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The Characterization of Reaction-Convection-Diffusion Processes by Travelling Waves

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It has long been known that the heat equation displays infinite speed of propagation. This is to say that if the initial data are nonnegative and have nonempty compact support, the solution of an initial-value problem is positive everywhere after any infinitesimal time. However, since the nineteen-fifties it has also been known that certain nonlinear diffusion equations of degenerate parabolic type do not display this phenomenon. For these equations, the (generalized) solution of an initial-value problem with compactly-supported initial data will have bounded support with respect to the spatial variable at all times. In this paper the necessary and sufficient criterion for finite speed of propagation for the general nonlinear reaction-convection-diffusion equation

$$u_t = (a(u))_{xx} + (b(u))_x + c(u)$$

is determined. The assumptions on the coefficients a, b and c are such that the classification unifies and generalizes previously-known results. The technique employed is comparision of an arbitrary solution of the equation with suitably-constructed travelling-wave solutions and subsolutions. Basically the central conclusion is that the equation exhibits finite speed of propagation if and only if it admits a travelling-wave solution with bounded support. Concurrently, the search for a travelling-wave solution with bounded support can be reduced to the study of a singular nonlinear integral equation whose solution must satisfy a certain constraint.

INTRODUCTION

Special solutions play an important role in the study of nonlinear partial differential equations.

Confronted with a mathematical model in the form of an initial or a boundary value problem for a partial differential equation or system of

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such equations, the foremost desire of an engineer, physicist or other practitioner is to solve the problem explicitly. If little theory is available and no explicit solution is readily obtainable, generally the ensuing line of attack is to identify circumstances under which the complexity of the problem may be reduced. In this respect, similarity analysis is a pre-eminent tool for reducing the number of variables involved [4, 5, 14–16, 18, 24].

At first sight, the main goal of identifying self-similar solutions is merely to reduce the original problem to a set of equations and initial or boundary conditions which is easier to analyse. Nevertheless, it is currently recognized that self-similar solutions are important in describing the intermediate asymptotic behavior of classes of solutions of the original problem with arbitrary initial and boundary conditions. Furthermore, their analysis can consequently be of significance for the design of adequate numerical computational schemes for the problem in hand [4, 5, 14, 15, 17, 18, 143].

Most remarkable is that particular self-similar solutions have proved to play a vital role in the development of mathematical theories for nonlinear partial differential equations [3–5, 34, 47, 48, 71, 73, 94, 96, 104, 117–119, 124, 135]. As most noteworthy illustrations, we mention the place of solitons in the theory of the Korteweg-de Vries equation [45, 46, 67, 106], the use of travelling-wave solutions for the Fisher and the KPP equations [8, 12, 44, 52–56, 105, 111], and the role of the instantaneous point-source solution commonly referred to as the Barenblatt-Pattle solution in the theory of the porous media equation [10, 11, 36, 88, 94, 95, 123, 138–140, 142].

In the present paper, we reinforce this vital role with respect to travelling waves. Roughly speaking we are going to show that the occurrence of a free boundary (front or interface) in solutions of a nonlinear reaction-convection-diffusion equation is equivalent to the admission of a particular type of travelling-wave solution.

In the next section we state precisely the question we consider. We indicate the nonlinear reaction-convection-diffusion processes involved and the free boundary whose occurrence we shall characterize. Thereafter, we review the earlier work on the question, and state our main results. The subsequent section is devoted to the study of a singular nonlinear integral equation of Volterra type whose solution is tantamount to the admission of a travelling-wave solution possessing the sought-after interface. Using a simple regularization technique and the concept of a maximal solution, difficulties associated with the singularity of this integral equation can be avoided. The ensuing section discusses the connection between the integral equation and the travelling waves, and contains the proof of our main results. In the last section we relate our results to earlier work on the topic and discuss their application to a number of specific nonlinear diffusion-convection-reaction equations. An announcement of the principal results under less general conditions than those considered in the present paper was published in [77].

1. The Central Question and Main Results

1.1. *Finite Speed of Propagation*

We consider the nonlinear equation

$$u_t = (a(u))_{xx} + (b(u))_x + c(u) \tag{1.1}$$

in which subscripts denote partial differentiation. About the coefficients in this equation we assume the following.

Hypothesis 1. The functions $a(s)$, $b(s)$ and $c(s)$ are defined and real for $0 \leqslant s < \infty$. Moreover, a is continuous and strictly increasing on $[0, \infty)$, b is continuous on $[0, \infty)$, c is integrable with respect to a in the sense of Lebesgue-Stieltjes on every compact subset of $[0, \infty)$, and

$$a(0) = b(0) = c(0) = 0. \tag{1.2}$$

Equation (1.1) models heat transfer in a medium where the thermal conductivity, convective transport, and sources or sinks of thermal energy may depend on the temperature but not on the place or time [4, 5, 38, 109, 113, 146, 147]. The equation also arises in various guises in numerous other fields [4, 5, 37, 38, 50, 113]—soil physics [19, 33, 74, 128, 137], population genetics [44, 53, 56, 83, 84, 110, 111, 114], fluid dynamics [25, 28–30], neurology [53, 112, 132], combustion theory [20, 26, 27, 144] and reaction chemistry [7, 8, 53], to name but a few. In these situations, the second-order term on the right-hand side of (1.1) describes a diffusive process, the first-order term corresponds to a convective or advective process, and the zero-order term is associated with reaction, sorption, sources or sinks.

