, \eta)$.*
2. *$Q_\Phi \overset{\mathfrak{B}}{>} 0$ iff $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$ and $\text{col}\left(D(\lambda), R(\lambda)\right)$ has full column rank for all $\lambda \in \mathbb{C}$.*

Proof. See Proposition 3.5 of [2]. □

2.3. Autonomous, oscillatory and lossless systems

An autonomous system is a system with no free variables. For such a system, the future of every trajectory is completely determined by its past. A formal definition for an autonomous behaviour is given below.

Definition 5. A behaviour $\mathfrak{B}$ is called *autonomous* if for all $w_1, w_2 \in \mathfrak{B}$,

$$[w_1(t) = w_2(t)\ \forall t \leq 0] \implies [w_1 = w_2].$$

The following proposition relates the property of autonomy of a multi-variable behaviour to algebraic properties of a matrix R inducing a kernel representation of the behaviour.

Proposition 6. *Let $\mathfrak{B} = \ker\left(R(\frac{d}{dt})\right)$, with $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$, be a kernel representation of $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$. Then $\mathfrak{B}$ is autonomous iff R has full column rank.*

The *invariant polynomials* of a polynomial matrix $P \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ are the diagonal elements of the Smith form (see Section 6.3-3, [9] for a definition) of P. Let $\mathfrak{B} = \ker\left(R(\frac{d}{dt})\right)$ be a minimal kernel representation of an autonomous behaviour $\mathfrak{B}$. Then the invariant polynomials of R are also called the invariant polynomials of $\mathfrak{B}$.

An oscillatory behaviour is defined below.

Definition 7. A behaviour $\mathfrak{B}$ defines an oscillatory system if every solution $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$ of $\mathfrak{B}$ is bounded on $\mathbb{R}$.

From the definition, it follows that an oscillatory system is necessarily autonomous: if there were any input variables in w, then those components of w could be chosen to be unbounded. It was proved in Proposition 2 of [7] that any behaviour $\mathfrak{B}$ is oscillatory if and only if every nonzero invariant polynomial of $\mathfrak{B}$ has distinct and purely imaginary roots. In the following, a polynomial matrix will be called oscillatory if all its invariant polynomials have distinct and purely imaginary roots.

The notion of a *conserved quantity* was first defined in [7], and it is used for defining lossless systems. This definition is given below.

Definition 8. Let $\mathfrak{B}$ be an autonomous behaviour. A QDF Q_Φ is a *conserved quantity* for $\mathfrak{B}$ if

$$\frac{d}{dt}Q_\Phi(w) = 0 \quad \forall w \in \mathfrak{B}.$$

Thus, conserved quantity is a QDF, whose derivative is zero along the trajectories of a given behaviour. The notion of an autonomous lossless system as in [8], is defined below.

Definition 9. An autonomous behaviour $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ is *lossless* if there exists a conserved quantity Q_E associated with $\mathfrak{B}$, such that $Q_E > 0$. Such a Q_E is called an *energy function* for the system.

The main result of [8] which is used in this paper, is given below.

Theorem 10. *An autonomous behaviour $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ is lossless if and only if it is oscillatory.*

Proof. See proof of Theorem 3, p. 1529, [8]. □

Given below is a modified version of Theorem 10 which is used to prove the main result of this paper.

Theorem 11. *A behaviour $\mathfrak{B}$ is oscillatory iff there exists a QDF Q_E, such that $Q_E \overset{\mathfrak{B}}{>} 0$ and $\frac{d}{dt}Q_\Phi(w) = 0 \, \forall w \in \mathfrak{B}$.*

Proof. (*Only if*): If $\mathfrak{B}$ is oscillatory, then from Theorem 10, it follows that there exists an energy function Q_E which is conserved along $\mathfrak{B}$, and $Q_E > 0$. This implies that $Q_E \overset{\mathfrak{B}}{>} 0$ and $\frac{d}{dt}Q_\Phi(w) = 0 \, \forall w \in \mathfrak{B}$. Hence the proof.

(*If*): Now assume that there exists a QDF Q_E, such that $Q_E \overset{\mathfrak{B}}{>} 0$ and $\frac{d}{dt}Q_\Phi(w) = 0 \, \forall w \in \mathfrak{B}$. It is first proved that $\mathfrak{B}$ is autonomous. Let $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$. Let $R = U\Delta V$ be a Smith form decomposition of R. Define $R_1 := \Delta V$. Since U is unimodular, $\mathfrak{B} = \ker\big(R_1(\frac{d}{dt})\big)$ is another kernel representation of $\mathfrak{B}$. Since $Q_E \overset{\mathfrak{B}}{>} 0$, from Proposition 4, it follows that there exist $F \in \mathbb{R}^{\bullet\times\mathtt{w}}[\zeta,\eta]$ and $D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$, such that

$$E(\zeta,\eta) = D(\zeta)^\top D(\eta) + F(\eta,\zeta)^\top R_1(\eta) + R_1(\zeta)^\top F(\zeta,\eta)$$

and $\mathrm{col}(D(\lambda), R_1(\lambda))$ has full column rank for all $\lambda \in \mathbb{C}$. Define $V_1 := V^{-1}$ and $D_1 := DV_1$. Then it is easy to see that $\mathrm{col}(D_1(\lambda), \Delta(\lambda))$ has full column rank for all $\lambda \in \mathbb{C}$. Now assume by contradiction that $\mathfrak{B}$ is not autonomous. Then from Proposition 6, it follows that R_1 does not have full column rank. It follows that Δ has its last column full of zeroes. Let $D_{\mathtt{w}} \in \mathbb{R}^\bullet[\xi]$ denote the last column of D_1. Since $\mathrm{col}(D_1(\lambda), \Delta(\lambda))$ has full column rank for all $\lambda \in \mathbb{C}$, it follows that $D_{\mathtt{w}}(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}$. Define $w_1 := \mathrm{col}(0_{\mathtt{w}-1}, e^{\lambda_1 t})$ where $\lambda_1 \in \mathbb{R}$ is nonzero and $w_2 := V_1(\frac{d}{dt})w_1$. Observe that $w_2 \in \mathfrak{B}$, and hence

$$\begin{aligned} Q_E(w_2) &= \left(D\left(\frac{d}{dt}\right)w_2\right)^\top \left(D\left(\frac{d}{dt}\right)w_2\right) \\ &= \left(D_1\left(\frac{d}{dt}\right)w_1\right)^\top \left(D_1\left(\frac{d}{dt}\right)w_1\right) \\ &= \left(D_{\mathtt{w}}\left(\frac{d}{dt}\right)e^{\lambda_1 t}\right)^\top \left(D_{\mathtt{w}}\left(\frac{d}{dt}\right)e^{\lambda_1 t}\right) \\ &= D_{\mathtt{w}}(\lambda_1)^\top D_{\mathtt{w}}(\lambda_1)e^{2\lambda_1 t}. \end{aligned}$$

Differentiating the above equation with respect to time, we get

$$\frac{d}{dt}Q_E(w_2) = 2\lambda_1 D_{\mathtt{w}}(\lambda_1)^\top D_{\mathtt{w}}(\lambda_1)e^{2\lambda_1 t}.$$

Since $w_2 \in \mathfrak{B}$, $\frac{d}{dt}Q_E(w_2) = 0$. This implies that $D_{\mathtt{w}}(\lambda_1)^\top D_{\mathtt{w}}(\lambda_1) = 0$, which in turn implies that $D_{\mathtt{w}}(\lambda_1) = 0$ as $D_{\mathtt{w}}(\lambda_1)^\top D_{\mathtt{w}}(\lambda_1)$ is a sum of squares of real numbers. This is a contradiction. This proves that $\mathfrak{B}$ is autonomous.