Let D denote a domain in $\mathbb{R}^2$ of the form

$$D = (\eta_1, \eta_2) \times (\tau_1, \tau_2] \tag{1.3}$$

with

$$-\infty \leqslant \eta_1 < \eta_2 \leqslant \infty \qquad \text{and} \qquad -\infty < \tau_1 < \tau_2 < \infty. \tag{1.4}$$

For any $\iota > 0$ define

$$c_\iota(s) := \begin{cases} c(s) & \text{for} \quad s > \iota \\ c(0) & \text{for} \quad s \leqslant \iota. \end{cases}$$

DEFINITION 1. A function $u(x, t)$ is said to be a generalized subsolution of Eq. (1.1) in D if it is defined, real, nonnegative and continuous in $\bar{D}$, and given any bounded rectangle of the form

$$R := (x_1, x_2) \times (t_1, t_2] \subseteq D$$

and any nonnegative function $\phi \in C^{2,1}(\bar{R})$ such that

$$\phi(x_1, t) = \phi(x_2, t) = 0 \qquad \text{for all} \quad t \in [t_1, t_2] \tag{1.5}$$

there holds

$$c_\iota(u)\, \phi \in L^1(R) \qquad \text{for all} \quad \iota > 0, \tag{1.6}$$

$$\iint_R c(u)\, \phi\, dx\, dt := \lim_{\iota \downarrow 0} \iint_R c_\iota(u)\, \phi\, dx\, dt \qquad \text{exists and is finite,} \tag{1.7}$$

and

$$\begin{aligned} &\iint_R [a(u)\, \phi_{xx} - b(u)\, \phi_x + c(u)\, \phi + u\phi_t]\, dx\, dt \\ &\quad \geqslant \int_{x_1}^{x_2} [u(x, t_2)\, \phi(x, t_2) - u(x, t_1)\, \phi(x, t_1)]\, dx \\ &\qquad + \int_{t_1}^{t_2} [a(u(x_2, t))\, \phi_x(x_2, t) - a(u(x_1, t))\, \phi_x(x_1, t)]\, dt. \end{aligned} \tag{1.8}$$

A function $u(x, t)$ is defined to be a generalized solution of Eq. (1.1) in D if it satisfies the previous requirements with equality in (1.8). Any generalized subsolution which is not a generalized solution is referred to as a generalized strict subsolution.

With the above notion of a generalized solution and more restricted notions of a solution, various existence and uniqueness theorems for the Cauchy problem, the Cauchy-Dirichlet problem, the Cauchy-Neumann problem and mixed-type boundary value problems for Eq. (1.1) have been proven [6, 9, 10, 36, 41, 57, 69, 72, 85, 91, 92, 94, 98, 101, 115, 117–119, 123, 136, 148].

We shall consider a generalized solution u of Eq. (1.1) in the half-strip

$$H := (0, \infty) \times (0, T] \qquad \text{with} \quad 0 < T < \infty$$

and define

$$P[t] := \{x \in (0, \infty) : u(x, t) > 0\} \tag{1.9}$$

and

$$\zeta(t) := \sup\{x \in (0, \infty) : u(x, t) > 0\} \tag{1.10}$$

for $t \in [0, T]$. We address the following question. *Suppose that the initial data function* $u_0(x) := u(x, 0)$ *is such that* $P[0]$ *is bounded and nonempty, i.e.,*

$$0 < \zeta(0) < \infty.$$

Then under what conditions is

$$\sup\{\zeta(t) : 0 \leqslant t \leqslant \tau\} < \infty \qquad \textit{for some} \quad \tau \in (0, T]?$$

DEFINITION 2 [10, 71, 89, 91, 93, 94, 102, 122, 142]. When conditions are such that the above question can be answered affirmatively, Eq. (1.1) is said to display finite speed of propagation of perturbations.

When (1.1) admits finite speed of propagation, ζ is a free boundary demarcating the support of the generalized solution u of (1.1). The occurrence of such an interface is of considerable interest with regard to the phenomena modelled by the equation. For instance, when the equation models the spreading of a biological population such an interface denotes the extent of migration [53, 83, 84, 110, 111, 114], when the equation models chemical kinetics it denotes the boundary of the reaction zone [7, 8, 53], when the equation models thin viscous fluid flow over a plate it constitutes a leading edge for the fluid flow [25], whilst, when the equation models unsaturated soil-moisture flow it denotes a wetting-front [19, 74]. In those circumstances in which the question addressed cannot be answered affirmitively, the equation is said to exhibit infinite velocity of propagation of disturbances [10, 71, 94, 102, 122, 142].

1.2. *Previous Work*

In his treatise on the theory of heat published in 1835, Poisson [126] observed that the heat equation

$$u_t = u_{xx}$$

propagates disturbances with infinite speed. This he deduced from the integral formula for solutions of the Cauchy problem which today bears his name. To quote: "Supposons que le barre n'a été échauffée primitivement que dans une portion limitée qui s'entendait depuis $x = -\varepsilon$ jusqu'à $x = \varepsilon$, de sorte qu'en dehors de ces limites sa température initiale *fx* était zéro, comme la température extérieure..... Cette expression (the Poisson formula) de u nous montre que la chaleur communiquée à une portion de la barre se répand instantanément dans toute sa longueur; car, quelque grande que soit la distance x, et quelque petit que soit le temps t, il y aura toujours une valeur de u qui ne sera pas rigoureusement nulle. Ce résultat tient à ce qu'en formant l'équation du mouvement de la chaleur, nous avons supposé instantanés les échanges de chaleur entre les tranches de la barre comprises dans l'étendue du rayonnement intérieur. Or, quelque rapides que soient ces échanges, ils ne peuvent avoir lieu dans la nature qu'en des intervalles de temps de grandeur finie; et si nous avions eu égard à cette circonstance, la conductibilité k et par suite la quantité a (the diffusion coefficient) ne seraient plus rigoureusement constantes .....”