Next in order to prove that $\mathfrak{B}$ is oscillatory, the case of $\mathfrak{B} \in \mathcal{L}^1$ is first dealt with. Consider the (*Only If*) part of the proof of Theorem 1, p. 1524, [8]. Here it has been proved that if an autonomous behaviour $\mathfrak{B} \in \mathcal{L}^1$ is not oscillatory then there does not exist a conserved quantity Q_E, such that $Q_E \overset{\mathfrak{B}}{>} 0$. This proves the theorem for any autonomous $\mathfrak{B} \in \mathcal{L}^1$. The theorem can now be proved for the case of a multi-variable autonomous behaviour $\mathfrak{B}$ along the same lines as the (*Only If*) part of the proof of Theorem 3, p. 1529, [8]. □

3. *J*-lossless behaviours

In this section, the notion of a J-lossless behaviour is introduced and its properties are discussed.

Definition 12 (*J-Lossless Behaviour*)**.** Let $J \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be such that $J \neq 0$. A behaviour $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ is said to be *J-lossless* if there exists a QDF $Q_E \overset{\mathfrak{B}}{>} 0$ with $E \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, such that for every trajectory $w \in \mathfrak{B}$, $Q_J(w) = \frac{d}{dt}Q_E(w)$.

Remark 13. The concept of J-losslessness is related to the concept of *half-line nonnegativity* of QDFs described in pp. 1725–1726 of [2]. In [2], the concept of average nonnegativity of a QDF is also discussed. Here, it is proved that a QDF Q_Φ being average nonnegative is equivalent with the existence of another QDF Q_Ψ called as a *storage function*, which obeys

$$\frac{d}{dt}Q_\Psi \le Q_\Phi$$

and Q_Φ being half-line nonnegative is equivalent with the existence of a storage function Q_Ψ, which obeys

$$\frac{d}{dt}Q_\Psi \le Q_\Phi$$

and $Q_\Psi \ge 0$. In Remark 5.9, p. 1723 of [2], a behaviour $\mathfrak{B}$ is said to be lossless or *conservative* with respect to a *supply rate* Q_Φ if there exists a storage function Q_Ψ, such that $\frac{d}{dt}Q_\Psi(w) = Q_\Phi(w) \forall w \in \mathfrak{B}$. Assume that a given behaviour $\mathfrak{B}$ is conservative as per this definition, and controllable. This implies that $\mathfrak{B}$ has an observable image representation given by $\mathfrak{B} = \mathrm{Im}\big(M(\frac{d}{dt})\big)$. Define $\Psi'(\zeta,\eta) := M(\zeta)^\top\Psi(\zeta,\eta)M(\eta)$ and $\Phi'(\zeta,\eta) = M(\zeta)^\top\Phi(\zeta,\eta)M(\eta)$. Then it follows that

$$\frac{d}{dt}Q_{\Psi'} = Q_{\Phi'}.$$

This implies that $Q_{\Phi'}$ is average nonnegative. Extending the definition of conservative behaviours so as to include the concept of *half-line nonnegativity*, one can say that a behaviour $\mathfrak{B}$ is *half-line conservative* with respect to a *supply rate* Q_Φ if there exists a storage function $Q_\Psi \overset{\mathfrak{B}}{\ge} 0$, such that $\frac{d}{dt}Q_\Psi(w) = Q_\Phi(w) \forall w \in \mathfrak{B}$. Observe that if $\mathfrak{B}$ is half-line conservative and controllable with an observable image representation given by $\mathfrak{B} = \mathrm{Im}\big(M(\frac{d}{dt})\big)$, then $Q_{\Phi'}$ with $\Phi'(\zeta,\eta) := M(\zeta)^\top\Phi(\zeta,\eta)M(\eta)$, is half-line nonnegative. According to Definition 12, $\mathfrak{B}$ is Φ-lossless if there exists a QDF $Q_\Psi \overset{\mathfrak{B}}{>} 0$, such that $\frac{d}{dt}Q_\Psi(w) = Q_\Phi(w) \forall w \in \mathfrak{B}$. Observe that definition of a J-lossless behaviour is related to the definition of a half-line conservative behaviour with respect to Q_J, but not exactly the same.

The next Lemma gives algebraic conditions on an observable image representation of a controllable behaviour $\mathfrak{B}$ for it to be J-lossless.

Lemma 14. *Consider a controllable behaviour $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ for which an observable image representation is $\mathfrak{B} = Im\,\big(M(\frac{d}{dt})\big)$ with $M \in \mathbb{R}^{\mathtt{w}\times l}[\xi]$. Let $J \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be such that $J \neq 0$. $\mathfrak{B}$ is J-lossless if and only if the following hold:*

1. $M(-\xi)^\top JM(\xi) = 0$.
2. $\Phi(\zeta,\eta) := \frac{M(\zeta)^\top JM(\eta)}{\zeta+\eta}$ *is such that* $Q_\Phi > 0$.

Proof. (*If*): Assume that $M(-\xi)^\top JM(\xi) = 0$. Consider a trajectory $w = M(\frac{d}{dt})\ell$. By assumption, ℓ is observable from w. Consequently, there exists $F \in \mathbb{R}^{l\times\mathtt{w}}[\xi]$, such that $\ell = F(\frac{d}{dt})w$. Now define $E(\zeta,\eta) := F(\zeta)^\top\Phi(\zeta,\eta)F(\eta)$. Then

$$Q_E(w) = Q_\Phi\left(F\left(\frac{d}{dt}\right)w\right) = Q_\Phi(\ell).$$

It can be easily verified that $\frac{\mathrm{d}}{\mathrm{d}t}Q_E(w) = Q_J(w)$. Now assume that $Q_\Phi > 0$. Then it is easy to see that $Q_E(w) > 0$ for any nonzero trajectory $w \in \mathfrak{B}$, and $Q_E(w) = 0$ implies that $w = 0$. Hence $\mathfrak{B}$ is J-lossless if $Q_\Phi > 0$.

(Only If): Assume that $\mathfrak{B}$ is J-lossless. Consider a trajectory $w \in \mathfrak{B}$ given by $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$. Since $\mathfrak{B}$ is J-lossless, there exists $E \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, such that $Q_E \overset{\mathfrak{B}}{>} 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}Q_E(w) = Q_J(w)$. Define $J'(\zeta,\eta) := M(\zeta)^\top J M(\eta)$ and $\Phi(\zeta,\eta) := M(\zeta)^\top E(\zeta,\eta)M(\eta)$ and observe that

$$Q_{J'}(\ell) = \frac{\mathrm{d}}{\mathrm{d}t}Q_\Phi(\ell). \tag{7}$$

We have from Eq. (7),

$$M(\zeta)^\top J M(\eta) = (\zeta+\eta)\Phi(\zeta,\eta).$$

From the above equation, it follows that $M(-\xi)^\top J M(\xi) = 0$ and $\Phi(\zeta,\eta) = \frac{M(\zeta)^\top J M(\eta)}{\zeta+\eta}$. We have $Q_E(w) = Q_\Phi(\ell)$ for all (w,ℓ), such that $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$. Since $Q_E \overset{\mathfrak{B}}{>} 0$, it follows that $Q_\Phi > 0$. Hence the claim. □

Given below is another property of J-lossless behaviours which will be useful in proving some of the results of this paper.