Considering the general linear parabolic equation

$$u_t = A(x, t)\, u_{xx} + B(x, t)\, u_x + C(x, t)\, u,$$

research carried out between 1905 and 1960 under successively weaker assumptions on the coefficients A, B and C showed that if these coefficients are bounded and uniformly Hölder continuous and if $A(x, t)$ is bounded away from zero, then this equation possesses a fundamental solution. Moreover the fundamental solution is positive [58, 107]. It follows that if the functions a, b and c in Eq. (1.1) have sufficient regularity to ensure that $A(x, t) := (a'(u))(x, t)$, $B(x, t) := (a''(u)\, u_x + b'(u))(x, t)$ and $C(x, t) := (c'(u))(x, t)$ satisfy the aforementioned conditions for any generalized solution $u(x, t)$ of Eq. (1.1), then the theory of fundamental solutions for linear parabolic equations implies that (1.1) displays infinite speed of propagation. Consequently, since the establishment of existence theorems for classical solutions of equations of the form (1.1) in the 1950s and 1960s [58, 107, 116], it has been known that when $a \in C^2([0, \infty))$, $b \in C^1([0, \infty)) \cap C^2(0, \infty)$, $c \in C^1([0, \infty))$, a'', b'' and c' are locally Hölder continuous on $(0, \infty)$, and, last but not least, $a'(s) > 0$ for all $s \geqslant 0$, Eq. (1.1) has infinite speed of propagation.

In 1950 though, Zel'dovich and Kompaneets [145] published an explicit solution of the porous media equation,

$$u_t = (u^m)_{xx} \qquad \text{with} \quad m > 1,$$

which indicated that this equation does not display infinite speed of propagation. Besides exhibiting the aforementioned characteristic, the solution was also not a classical solution of the equation. It failed to be a classical solution at precisely those points $(\zeta(t), t)$ on the interface of the solution. Today this solution corresponding to initial data representing an instantaneous point-source is commonly referred to as the Barenblatt-Pattle [13, 120] solution and, as mentioned earlier, has played a fundamental role in the study of the porous media equation [10, 11, 36, 88, 94, 95, 123, 138–140, 142].

Ostensibly, the finite speed of propagation and associated non-admittance of classical solutions of the porous media equation results from the property that $a'(0)=0$ when the equation is cast in the form (1.1), and, consequently for $u=0$ the equation degenerates from parabolic type. Nevertheless, for equations of the form

$$u_t=(a(u))_{xx} \tag{1.11}$$

with $a\in C^1([0,\infty))\cap C^2(0,\infty)$,

$$a'(s)>0 \qquad \text{for all} \quad s>0, \tag{1.12}$$

and a'' locally Hölder continuous on $(0,\infty)$, the condition $a'(0)=0$ is not the definitive criterion for finite speed of propagation. The true criterion [89, 115, 121, 122] is whether or not

$$a'(s)/s\in L^1(0,\delta) \qquad \text{for some} \quad \delta\in(0,\infty). \tag{1.13}$$

The theory for nonlinear degenerate parabolic equations of the type (1.1) was given a permanent basis in 1958 in a now renowned paper by Oleinik *et al.* [115]. These authors formulated the concept of a weak solution of (1.11) (such a solution is a generalized solution in the sense of our definition), and, under the assumptions that $a\in C^1([0,\infty))\cap C^2(0,\infty)$, (1.12) holds, and a'' is locally Hölder continuous on $(0,\infty)$, subsequently established existence and uniqueness theorems for diverse boundary-value problems. Their theory more than adequately covered the instantaneous point-source solutions of the porous media equation. It showed that in general weak solution of (1.11) were classical solutions as long as they were positive, and, thereby that the non-admittance of classical solutions of (1.11) was intricately bound up with the existence of interfaces like (1.10). Furthermore, Oleinik *et al.* were able to show that (1.13) was sufficient for finite speed of propagation of solutions of (1.11). The necessity of this condition under the restriction

$$sa''(s)\leqslant\gamma a'(s) \qquad \text{for all} \quad s>0 \qquad \text{and some} \quad \gamma\in(0,1/3) \tag{1.14}$$

was later proved by Kalashnikov [89], and, independently without this restriction by Peletier [121, 122]. For Eq. (1.1) under the aforestated conditions on a, it follows that $a'(0)=0$ is a necessary condition for finite speed of propagation but not a sufficient one.

The theory of Oleinik *et al.* [115] has since been extended to cover Eq. (1.1) under increasingly more general conditions on the coefficients in the equation. See [6, 9, 10, 36, 41, 57, 69, 72, 85, 91, 93, 94, 98, 101, 117, 118, 136, 148] for instance. For equations of the form

$$u_t=(a(u))_{xx}+(b(u))_x \tag{1.15}$$

where $a, b \in C^1([0, \infty))$, and of the form

$$u_t = (a(u))_{xx} + c(u) \tag{1.16}$$

where $a, c \in C^1([0, \infty))$ and

$$c(s) \leqslant 0 \qquad \text{for all} \quad s > 0, \tag{1.17}$$

it is now more or less clear that if (1.12) holds and the coefficients in these equations have sufficient additional regularity to justify the existence of weak solutions in the sense of Oleinik *et al.*, (1.13) is still the necessary and sufficient criterion for finite speed of propagation [69, 71, 91, 102].

The situation becomes less clear-cut however when the coefficients in Eq. (1.1) become singular. For instance, Martinson and Pavlov [108] constructed solutions for the equation

$$u_t = (u^m)_{xx} + c_0 u^p \tag{1.18}$$

with $c_0 < 0$ and $1 > p > 0$ which show that this equation admits finite speed of propagation whenever $m \geqslant 1$. Whilst, for the equation

$$u_t = (u^m)_{xx} + b_0 (u^n)_x \tag{1.19}$$

with $b_0 \neq 0$ and $m \geqslant 1 > n > 0$ it was shown by Diaz and Kersner [40] that one has finite speed of propagation in terms of the interface defined by (1.10) if and only if $b_0 > 0$. In both these instances, the conclusion is valid irrespective of whether the parameter $m > 1$ or $m = 1$, i.e. irrespective of whether (1.13) holds or not.

Under the assumptions that $a, b \in C([0, \infty)) \cap C^2(0, \infty)$, (1.12) holds and a'' and b'' are locally Hölder continuous on $(0, \infty)$, which conditions are used to prove the existence of generalized solutions of Eq. (1.15), it is now known [71] that this equation admits finite speed of propagation if and only if there exists a real parameter λ such that

$$\lambda s + b(s) > 0 \qquad \text{for all} \quad s \in (0, \delta)$$

and

$$a'(s)/[\lambda s + b(s)] \in L^1(0, \delta) \qquad \text{for some} \quad \delta \in (0, \infty).$$

Plainly, when $b \equiv 0$ this condition is equivalent to (1.13). For the Eq. (1.19) with $m > 0$ and $n > 0$ the above condition means that when $b_0 < 0$ the interface ζ exists if and only if $m > 1$ and $n \geqslant 1$, whilst when $b_0 > 0$ it exists if and only if $m > \min\{n, 1\}$.

Under the assumptions $a \in C([0, \infty)) \cap C^2(0, \infty)$, $c \in C([0, \infty)) \cap C^1(0, \infty)$, (1.12) and (1.17) hold, and, a'' and c' are locally Hölder continuous on $(0, \infty)$, which conditions are again used for the existence of generalized solutions, Kalashnikov [91] has shown that when

$$a''(s) \geqslant 0 \qquad \text{for all} \quad s > 0 \tag{1.20}$$

and

$$c'(s) < 0 \qquad \text{for all} \quad s > 0, \tag{1.21}$$

Eq. (1.16) has infinite speed of propagation if (1.14) holds,

$$c(s)a'(s) = \mathcal{O}(s) \qquad \text{as} \quad s \downarrow 0 \tag{1.22}$$

and (1.13) is negated, whereas the equation has finite speed of propagation if

$$a'(s) \Big/ \left| \int_0^s c(r)a'(r)\, dr \right|^{1/2} \in L^1(0, \delta) \qquad \text{for some} \quad \delta \in (0, \infty). \tag{1.23}$$

Under the same basic assumptions, (1.20), (1.21), and the hypotheses that the functions $c(s)\, a'(s)/s$, $c(s)/s$, $c'(s)$ and $a'(s)/s$ are monotonic near $s = 0$, Kersner [102] later proved that if (1.22) holds then (1.13) is necessary and sufficient for finite speed of propagation, whilst if

$$s = \mathcal{O}(c(s)a'(s)) \qquad \text{as} \quad s \downarrow 0 \tag{1.24}$$

then

$$a'(s) \Big/ \left[\int_0^s |c(r)a'(r)/r|^{1/2}\, dr \right] \in L^1(0, \delta) \qquad \text{for some} \quad \delta \in (0, \infty)$$

is necessary and sufficient. Actually, to be pedantic, Kersner needed some additional technical restrictions on the coefficients a and c to be able to prove necessity in both instances. Apparently unaware of Kernser's results, Chen [31], generalizing Kalashnikov's work, showed that if

$$c(s)\, a'(s) = \mathcal{O}(\min\{a(s), s\}) \qquad \text{as} \quad s \downarrow 0$$

even when (1.20) and (1.21) do not hold, (1.13) is necessary for finite speed of propagation. Recently, assuming that

$$sa'(s) = \mathcal{O}(s) \qquad \text{as} \quad s \downarrow 0, \tag{1.25}$$

$$a(s) = \mathcal{O}(sa'(s)) \qquad \text{as} \quad s \downarrow 0, \tag{1.26}$$

$$\int_0^s c(r)a'(r)\, dr = \mathcal{O}(sc(s)a'(s)) \qquad \text{as} \quad s \downarrow 0, \tag{1.27}$$

$$sc(s)a'(s) = \mathcal{O}\left(\int_0^s c(r)a'(r)\,dr\right) \qquad \text{as} \quad s \downarrow 0, \tag{1.28}$$

$$s = \mathcal{O}(a(s)a'(s)) \qquad \text{or} \qquad a(s)a'(s) = \mathcal{O}(s) \qquad \text{as} \quad s \downarrow 0, \tag{1.29}$$

and

$$c(s) = \mathcal{O}(a(s)) \qquad \text{or} \qquad a(s) = \mathcal{O}(c(s)) \qquad \text{as} \quad s \downarrow 0, \tag{1.30}$$

Song [136] has proved that if (1.22) holds then (1.13) is necessary and sufficient for finite speed of propagation, whilst if (1.24) holds then (1.23) is the necessary and sufficient condition. With the different suppositions required by each author, these results overlap one other. Nonetheless, in no two cases do they do this totally. Furthermore, despite the apparent differences in the formulation of the conditions for and against finite speed of propagation, none of the results are contradictory. (We refer the reader to the last section of this paper for the justification of this remark.) Applied to Eq. (1.18) with $m > 0$, $c_0 < 0$ and $p > 0$, the results tell us that these is an interface of the type (1.10) if and only if $m > \min\{p, 1\}$.