Lemma 15. *Consider a J-lossless behaviour $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$. Let $A \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be an invertible matrix. Define $\tilde{\mathfrak{B}} := A^{-1}\mathfrak{B}$, and $\Phi(\zeta,\eta) := A^\top J(\zeta,\eta)A$. Then $\tilde{\mathfrak{B}}$ is Φ-lossless.*

Proof. Consider a trajectory $w \in \mathfrak{B}$. Define $\tilde{w} := A^{-1}w$, and note that $\tilde{w} \in \tilde{\mathfrak{B}}$. Since $\mathfrak{B}$ is J-lossless, there exists $E \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, such that $Q_E \overset{\mathfrak{B}}{>} 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}Q_E(w) = Q_J(w)$. Now observe that

$$Q_J(w) = Q_J(A\tilde{w}) = Q_\Phi(\tilde{w}).$$

Define $\tilde{E}(\zeta,\eta) := A^\top E(\zeta,\eta)A$ and observe that $Q_E(w) = Q_E(A\tilde{w}) = Q_{\tilde{E}}(\tilde{w})$. This implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_{\tilde{E}}(\tilde{w}) = Q_\Phi(\tilde{w}). \tag{8}$$

If $w \neq 0$, then $Q_E(w) > 0$. This implies that $Q_{\tilde{E}}(\tilde{w}) > 0$ if $\tilde{w} \neq 0$. Since $[Q_E(w) = 0] \Longrightarrow [w = 0]$ and A is invertible, it follows that $[Q_{\tilde{E}}(\tilde{w}) = 0] \Longrightarrow [\tilde{w} = 0]$. Hence from Definition 2, it follows that

$$Q_{\tilde{E}} \overset{\tilde{\mathfrak{B}}}{>} 0. \tag{9}$$

It now follows from Eqs. (8) and (9) that $\tilde{\mathfrak{B}}$ is Φ-lossless. □

4. Interconnection of *J*-lossless behaviours

4.1. Main result

The main result of this paper is given below.

Theorem 16. *Define*

$$J := \begin{bmatrix} 0_{l\times l} & I_l \\ I_l & 0_{l\times l} \end{bmatrix} \tag{10}$$

$$\Sigma := \begin{bmatrix} -I_l & 0_{l\times l} \\ 0_{l\times l} & I_l \end{bmatrix}. \tag{11}$$

Consider two J-lossless behaviours $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}^{2l}$. Define $\mathfrak{B}_2' := \Sigma\mathfrak{B}_2$. Then the behaviour $\mathfrak{B} = \mathfrak{B}_1 \cap \mathfrak{B}_2'$ is oscillatory.

Proof. Consider a trajectory $w = \mathrm{col}(w_1, w_2) \in \mathfrak{B}$, where $w_1, w_2 \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^l)$. It is easy to see that $w \in \mathfrak{B}_1$ and $\Sigma w \in \mathfrak{B}_2$, or $\mathrm{col}(-w_1, w_2) \in \mathfrak{B}_2$. Since $\mathfrak{B}_1$ is J-lossless, there exists $E_1 \in \mathbb{R}_s^{2l\times 2l}[\zeta,\eta]$, such that $Q_{E_1} \overset{\mathfrak{B}_1}{>} 0$, and

$$Q_J(w) = w_1^\top w_2 + w_2^\top w_1 = \frac{\mathrm{d}}{\mathrm{d}t}Q_{E_1}(w). \tag{12}$$

Since $\mathfrak{B}_2$ is also J-lossless, there exists $E_2 \in \mathbb{R}_s^{2l\times 2l}[\zeta,\eta]$, such that $Q_{E_2} \overset{\mathfrak{B}_2}{>} 0$, and

$$Q_J(\Sigma w) = -w_1^\top w_2 - w_2^\top w_1 = \frac{\mathrm{d}}{\mathrm{d}t}Q_{E_2}(\Sigma w) = \frac{\mathrm{d}}{\mathrm{d}t}Q_{E_2'}(w) \tag{13}$$

where $E_2'(\zeta,\eta) := \Sigma^\top E_2(\zeta,\eta)\Sigma$. Define $E(\zeta,\eta) := E_1(\zeta,\eta) + E_2'(\zeta,\eta)$. Adding Eqs. (12) and (13), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_E(w) = 0. \tag{14}$$

Note that $\Sigma^\top = \Sigma^{-1} = \Sigma$. Let $\mathfrak{B}_2 = \ker\big(R_2(\frac{\mathrm{d}}{\mathrm{d}t})\big)$ denote a minimal kernel representation for $\mathfrak{B}_2$. Since $Q_{E_2} \overset{\mathfrak{B}_2}{>} 0$, from Proposition 4, it follows that there exist $F \in \mathbb{R}^{\bullet\times 2l}[\zeta,\eta]$ and $D_2 \in \mathbb{R}^{\bullet\times 2l}[\xi]$, such that

$$E_2(\zeta,\eta) = D_2(\zeta)^\top D_2(\eta) + F(\eta,\zeta)^\top R_2(\eta) + R_2(\zeta)^\top F(\zeta,\eta)$$

and $\mathrm{col}(D_2(\lambda), R_2(\lambda))$ has full column rank for all $\lambda \in \mathbb{C}$. It follows that

$$E_2'(\zeta,\eta) = \big(D_2(\zeta)\Sigma\big)^\top\big(D_2(\eta)\Sigma\big) + \big(F(\eta,\zeta)\Sigma\big)^\top\big(R_2(\eta)\Sigma\big) + \big(R_2(\zeta)\Sigma\big)^\top\big(F(\zeta,\eta)\Sigma\big).$$

Observe that $\mathfrak{B}_2' = \ker\big(R_2(\frac{\mathrm{d}}{\mathrm{d}t})\Sigma\big)$ is a minimal kernel representation for $\mathfrak{B}_2'$. Since Σ is nonsingular, $\mathrm{col}(D_2(\lambda)\Sigma, R_2(\lambda)\Sigma)$ has full column rank for all $\lambda \in \mathbb{C}$. From Proposition 4, it follows that $Q_{E_2'} \overset{\mathfrak{B}_2'}{>} 0$. Now observe that

$$Q_E(w) = Q_{E_1}(w) + Q_{E_2'}(w) \tag{15}$$

for any trajectory $w \in \mathfrak{B}$. Observe that

$$[Q_{E_1} \overset{\mathfrak{B}_1}{>} 0] \Longrightarrow [Q_{E_1} \overset{\mathfrak{B}}{>} 0] \quad \text{and} \quad [Q_{E_2'} \overset{\mathfrak{B}_2'}{>} 0] \Longrightarrow [Q_{E_2'} \overset{\mathfrak{B}}{>} 0].$$

This implies that for any nonzero trajectory $w \in \mathfrak{B}$, the right hand side of Eq. (15) is positive. If $Q_E(w) = 0$, then $Q_{E_1}(w) = Q_{E_2'}(w) = 0$, which imply that $w = 0$. Hence $Q_E(w) \overset{\mathfrak{B}}{>} 0$. Since Eq. (14) holds for every trajectory $w \in \mathfrak{B}$, from Theorem 11, it follows that $\mathfrak{B}$ is oscillatory. □