Under a combination of the underlying assumptions in both of the previous two paragraphs, Song [136] has also established some necessary and sufficient conditions for finite speed of propagation of the full Eq. (1.1) when (1.17) holds. Noting the hypotheses (1.22) or (1.24) and (1.25)–(1.30) which Song required for Eq. (1.16), these results are only obtained under a large number of additional analogous hypotheses on the coefficient b. Nevertheless, the results do give a complete picture for the equation.

$$u_t = (u^m)_{xx} + b_0(u^n)_x + c_0 u^p \tag{1.31}$$

with $m > 0$, $b_0 \neq 0$, $n > 0$, $c_0 < 0$ and $p > 0$. For $b_0 < 0$ and $n \geqslant 1$ the interface (1.10) exists if and only if $m > \min\{p, 1\}$. For $b_0 < 0$ and $n < 1$ this interface exists if and only if $\min\{m, n\} > p$. Whilst for $b_0 > 0$ it exists if and only if $m > \min\{n, p, 1\}$. We refer the reader to Section 4 of this paper for a further discussion of Song's results and naturally to the original reference [136] for all the particulars.

For Eq. (1.16) with $a \in C([0, \infty)) \cap C^2(0, \infty)$, (1.12) holding, $c \in C(0, \infty)$ and

$$c(s) > 0 \qquad \text{for all} \quad s > 0,$$

Galaktionov [60] has indicated that when

$$\lim_{s \downarrow 0} \frac{1}{s} \int_0^s \frac{c(r)a'(r)}{r}\,dr \qquad \text{exists and is finite}$$

and (1.13) holds, (1.16) admits finite speed of propagation. Applied to the Eq. (1.18) with $c_0 > 0$, these conditions are equivalent to $m > 1$ and $m + p \geqslant 2$. More recently, de Pablo and Vazquez [117, 118] have shown that for (1.18) with $m > 1$ and $c_0 > 0$, the constraint $m + p \geqslant 2$ is necessary and sufficient for finite speed of propagation.

Lastly, we mention that the property of finite speed of propgation for Eq. (1.31) with $b_0 \neq 0$, $c_0 \neq 0$ and with m, n and p arbitrary real numbers has been investigated by formal methods by Pokrovskii and Taranenko [127].

1.3. *Our Hypotheses and Tools*

In this paper we shall establish the necessary and sufficient condition for the existence of the interface (1.10) for the full Eq. (1.1) with hypotheses on the coefficients which cover and generalize the previous results.

Our basic hypotheses is Hypothesis 1. We note that condition (1.2) in this hypothesis does not represent an essential restriction on the admissibility of the coefficients in (1.1). For a start, the assumption $a(0) = b(0) = 0$ involves no loss of generality, for in general in (1.1) one can always replace $a(u)$ by $a(u) - a(0)$ and $b(u)$ by $b(u) - b(0)$ so that a new equation with coefficients which do conform to this assumption is obtained. With regard to $c(0) = 0$, we observe that should (1.1) display finite speed of propagation there must be a nonempty bounded rectangle $R := (x_1, x_2) \times (t_1, t_2] \subseteq H$ such that

$$u(x, t) = 0 \qquad \text{for all} \quad (x, t) \in \bar{R}.$$

Whence, by the integral identity (1.8) of the definition of a generalized solution of (1.1), there holds

$$c(0) \iint_R \phi(x, t)\, dx\, dt = 0$$

for any nonnegative function $\phi \in C^{2,1}(\bar{R})$ which satisfies (1.5). From this it follows that necessarily $c(0) = 0$. So, even setting aside questions of the existence of solutions, when $c(0) \neq 0$ there is no finite speed of propagation anyway.

The main tool in our investigation of the phenomenon of finite speed of propagation is the following comparison principle.

Hypothesis 2 (*The Comparison Principle*). (i) Given any generalized solution $v(x, t)$ of Eq. (1.1) in a domain $D \subseteq H$ of the form (1.3), (1.4) with $v(x, t) \geqslant u(x, t)$ for all $(x, t) \in \bar{D} \backslash D$ there holds $v(x, t) \geqslant u(x, t)$ for all $(x, t) \in \bar{D}$.

(ii) There exists a sequence of functions $\{c_k\}_{k=1}^{\infty}$ such that Hypothesis 1 holds with c_k in place of c for each $k \geqslant 1$,

$$\int_0^{\infty} \max\{c_k(w) - c_{k+1}(w), 0\}\, da(w) = 0 \qquad \text{for each} \quad k \geqslant 1,$$

$$\int_0^{s} |c_k(w) - c(w)|\, da(w) \to 0 \qquad \text{as} \quad k \to \infty$$

for all $s \in (0, \infty)$, and, given any generalized strict subsolution $v(x, t)$ of Eq. (1.1) with c_k in place of c in a domain $D \subseteq H$ of the form (1.3), (1.4) with $v(x, t) \leqslant u(x, t)$ for all $(x, t) \in \bar{D} \backslash D$ there holds $v(x, t) \leqslant u(x, t)$ for all $(x, t) \in \bar{D}$.