Remark 17. With reference to Theorem 16, let

$$\mathfrak{B}_1 = \left\{\mathrm{col}(\mathcal{I}_1, V_1) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{2l}) \mid D_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathcal{I}_1 = N_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)V_1\right\}$$

$$\mathfrak{B}_2 = \left\{\mathrm{col}(\mathcal{I}_2, V_2) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{2l}) \mid D_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathcal{I}_2 = N_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)V_2\right\}$$

be minimal kernel representations of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ where $D_1, N_1, D_2, N_2 \in \mathbb{R}^{\bullet\times l}$. Define

$$R := \begin{bmatrix} D_1 & -N_1 \\ D_2 & N_2 \end{bmatrix}.$$

Then a kernel representation for $\mathfrak{B}$ is

$$\mathfrak{B} = \left\{w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{2l}) \mid R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = 0\right\}.$$

Given below is a corollary of Theorem 16. It will be shown later that this result can be used to prove that interconnection of two lossless networks leads to an oscillatory system whenever the networks are described by scattering representations.

Corollary 18. *Let J and Σ be defined by Eqs. (10) and (11) respectively. Consider two Σ-lossless behaviours $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}^{2l}$. Define $\mathfrak{B}_2' := J\mathfrak{B}_2$. Then the behaviour $\mathfrak{B} = \mathfrak{B}_1 \cap \mathfrak{B}_2'$ is oscillatory.*

Proof. Define $\tilde{\mathfrak{B}}_i := \Xi\mathfrak{B}_i$ for $i = 1, 2$, where

$$\Xi := \frac{1}{\sqrt{2}}\begin{bmatrix} -I_l & I_l \\ I_l & I_l \end{bmatrix}.$$

Observe that $\Xi^\top = \Xi^{-1} = \Xi$, and $\Xi\Sigma\Xi = J$. From Lemma 15, it follows that $\tilde{\mathfrak{B}}_1$ and $\tilde{\mathfrak{B}}_2$ are J-lossless. Define $\tilde{\mathfrak{B}} := \Xi\mathfrak{B}$, $\tilde{\mathfrak{B}}_2' := \Xi\mathfrak{B}_2'$. It is easy to see that $\tilde{\mathfrak{B}} = \tilde{\mathfrak{B}}_1 \cap \tilde{\mathfrak{B}}_2'$. Now observe that

$$\tilde{\mathfrak{B}}_2' = \Xi\mathfrak{B}_2' = \Xi J\mathfrak{B}_2 = \Xi J\Xi\tilde{\mathfrak{B}}_2 = \Sigma\tilde{\mathfrak{B}}_2.$$

It now follows from Theorem 16 that $\tilde{\mathfrak{B}}$ is oscillatory. This implies that $\mathfrak{B}$ is also oscillatory. □

4.2. Relation with Kalman–Yakubovich–Popov lemma

In this section, it is shown that there is a relationship between the lossless version of Kalman–Yakubovich–Popov (KYP) lemma and Theorem 16. Throughout this section J and Σ are given by Eqs. (10) and (11) respectively. In the following, using KYP lemma, it is shown that certain systems with lossless positive real transfer functions (see Appendix for a definition) are J-lossless. The lossless version of KYP lemma (see [4], pp. 221–222) is given below:

Lemma 19. *Consider a multi-variable, linear, time-invariant system described by the following state-space representation:*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Bu(t) \qquad (16)$$
$$y(t) = Cx(t) + Du(t)$$

where $x(t) \in \mathbb{R}^{\mathtt{x}}$, $u(t), y(t) \in \mathbb{R}^l$ with $\mathtt{x} \geq l$. Suppose that the state-space representation (16) is minimal. Then the transfer function matrix $G(s) = C(sI_{\mathtt{x}} - A)^{-1}B + D$ corresponding to the system is lossless positive real if and only if there exists $P \in \mathbb{R}^{\mathtt{x}\times\mathtt{x}}$, such that $P = P^\top > 0$ and

$$A^\top P + PA = 0$$
$$B^\top P = C$$
$$D + D^\top = 0.$$

The following lemma gives a condition under which a behaviour with a lossless positive real transfer function is J-lossless.

Lemma 20. *Consider a behaviour $\mathfrak{B} \in \mathcal{L}^{2l}$ with an output–input partition $\mathrm{col}(y, u)\big(y, u \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^l)\big)$ and a minimal state-space representation*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + Bu \qquad (17)$$
$$y = Cx + Du.$$

Assume that B has full column rank and the transfer function matrix of $\mathfrak{B}$ given by $G(\xi) = C(\xi I_{\mathtt{x}} - A)^{-1}B + D$ is lossless positive real. Then $\mathfrak{B}$ is J-lossless.

Proof. Since G is lossless positive real, from KYP lemma, it follows that $D + D^\top = 0$ and there exists $P \in \mathbb{R}^{\mathtt{x}\times\mathtt{x}}$, such that $P = P^\top > 0$, $A^\top P + PA = 0$ and $B^\top P = C$. Define $S = x^\top Px$. Then it can be verified that

$$\frac{\mathrm{d}S}{\mathrm{d}t} = u^\top y + y^\top u.$$

Define $w := \mathrm{col}(y, u)$. Since the state-space representation (17) is minimal, the system is state-observable, or that $[w = 0] \Longrightarrow [x = 0]$. This implies that there exists a matrix $X \in \mathbb{R}^{\mathtt{x}\times l}[\xi]$, such that $x = X(\frac{\mathrm{d}}{\mathrm{d}t})w$. Define $E(\zeta, \eta) := X(\zeta)^\top PX(\eta)$ and observe that

$$S = x^\top Px = \left(X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w\right)^\top PX\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = Q_E(w).$$

Observe that $\frac{\mathrm{d}}{\mathrm{d}t}Q_E(w) = Q_J(w)$. Since B has full column rank, it follows that there exists $B_1 \in \mathbb{R}^{l\times\mathtt{x}}$, such that $B_1B = I_l$. Since P is positive definite, it follows that $[S = x^\top Px = 0] \Longrightarrow [x = 0] \Longrightarrow [u = 0, y = 0] \Longrightarrow [w = 0]$. From Definition 3, it follows that $Q_E \overset{\mathfrak{B}}{>} 0$. From Definition 12, it now follows that $\mathfrak{B}$ is J-lossless. □

In [4,10], it has been proved that G is a hybrid transfer function matrix (see Appendix D) of a multi-port linear electrical network consisting of a finite number of ideal capacitors, inductors, transformers and gyrators if and only if it is lossless positive real. Now consider such a multi-port electrical network for which $V_1 = \mathrm{col}(V_{11}, V_{12}, \ldots, V_{1l})$ denotes the vector of port voltages and $\mathcal{I}_1 = \mathrm{col}(\mathcal{I}_{11}, \mathcal{I}_{12}, \ldots, \mathcal{I}_{1l})$ denotes the vector of respective port-currents. Consider partitions of V_1 and $\mathcal{I}_1$ given by $V_1 = \mathrm{col}(\tilde{V}_{11}, \tilde{V}_{12})$ and $\mathcal{I}_1 = \mathrm{col}(\tilde{\mathcal{I}}_{11}, \tilde{\mathcal{I}}_{12})$ such that $\dim(\tilde{V}_{11}) \in \{0, 1, \ldots, l\}$, $\dim(\tilde{\mathcal{I}}_{11}) = \dim(\tilde{V}_{11})$ and the transfer function from $u_1 := \mathrm{col}(\tilde{V}_{11}, \tilde{\mathcal{I}}_{12})$ to $y_1 := \mathrm{col}(\tilde{\mathcal{I}}_{11}, \tilde{V}_{12})$ exists and is a hybrid transfer function for the network. Let $G_1 \in \mathbb{R}^{l\times l}(\xi)$ denote this transfer function matrix. Then G_1 is lossless positive real. Let $\mathfrak{B}_1$ denote the space of all admissible trajectories $\mathrm{col}(y_1, u_1)$. From KYP lemma, a minimal state-space representation for the network given by

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = A_1x_1 + B_1u_1 \qquad (18)$$
$$y_1 = C_1x_1 + D_1u_1$$

is such that there exists $P_1 \in \mathbb{R}^{\mathtt{x}_1\times\mathtt{x}_1}$ with $P_1 = P_1^\top > 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_1^\top P_1x_1) = u_1^\top y_1 + y_1^\top u_1 = \tilde{V}_{11}^\top\tilde{\mathcal{I}}_{11} + \tilde{\mathcal{I}}_{11}^\top\tilde{V}_{11} + \tilde{V}_{12}^\top\tilde{\mathcal{I}}_{12} + \tilde{\mathcal{I}}_{12}^\top\tilde{V}_{12}$$
$$= V_1^\top\mathcal{I}_1 + \mathcal{I}_1^\top V_1.$$

Define $l_1 := \dim(\tilde{V}_{11})$. Define

$$P_1 := \begin{bmatrix} I_{l_1} & 0_{l_1\times(l-l_1)} & 0_{l_1\times l_1} & 0_{l_1\times(l-l_1)} \\ 0_{(l-l_1)\times l_1} & 0_{(l-l_1)\times(l-l_1)} & 0_{(l-l_1)\times l_1} & I_{l-l_1} \\ 0_{l_1\times l_1} & 0_{l_1\times(l-l_1)} & I_{l_1} & 0_{l_1\times(l-l_1)} \\ 0_{(l-l_1)\times l_1} & I_{l-l_1} & 0_{(l-l_1)\times l_1} & 0_{(l-l_1)\times(l-l_1)} \end{bmatrix}.$$

Define $H_1 := \begin{bmatrix} I_l & 0_l \end{bmatrix}P_1$ and $K_1 := \begin{bmatrix} 0_l & I_l \end{bmatrix}P_1$, $w := \mathrm{col}(\mathcal{I}_1, V_1)$ and observe that $y_1 = H_1w$ and $u_1 = K_1w$. Eq. (18) can thus be written as

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = A_1x_1 + B_1K_1w \qquad (19)$$
$$H_1w = C_1x_1 + D_1K_1w.$$

Now consider another multi-port linear electrical network consisting of a finite number of capacitors, inductors, transformers and gyrators for which $V_2 = \mathrm{col}(V_{21}, V_{22}, \ldots, V_{2l})$ denotes the vector of port voltages and $\mathcal{I}_2 = \mathrm{col}(\mathcal{I}_{21}, \mathcal{I}_{22}, \ldots, \mathcal{I}_{2l})$ denotes the vector of respective port-currents. Consider partitions of V_2 and $\mathcal{I}_2$ given by $V_2 = \mathrm{col}(\tilde{V}_{21}, \tilde{V}_{22})$ and $\mathcal{I}_2 = \mathrm{col}(\tilde{\mathcal{I}}_{21}, \tilde{\mathcal{I}}_{22})$ such that

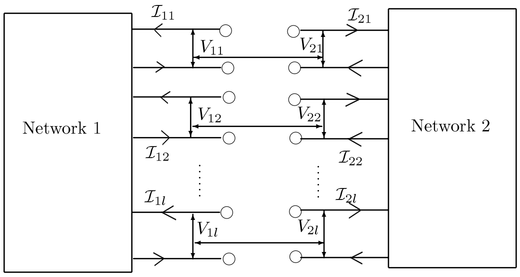

Fig. 2. Interconnection of multi-port electrical networks.

$\dim(\tilde{V}_{21}) \in \{0, 1, \ldots, l\}$, $\dim(\tilde{\mathcal{I}}_{21}) = \dim(\tilde{V}_{21})$ and the transfer function from $u_2 := \mathrm{col}(\tilde{V}_{21}, \tilde{\mathcal{I}}_{22})$ to $y_2 := \mathrm{col}(\tilde{\mathcal{I}}_{21}, \tilde{V}_{22})$ exists and is a hybrid transfer function for the network. Let $G_2 \in \mathbb{R}^{l\times l}(\xi)$ denote this transfer function matrix. Then G_2 is lossless positive real. Let $\mathfrak{B}_2$ denote the space of all admissible trajectories $\mathrm{col}(y_2, u_2)$. Define $l_2 := \dim(\tilde{V}_{21})$. Define

$$P_2 := \begin{bmatrix} I_{l_2} & 0_{l_2\times(l-l_2)} & 0_{l_2\times l_2} & 0_{l_2\times(l-l_2)} \\ 0_{(l-l_2)\times l_2} & 0_{(l-l_2)\times(l-l_2)} & 0_{(l-l_2)\times l_2} & I_{l-l_2} \\ 0_{l_2\times l_2} & 0_{l_2\times(l-l_2)} & I_{l_2} & 0_{l_2\times(l-l_2)} \\ 0_{(l-l_2)\times l_2} & I_{l-l_2} & 0_{(l-l_2)\times l_2} & 0_{(l-l_2)\times(l-l_2)} \end{bmatrix}.$$

Define $H_2 := \begin{bmatrix} I_l & 0_l \end{bmatrix} P_2$and $K_2 := \begin{bmatrix} 0_l & I_l \end{bmatrix} P_2$and observe that $y_2 = H_1\mathrm{col}(\mathcal{I}_2, V_2)$ and $u_2 = K_2\mathrm{col}(\mathcal{I}_2, V_2)$. Let a minimal state-space representation for the network be given by

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = A_2x_2 + B_2u_2 \qquad (20)$$
$$y_2 = C_2x_2 + D_2u_2.$$

Now consider the interconnection of the two networks as depicted in Fig. 2. From Kirchhoff's voltage and current laws, we obtain $V_1 = V_2$ and $\mathcal{I}_1 = -\mathcal{I}_2$. Eq. (20) can then be written as

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = A_2x_2 + B_2K_2\Sigma w \qquad (21)$$
$$H_2\Sigma w = C_2x_2 + D_2K_2\Sigma w.$$

From KYP lemma, it follows that there exists $P_2 \in \mathbb{R}^{x_2\times x_2}$ such that $P_2 = P_2^\top > 0$ and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(x_2^\top P_2x_2) &= u_2^\top y_2 + y_2^\top u_2 \\ &= \tilde{V}_{21}^\top\tilde{\mathcal{I}}_{21} + \tilde{\mathcal{I}}_{21}^\top\tilde{V}_{21} + \tilde{V}_{22}^\top\tilde{\mathcal{I}}_{22} + \tilde{\mathcal{I}}_{22}^\top\tilde{V}_{22} \\ &= V_2^\top\mathcal{I}_2 + \mathcal{I}_2^\top V_2 = -V_1^\top\mathcal{I}_1 - \mathcal{I}_1^\top V_1. \end{aligned}$$

Define

$$\mathfrak{B}_x := \{\mathrm{col}(x_1, x_2) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{x_1+x_2}) \mid \exists\, w \text{ such that (19) and (21) hold}\}.$$

Define $E := \mathrm{diag}(P_1, P_2)$ and observe that $Q_E \overset{\mathfrak{B}_x}{>} 0$ since P_1 and P_2 are positive definite. Also observe that $\frac{\mathrm{d}}{\mathrm{d}t}Q_E(x) = 0$ for all $x \in \mathfrak{B}_x$. Hence from Theorem 11, it follows that $\mathfrak{B}_x$ is oscillatory, i.e the interconnection of two multi-port lossless networks leads to an oscillatory behaviour.