When $a, b \in C^1([0, \infty)) \cap C^2(0, \infty)$, $c \in C^1([0, \infty))$, (1.12) and (1.17) hold, and a'', b'' and c' are locally Hölder continuous on $(0, \infty)$, in which case one may take $c_k \equiv c$ for all $k \geqslant 1$ in this hypothesis, this comparison principle may be proved by a direct extension [6, 22, 41, 72, 85, 91, 92, 94, 98, 101, 117, 118, 136] of the work of Oleinik *et al.* [115]. Alternative generalizations can be found in [1, 2, 9, 22, 36, 117, 118].

The deposition of this comparison principle as a hypothesis may be viewed from different angles. Purists may look upon it as inferring that we consider only those solutions of initial and boundary value problems for equations of the form (1.1) for which it has been proven. We prefer to accept it as a logical datum. It comprises precisely what is essential for the obtainment of our results. In taking this approach, we follow Evans and Knerr [51]. Motivated by a body of recent work on Eq. (1.1) in which the existence and uniqueness of alternatively-defined solutions has been proven under extremely weak conditions on the coefficients in (1.1) [2, 21, 23, 43, 130, 134], we anticipate that in the future the conditions under which the comparison principle will have been established will be relaxed.

N.B. For those equations of the type (1.1) for which initial and boundary value problems are not uniquely solvable [1, 117–119], Hypothesis 2 implicitly means that our theorems concern the *minimal solution* of the appropriate problem.

1.4. *The Main Results*

Two distinct schools of approach can be discerned in those articles which have previously been concerned with the existence of the interface defined by (1.10). The first school is composed of authors who sought to establish criteria for finite speed of propagation using explicitly-constructed generalized subsolutions and supersolutions of Eq. (1.1). The other school consists of authors who have obtained their results using self-similar solutions. This distinction is reflected in the number of technical assumptions imposed on the coefficients. As an illustration, we mention the demonstration of the necessity of (1.13) for finite speed of propagation of (1.11). Using an explicitly-constructed subsolution Kalashnikov [89] needed (1.14) to obtain this result, whereas Peletier [121, 122] did not. The latter proved the necessity of (1.13) using self-similar solutions of (1.11) derived through application of the well-known Boltzmann transformation [4, 37]. Likewise, in the review in the previous subsection, we may compare the results which were obtained for Eq. (1.16) with the aid of explicit generalized subsolutions and supersolutions with those obtained for (1.15) using self-similar solutions of travelling-wave type. This observation lies at the foundation of our theory. We turn to the only self-similar solutions of (1.1) which are available when the equation has arbitrary coefficients—the travelling waves.

Suppose for arguments sake that in the classical sense (1.1) admits a travelling-wave solution of the form

$$U(x, t) = f(\xi) \qquad \text{with} \quad \xi = x - \lambda t.$$

Suppose, furthermore that this solution possesses an interface of the form (1.10) via the conditions

$$f(\xi) = 0 \qquad \text{for} \quad \xi \geqslant \xi^*, \tag{1.32}$$

$$f(\xi) > 0 \qquad \text{for} \quad \xi < \xi^* \tag{1.33}$$

and

$$f \text{ is strictly decreasing on } (-\infty, \xi^*] \tag{1.34}$$

for some $\xi^* \in (-\infty, \infty)$. Substituting U in (1.1) one expects

$$-\lambda f' = (a(f))'' + (b(f))' + c(f). \tag{1.35}$$

Whence, recalling that we are assuming that U is a classical solution of (1.1), so that (1.32) implies $(a(f))'(\xi^*) = 0$, integrating (1.35) yields

$$-(a(f))'(\xi) = \lambda f(\xi) + b(f(\xi)) + \int_{\xi^*}^{\xi} c(f(v))\, dv \tag{1.36}$$

for all $\xi \in (-\infty, \infty)$. Now, in view of (1.32)–(1.34) and the hypothesis that a is strictly increasing on $[0, \infty)$, we may define a nonnegative function θ in a right neighbourhood of zero by

$$\theta(f(\xi)) = -(a(f))'(\xi) \qquad \text{for all} \quad \xi \in (-\infty, \xi^*].$$

In this case though

$$\int_0^{f(\xi)} 1/\theta(s)\, da(s) = \xi^* - \xi \qquad \text{for} \quad \xi \in (-\infty, \xi^*)$$

and by substitution in (1.36), θ satisfies the identity

$$\theta(s) = \lambda s + b(s) - \int_0^s c(r)/\theta(r)\, da(r). \tag{1.37}$$

Hence, formally, if (1.1) has an appropriate travelling-wave solution with an interface then the nonlinear Volterra integral Eq. (1.37) must have a solution on an interval $[0, \delta)$ such that

$$\int_0^{\delta} 1/\theta(s)\, da(s) < \infty \qquad \text{for some} \quad \delta \in (0, \infty). \tag{1.38}$$

Retracing the above argument we also find that should (1.37) have a continuous nonnegative solution which satisfies (1.38), then we could construct a travelling-wave solution of (1.1) in some subdomain of H with an interface of the type (1.10). Thus formally in (1.37) and (1.38) we have a necessary and sufficient criterion for the local existence of a particular travelling-wave solution of (1.1) with an interface. Our point is that in turn the existence of such a travelling wave is necessary and sufficient for finite speed of propagation of Eq. (1.1).

The bulk of this paper is essentially devoted to proving the verity of the above heuristic observations. Supposing that Hypotheses 1 and 2 and two additional technical hypotheses which we state later hold, our principal results are the following. Recall that $P[t]$ is defined by (1.9).

THEOREM 1. *Suppose that there is a real parameter λ for which* (1.37) *has a continuous nonnegative solution whose reciprocal is integrable with respect to a in the sense of Lebesgue–Stieltjes in a right neighbourhood of zero. Then, if $P[0]$ is bounded, there exists a $\tau \in (0, T]$ such that $P[t]$ is uniformly bounded for all $t \in [0, \tau]$.*

THEOREM 2. *Suppose that there is no real parameter λ for which* (1.37) *has a continuous nonnegative solution whose reciprocal is integrable with respect to a in the sense of Lebesgue–Stieltjes in a right neighbourhood of zero. Then, if $P[0]$ is not empty, there exists a $\tau \in (0, T]$ such that $P[t]$ is nonempty, connected and unbounded for all $t \in (0, \tau]$.*

From these two theorems we conclude that under the stated hypotheses, Eq. (1.1) displays finite speed of propagation if and only if there is a real parameter λ such that (1.37) has a solution θ on an interval $[0, \delta)$ satisfying (1.38).

Remark. If Eq. (1.37) has a continuous nonnegative solution whose reciprocal is integrable with respect to a in the sense of Lebesgue–Stieltjes in a right neighbourhood of zero for some parameter value λ^* then the same can be said for all $\lambda \geqslant \lambda^*$.

This remark is justified in the next section.

In Section 3, we explain the exact link between finite speed of propagation, the travelling-wave solutions and the integral Eq. (1.37). In Section 4 we elaborate on the main conclusions.

2. The Integral Equation

2.1. *General Theory*

The difficulty with the study of the nonlinear Volterra integral Eq. (1.37) is connected with its singular kernel. In general this equation admits neither existence nor uniqueness [75]. Moreover, it is possible that the equation admits at least two solutions one of which satisfies (1.38) and one which does not. By way of illustration consider

$$a(s) = s^m,$$

$$b(s) = s^p + (1 + p/q)\, s^q$$

and

$$c(s) = ps^{q-m}(s^p + s^q)/m,$$

with

$$0 < p < m < q.$$

It can be checked that with this combination of coefficients and $\lambda = 0$ Eq. (1.37) admits the solution $\theta_1(s) := s^p + s^q$ which satisfies (1.38) and $\theta_2(s) := s^q$ which does not.

To avoid the difficulties associated with the singular kernel in (1.37) we study this equation as the limit as $\varepsilon \downarrow 0$ of the regularized equation

$$\theta(s) = \varepsilon + \lambda s + b(s) - \int_0^s c(r)/\theta(r)\, da(r). \tag{2.1}$$

Without further repetition, we assume that the functions a, b and c satisfy Hypothesis 1.

Definition 3. A function θ is a solution of Eq. (2.1) if it is defined, real, nonnegative and continuous in a right neighbourhood of zero $[0, l)$ with $0 < l \leqslant \infty$, setting

$$I(r, \theta) = \begin{cases} c(r)/\theta & \text{if} \quad \theta > 0 \\ \infty & \text{if} \quad c(r) > 0 \quad \text{and} \quad \theta = 0 \\ 0 & \text{if} \quad c(r) = 0 \quad \text{and} \quad \theta = 0 \\ -\infty & \text{if} \quad c(r) < 0 \quad \text{and} \quad \theta = 0 \end{cases}$$

the function $I(r, \theta(r))$ is integrable with respect to a in the sense of Lebesgue–Stieltjes on every compact subset of $(0, l)$,

$$\int_0^s I(r, \theta(r))\, da(r) := \lim_{\delta \downarrow 0} \int_\delta^s I(r, \theta(r))\, da(r) \quad \text{exists}$$

and satisfies

$$\theta(s) = \varepsilon + \lambda s + b(s) - \int_0^s I(r, \theta(r))\, da(r) \qquad \text{for all} \quad s \in (0, l).$$

Equation (2.1) with $a(s) \equiv s$ has been studied in some detail in [75]. Reproducing the arguments there with standard Lebesgue integration replaced by Lebesgue–Stieltjes integration with respect to a the following five lemmas can readily be proven.

Lemma 1. (i) *For any* $\varepsilon > 0$ *Eq.* (2.1) *has a maximal solution* $\theta(s; \lambda, \varepsilon)$ *defined on a maximal interval of existence* $[0, \tilde{M}(\lambda, \varepsilon))$.

(ii) *If* $0 < \varepsilon_1 < \varepsilon_2$ *there holds* $\tilde{M}(\lambda, \varepsilon_1) \leqslant \tilde{M}(\lambda, \varepsilon_2)$ *and* $\theta(s; \lambda, \varepsilon_1) \leqslant \theta(s; \lambda, \varepsilon_2)$ *for all* $s \in [0, \tilde{M}(\lambda, \varepsilon_1))$.

(iii) *Setting*

$$\tilde{N}(\lambda) := \inf\{\tilde{M}(\lambda, \varepsilon) : \varepsilon > 0\}, \tag{2.2}$$

and

$$\theta(s; \lambda, 0) := \inf\{\theta(s; \lambda, \varepsilon) : \varepsilon > 0\}, \tag{2.3}$$

the function $\theta(s; \lambda, 0)$ *is continuous on* $[0, \tilde{N}(\lambda))$.

(iv) *Setting*

$$\tilde{M}(\lambda, 0) := \sup \left\{ s \in [0, \tilde{N}(\lambda)) : \int_\delta^s |c(r)/\theta(r; \lambda, 0)|\, da(r) < \infty \right.$$
$$\left. \textit{for all } \delta \in (0, s) \right\}$$

with the convention that $\tilde{M}(\lambda, 0) = 0$ if this supremum is taken over an empty set, Eq. (1.37) has a solution if and only if $\tilde{M}(\lambda, 0) > 0$. Moreover, in this event, $\theta(s; \lambda, 0)$ defines the maximal solution of (1.37) with $[0, \tilde{M}(\lambda, 0))$ its maximal interval of existence.