Now assume that B_1 and B_2 in Eqs. (18) and (20) respectively have full column rank. From Lemma 20, it follows that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are J-lossless. If $\mathfrak{B}$ denotes the space of all admissible trajectories $\mathrm{col}(\mathcal{I}_1, V_1)$ corresponding to the interconnected network, then it is easy to see that $\mathfrak{B} = \left(P_1^\top\mathfrak{B}_1\right) \cap \left(\Sigma P_2^\top\mathfrak{B}_2\right)$. Define $\mathfrak{B}_1' := P_1^\top\mathfrak{B}_1$ and $\mathfrak{B}_2' := P_2^\top\mathfrak{B}_2$. It can be verified that $P_1^\top = P_1^{-1} = P_1$, $P_2^\top = P_2^{-1} = P_2$ and $P_1^\top JP_1 = P_2^\top JP_2 = J$. Consequently from Lemma 15, it follows that $\mathfrak{B}_1'$ and $\mathfrak{B}_2'$ are J-lossless. Since $\mathfrak{B} = \mathfrak{B}_1' \cap (\Sigma\mathfrak{B}_2')$, from Theorem 16, it now follows that $\mathfrak{B}$ is oscillatory.

Remark 21. The significance of Corollary 18 is now explained. Consider a multi-port linear electrical network consisting of a finite number of ideal capacitors, inductors, transformers and gyrators with a given *scattering description* (see pp. 30–31 of [4]). Let V_1 and $\mathcal{I}_1$ denote the vectors of port voltages and currents respectively with equal dimension l. In a scattering description of the network, the external variables are the *incident voltage* $V_1^i := \frac{1}{2}(V_1 + \mathcal{I}_1)$ and the *reflected voltage* $V_1^r = \frac{1}{2}(V_1 - \mathcal{I}_1)$ and the transfer function $S \in \mathbb{R}^{l\times l}(\xi)$ from V_1^i to V_1^r is called the *scattering matrix* of the network, i.e if the governing law of the network is $P(\frac{\mathrm{d}}{\mathrm{d}t})V_1^i = Q(\frac{\mathrm{d}}{\mathrm{d}t})V_1^r$, then the scattering matrix is $S := PQ^{-1}$. Let $\mathfrak{B}_1$ be the space of all possible trajectories $\mathrm{col}(\sqrt{2}V_1^r, \sqrt{2}V_1^i)$ corresponding to the given network. Define

$$\Xi := \frac{1}{\sqrt{2}}\begin{bmatrix} -I_l & I_l \\ I_l & I_l \end{bmatrix}$$

and $\tilde{\mathfrak{B}}_1 := \Xi\mathfrak{B}_1$. It is easy to see that $\tilde{\mathfrak{B}}_1$ consists of all admissible trajectories $\mathrm{col}(\mathcal{I}_1, V_1)$ corresponding to the network. Assume that the given network is such that $\tilde{\mathfrak{B}}_1$ is J-lossless. Observe that $\Xi = \Xi^{-1} = \Xi^\top$ and $\Xi J\Xi = \Sigma$. From Lemma 15, it follows that $\mathfrak{B}_1$ is Σ-lossless.

Now consider another network for which V_2^r and V_2^i denote the reflected and incident voltages. Assume that like in the case of the earlier network, the space $\mathfrak{B}_2$ of all admissible trajectories $\mathrm{col}(\sqrt{2}V_2^r, \sqrt{2}V_2^i)$ is Σ-lossless. Now consider the interconnection of the two networks as depicted in Fig. 2, i.e interconnection in such a way that $V_1^r = V_2^i$ and $V_1^i = V_2^r$. If $\mathfrak{B}$ denotes the space of all admissible trajectories $\mathrm{col}(\sqrt{2}V_1^r, \sqrt{2}V_1^i)$ corresponding to the interconnected network, then it is easy to see that $\mathfrak{B} = \mathfrak{B}_1 \cap (J\mathfrak{B}_2)$. From Corollary 18, it follows that $\mathfrak{B}$ is oscillatory.

4.3. Interconnection of one-port lossless electrical networks

Consider a lossless one-port electrical network, for which the system equation is

$$n\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)V = d\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathcal{I} \qquad (22)$$

where V and $\mathcal{I}$ denote the voltage across the port and the current through the network respectively. Define $\mathfrak{B}$ as the set of all admissible trajectories $\mathrm{col}(\mathcal{I}, V) : \mathbb{R} \to \mathbb{R}^2$ that obey Eq. (22). From the theory of electrical networks, it follows that Z defined by $Z := \frac{n}{d}$ is lossless positive real. Assume that $\mathfrak{B}$ is controllable. KYP lemma can now be used in order to prove that $\mathfrak{B}$ is J-lossless where

$$J := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Consider first the case, where $\deg(n) < \deg(d)$. Since $\frac{n}{d}$ is lossless positive real, observe that a minimal state-space representation of $\mathfrak{B}$ given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + BV$$
$$\mathcal{I} = Cx + DV$$

will be such that $D = 0$, $B \in \mathbb{R}^{x}$ and $B \neq 0$, because from KYP Lemma, it follows that $[B = 0] \Longrightarrow [C = 0] \Longrightarrow [\mathcal{I} = 0]$, which is not true. From Lemma 20, it follows that $\mathfrak{B}$ is J-lossless. In a similar way, it can be proved that $\mathfrak{B}$ is J-lossless also if $\deg(n) > \deg(d)$.

J-losslessness of $\mathfrak{B}$ can also be proved by another method which uses properties of Hurwitz polynomials and of positive QDFs. This method has the advantage that it does not refer to state-space realizations. This method is now given. From the controllability of $\mathfrak{B}$, it follows that n and d are co-prime. Since $\frac{n}{d}$ is lossless positive

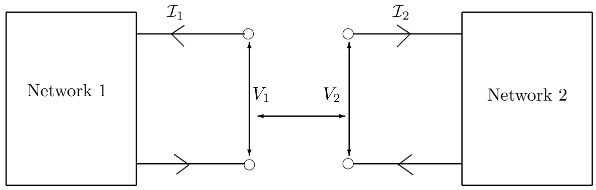

Fig. 3. Interconnection of one-port electrical networks.

real, from the material in Appendix C, it follows that both d and n are oscillatory, one of them is even and the other is odd and the purely imaginary roots of one are interlaced between those of the other. From Theorem 24, Appendix, it follows that $n+d$ is Hurwitz. It is easy to see that $\mathfrak{B} = \mathrm{Im}\big(M(\frac{d}{dt})\big)$, where $M := \mathrm{col}(n, d)$. Define

$$\Phi(\zeta,\eta) := \frac{n(\zeta)d(\eta)+d(\zeta)n(\eta)}{\zeta+\eta}.$$

From Theorem 23, Appendix, it follows that $Q_\Phi > 0$. Consequently from Lemma 14, it follows that $\mathfrak{B}$ is J-lossless.

Now consider the interconnection of two lossless one-port electrical networks as depicted in Fig. 3. Let the system equations for the two networks be given by

$$n_1\left(\frac{d}{dt}\right)V_1 = d_1\left(\frac{d}{dt}\right)\mathcal{I}_1$$
$$n_2\left(\frac{d}{dt}\right)V_2 = d_2\left(\frac{d}{dt}\right)\mathcal{I}_2.$$

Assume that each of the sets $\{n_1, d_1\}$ and $\{n_2, d_2\}$ consists of coprime polynomials. When the two networks are interconnected as depicted in the figure, from Kirchhoff's voltage and current laws, we obtain $V_1 = V_2$ and $\mathcal{I}_1 = -\mathcal{I}_2$. From the discussion of Section 4.2, it follows that the behaviour $\mathfrak{B}$ consisting of all admissible trajectories $\mathrm{col}(\mathcal{I}_1, V_1)$ corresponding to the interconnected network is oscillatory. This result can also be proved by making use of properties of Hurwitz polynomials and of autonomous lossless behaviours as follows. Observe that the characteristic equation for the resulting autonomous system is $r := n_1d_2+n_2d_1$. It is now proved that r is oscillatory which implies that the resulting autonomous system is lossless.

It is known that (n_1+d_1) and (n_2+d_2) are both Hurwitz. Hence their product

$$p = (n_1+d_1)(n_2+d_2)$$
$$= (n_1n_2+d_1d_2)+(n_1d_2+n_2d_1)$$

is also Hurwitz. Consider four cases.

- Case 1: n_1 and n_2 are even and d_1 and d_2 are odd. In this case, r is the odd part of p and hence from Theorem 24, it is oscillatory.
- Case 2: n_1 and d_2 are even and n_2 and d_1 are odd. In this case, r is the even part of p and hence from Theorem 24, it is oscillatory.
- Case 3: d_1 and n_2 are even and n_1 and d_2 are odd. In this case, r is the even part of p and hence from Theorem 24, it is oscillatory.
- Case 4: d_1 and d_2 are even and n_1 and n_2 are odd. In this case, r is the odd part of p and hence from Theorem 24, it is oscillatory.

This proves that the interconnection of two lossless one-port networks of the type depicted in Fig. 3 always results in a lossless autonomous system.

The problem of decomposition of an oscillatory behaviour with a given characteristic polynomial as an interconnection of two SISO behaviours, such that one has a lossless positive real transfer function and the other has a lossless negative real transfer function is now considered and an algorithm is provided for the same. Note that this problem can be considered as an inverse problem to the one where an autonomous behaviour which is an interconnection of two SISO behaviours such that one has a positive real transfer function and the other has a negative real transfer function is analysed. The solution to this problem also provides ways of decomposing an autonomous electrical lossless circuit with a given characteristic polynomial as an interconnection of two one-port lossless electrical circuits.

Algorithm 22. *Data*: An oscillatory even polynomial $r \in \mathbb{R}[\xi]$ of degree 2m.

Output: Two J-lossless behaviours $\mathfrak{B}_1$ and $\mathfrak{B}_2$, such that $\mathfrak{B} = \mathfrak{B}_1 \cap \mathfrak{B}_2'$ has its characteristic polynomial equal to r, where

$$J := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$\mathfrak{B}_2' := \Sigma\mathfrak{B}_2$, with

$$\Sigma := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Step 1 Either choose an odd polynomial $r_1 \in \mathbb{R}[\xi]$ of degree $2m+1$ in such way that the roots of r are interlaced between those of r_1 or choose an odd polynomial of degree $2m-1$ in such a way that the roots of r_1 are interlaced between those of r.

Step 2 Factorize the polynomial $r + r_1$ into two factors $p, q \in \mathbb{R}[\xi]$, i.e $r + r_1 = pq$. Let p_e and p_o be the even and odd parts of p and let q_e and q_o be the even and odd parts of q.

Step 3 Output:

$$\mathfrak{B}_1 = \ker\left[p_e\left(\frac{d}{dt}\right) \quad -p_o\left(\frac{d}{dt}\right)\right]$$
$$\mathfrak{B}_2 = \ker\left[-q_o\left(\frac{d}{dt}\right) \quad q_e\left(\frac{d}{dt}\right)\right].$$

With reference to the above algorithm, observe that

$$r + r_1 = (p_eq_e + p_oq_o) + (p_eq_o + p_oq_e).$$

Define $s_1 := p_eq_e + p_oq_o$ and $s_2 := p_eq_o + p_oq_e$. Then it is easy to see that s_1 is even and s_2 is odd, and hence $s_1 = r$ which is the characteristic polynomial of $\mathfrak{B}_1 \cap \mathfrak{B}_2'$. Since r and r_1 obey interlacing property, from Theorem 24 it follows that $r+r_1$ is Hurwitz, and consequently both p and q are Hurwitz. Conclude from Theorem 24 that both the pairs (p_e, p_o) and (q_e, q_o) obey interlacing property. Define $M_1 := \mathrm{col}(p_o, p_e)$ and $M_2 := \mathrm{col}(q_e, q_o)$. It is easy to see that $\mathfrak{B}_1 = \mathrm{Im}\big(M_1(\frac{d}{dt})\big)$, and $\mathfrak{B}_2 = \mathrm{Im}\big(M_2(\frac{d}{dt})\big)$. Define

$$\Phi_1(\zeta,\eta) := \frac{p_e(\zeta)p_o(\eta)+p_o(\zeta)p_e(\eta)}{\zeta+\eta}$$
$$\Phi_2(\zeta,\eta) := \frac{q_e(\zeta)q_o(\eta)+q_o(\zeta)q_e(\eta)}{\zeta+\eta}.$$

From Theorem 23, it follows that Q_{Φ_1} and Q_{Φ_2} are both positive. Consequently by definition, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are J-lossless. This proves the correctness of Algorithm 22. Observe also that the transfer functions p_o/p_e and $-q_e/q_o$ corresponding to the behaviours $\mathfrak{B}_1$ and $\mathfrak{B}_2'$ are lossless positive real and lossless negative real respectively.

Observe that in step 1 of Algorithm 22, there are infinite number of ways of choosing r_1 such that $\frac{r}{r_1}$ is lossless positive real. For each of these ways, in step 2, there are a finite number of ways of factorizing $(r + r_1)$. Hence, there are infinite number of ways of choosing two behaviours with lossless positive real and lossless negative real transfer functions respectively, such that their intersection has for its characteristic polynomial, a given oscillatory polynomial.

5. Conclusion

The main result of this paper is Theorem 16 where it is proved that the interconnection of two J-lossless behaviours leads to an oscillatory behaviour. The properties J-lossless behaviours and of autonomous lossless behaviours have been used to prove this theorem in the multi-variable case. In the case of interconnection of SISO lossless behaviours, it has been shown in Section 4.3 that Theorem 16 can also be proved by making use of properties of Hurwitz polynomials and of positive QDFs. The relationship of Theorem 16 with KYP lemma has also been described in this paper. The algebra of two-variable polynomial matrices has been used throughout as a tool in proving many of the results in this paper.

Appendix A. Interlacing property and positive quadratic differential forms

Given below is a theorem that relates positivity of a quadratic differential form to an interesting property relevant to this paper, known as the *interlacing property*.

Theorem 23. *Let $r_1 \in \mathbb{R}[\xi]$ be given by $r_1(\xi) = (\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\dots(\xi^2+\omega_{n-1}^2)$, where $\omega_0 < \omega_1 \dots < \omega_{n-1} \in \mathbb{R}^+$ and n is a positive integer. Define $r'(\xi) := (\xi+\omega_0^2)(\xi+\omega_1^2)\dots(\xi+\omega_{n-1}^2)$; $r_2(\xi) := \xi r_1(\xi)$ and $\check{r}(\xi) := \xi r'(\xi)$. Then the following hold:*

1. *Let $f_1(\xi)$ be a polynomial of degree less than or equal to $n-1$. Define*
$$\phi_1(\zeta,\eta) := \frac{\eta r'(\zeta^2)f_1(\eta^2)+\zeta r'(\eta^2)f_1(\zeta^2)}{\zeta+\eta}.$$
Then $Q_{\phi_1} > 0$ if and only if $f_1(-\omega_0^2) > 0$ and the roots of f_1 are interlaced between those of r', i.e along the real axis, exactly one root of f_1 occurs between any two consecutive roots of r'.
2. *Let $f_2(\xi)$ be a polynomial of degree less than or equal to n. Define*
$$\phi_2(\zeta,\eta) := \frac{\zeta r'(\zeta^2)f_2(\eta^2)+\eta r'(\eta^2)f_2(\zeta^2)}{\zeta+\eta}.$$
Then $Q_{\phi_2} > 0$ if and only if $f_2(0) > 0$ and the roots of f_2 are interlaced between those of $\check{r}$.

Proof. See proof of Theorem 2, p. 1527, [8]. □

Appendix B. Hurwitz polynomial

A Hurwitz polynomial is a polynomial with all its roots in the open left half of the complex plane. Given below is an interesting property of a Hurwitz polynomial which is used in this paper.

Theorem 24. *Consider a polynomial $p(\xi) = p'(\xi^2)+\xi p''(\xi^2)$, where $p', p'' \in \mathbb{R}[\xi]$. Assume that the leading coefficient of p is positive. Define*

$$\omega := \begin{cases} \text{the root of } p' \text{ with the smallest absolute value if } \deg(p') \geq 1,\\ 0 \text{ if } \deg(p') = 0.\end{cases}$$

Define $p_1(\xi) := \xi p''(\xi)$. p is Hurwitz iff either one of the following holds:

- $\deg(p') > \deg(p'') \geq 1$; *$p'$ and p'' have distinct roots on the negative real axis; the roots of p'' are interlaced between those of p' and $p''(\omega) > 0$.*
- $\deg(p') = \deg(p'') \geq 1$; *$p'$ and p'' have distinct roots on the negative real axis; the roots of p' are interlaced between those of p_1 and $p'(0) > 0$.*
- $\deg(p'') = 0$, $\deg(p') = 1$, *p' has a root on the negative real axis and $p''(\omega) > 0$.*
- $\deg(p'') = \deg(p') = 0$ *and* $p'(0) > 0$.

Proof. See proof of Theorem 1, p. 107 of [11]. □

Given below are the definitions for the even and odd parts of a given polynomial.

Definition 25. The *even part* $p_e \in \mathbb{R}[\xi]$ and *odd part* $p_o \in \mathbb{R}[\xi]$ of a given polynomial $p \in \mathbb{R}[\xi]$ are defined as

$$p_e(\xi) := \frac{p(\xi)+p(-\xi)}{2}$$
$$p_o(\xi) := \frac{p(\xi)-p(-\xi)}{2}.$$

If $p_e, p_o \in \mathbb{R}[\xi]$ respectively denote the even and odd parts of a given Hurwitz polynomial $p \in \mathbb{R}[\xi]$, then with reference to Theorem 24, observe that

$$p_e(\xi) = p'(\xi^2)$$
$$p_o(\xi) = \xi p''(\xi^2).$$

From Theorem 24, it follows that $Z := p_e/p_o$ has one of the following two forms:

$$Z(\xi) = \frac{H(\xi^2+\omega_1^2)(\xi^2+\omega_3^2)\dots(\xi^2+\omega_{2m-1}^2)}{\xi(\xi^2+\omega_2^2)(\xi^2+\omega_4^2)\dots(\xi^2+\omega_{2m-2}^2)} \quad (23)$$

or

$$Z(\xi) = \frac{H(\xi^2+\omega_1^2)(\xi^2+\omega_3^2)\dots(\xi^2+\omega_{2m-1}^2)}{\xi(\xi^2+\omega_2^2)(\xi^2+\omega_4^2)\dots(\xi^2+\omega_{2m}^2)} \quad (24)$$

where $H \in \mathbb{R}^+$, $m \in \mathbb{N}$ and

$0 < \omega_1 < \omega_2 < \omega_3 < \omega_4 < \cdots$.

Appendix C. Positive real transfer functions

Definition 26. A rational matrix $B \in \mathbb{R}^{u\times u}(\xi)$ is called *positive real* if the following conditions hold

1. All elements of B are analytic in the open right half plane.
2. $B^*(\lambda)+B(\lambda) \geq 0$ for $\mathrm{Re}(\lambda) > 0$.

Definition 27. A rational matrix $B \in \mathbb{R}^{u\times u}(\xi)$ is called *lossless positive real* if the following conditions hold

1. B is positive real.
2. $B^*(j\omega)+B(j\omega) = 0$ for all $\omega \in \mathbb{R}$, with $j\omega$ not a pole of any element of B.

A matrix $A \in \mathbb{R}^{u\times u}(\xi)$ is called *lossless negative real* if $-A$ is lossless positive real. If $A \in \mathbb{R}(\xi)$ is lossless positive real, then it is shown in [12], pp. 49–50 that A has one of the four forms of which two are shown in the right hand side of Eqs. (23) and (24) and two other forms are their reciprocals.

Appendix D. Hybrid transfer function matrix of a multi-port electrical network

The following material is from [4]. Consider a multi-port electrical network consisting of a finite number of ideal resistors, capacitors, inductors, transformers and gyrators for which V denotes the vector of port voltages and $\mathcal{I}$ denotes the vector of respective port-currents, both having dimension equal to l. Consider partitions of V and $\mathcal{I}$ given by $V = \mathrm{col}(V_1, V_2)$ and $\mathcal{I} = \mathrm{col}(\mathcal{I}_1, \mathcal{I}_2)$ such that $\dim(V_1) \in \{0, 1, \dots, l\}$ and $\dim(\mathcal{I}_1) = \dim(V_1)$. Define $u := \mathrm{col}(V_1, \mathcal{I}_2)$ and $y := \mathrm{col}(\mathcal{I}_1, V_2)$. Assume that the transfer function G from u to y exists. G is called a *hybrid transfer function matrix* of the network if $\lim_{\xi\to\infty} G(\xi) < \infty$. In [13], it has been proved that there exists at least one hybrid transfer function matrix for a given multi-port network.

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