LEMMA 2. *If $0 < \tilde{M}(\lambda, \varepsilon) < \infty$ for some $\varepsilon \geqslant 0$ then $\theta(s; \lambda, \varepsilon) \to 0$ as $s \uparrow \tilde{M}(\lambda, \varepsilon)$.*

LEMMA 3. *Suppose that*

$$\int_0^l \min\{c(r), 0\}\, da(r) = 0 \tag{2.4}$$

for some $0 < l \leqslant \infty$. Set

$$Q(s) = \left| \int_0^s c(r)\, da(r) \right|^{1/2}. \tag{2.5}$$

(i) *If $\lambda s + b(s) \geqslant \sqrt{8}\, Q(s)$ for all $s \in [0, l)$ then $\tilde{M}(\lambda, 0) \geqslant l$ and $[\lambda s + b(s)]/2 \leqslant \theta(s; \lambda, 0) \leqslant \lambda s + b(s)$ for all $s \in [0, l)$.*

(ii) *If $\lambda s + b(s) \leqslant \sqrt{\kappa}\, Q(s)$ for all $s \in [0, l)$ for some $0 \leqslant \kappa < 8$ then $\tilde{M}(\lambda, 0) > 0$ only if $\theta(s; \lambda, 0) \equiv 0$ on $[0, \delta)$ for some $\delta \in (0, \min\{\tilde{M}(\lambda, 0), l\})$.*

(iii) *If $\lambda s + b(s)$ is nondecreasing on $[0, l)$ and $\lambda \delta + b(\delta) < \sqrt{2}\, Q(\delta)$ for some $\delta \in (0, l)$ then $\tilde{M}(\lambda, 0) < \delta$.*

LEMMA 4. *Suppose that*

$$\int_0^l \max\{c(r), 0\}\, da(r) = 0 \tag{2.6}$$

for some $0 < l \leqslant \infty$. Then Eq. (1.37) has at most one solution on $[0, l)$.

LEMMA 5. *Consider Eq. (2.1) with two sets of coefficients $a, b^{(i)}$ and $c^{(i)}$ satisfying Hypothesis 1 and parameters ε_i and λ_i for $i = 1, 2$. Suppose that (2.1) with $i = 1$ has a solution $\theta^{(i)}$ on an interval $[0, l)$. Suppose furthermore that $\varepsilon_2 \geqslant \varepsilon_1$,*

$$\lambda_2 s + b^{(2)}(s) - \lambda_1 s - b^{(1)}(s) \text{ is nondecreasing on } [0, l),$$

$$\int_0^l \max\{c^{(2)}(s) - c^{(1)}(s), 0\}\, da(s) = 0$$

and

$$\int_\delta^l |c^{(2)}(s)/\theta^{(1)}(s)|\, da(s) < \infty \qquad \text{for all} \quad \delta \in (0, l). \tag{2.7}$$

Then (2.1) *with* $i=2$ *has a solution* $\theta^{(2)}$ *on* $[0, l)$ *such that* $\theta^{(2)}(s) \geqslant \theta^{(1)}(s)$ *for all* $s \in [0, l)$.

Taking $b^{(1)} \equiv b^{(2)}$, $c^{(1)} \equiv c^{(2)}$ and $\varepsilon_1 = \varepsilon_2 = 0$ in this lemma justifies the remark following the statement of Theorems 1 and 2.

2.2. *Additional Theory*

Lemma 1 defines the notation of the maximal solution $\theta(s; \lambda, \varepsilon)$ of (2.1) with its maximal interval of existence $[0, \tilde{M}(\lambda, \varepsilon))$ which we shall use throughout the remainder of this paper. Moreover, invoking Lemma 2 we may assume that $\theta(s; \lambda, \varepsilon) \in C([0, \tilde{M}(\lambda, \varepsilon)))$ with $\theta(\tilde{M}(\lambda, \varepsilon); \lambda, \varepsilon) = 0$ if $\tilde{M}(\lambda, \varepsilon) < \infty$.

We introduce now some supplementary notation for the study of solutions of (1.37) which satisfy the constraint (1.38). For each λ and $\varepsilon \geqslant 0$ we define

$$M(\lambda, \varepsilon) := \sup \left\{ s \in [0, \tilde{M}(\lambda, \varepsilon)) : \int_0^s 1/\theta(r; \lambda, \varepsilon)\, da(r) < \infty \right\} \tag{2.8}$$

where once more we adopt the convention that the variable assumes the value zero if the supremum is taken over an empty set. We note that by a continuity argument

$$M(\lambda, \varepsilon) > 0 \qquad \text{if} \quad \varepsilon > 0$$

and

$$\theta(M(\lambda, \varepsilon); \lambda, \varepsilon) = 0 \qquad \text{if} \quad M(\lambda, \varepsilon) < \infty \qquad \text{for any} \quad \varepsilon \geqslant 0. \tag{2.9}$$

Furthermore, by Lemma 1(ii) we know $M(\lambda, \varepsilon_1) \leqslant M(\lambda, \varepsilon_2)$ for any λ and $0 < \varepsilon_1 < \varepsilon_2$. Whilst Lemma 5 implies $M(\lambda_1, \varepsilon) \leqslant M(\lambda_2, \varepsilon)$ for any $\lambda_1 \leqslant \lambda_2$ and $\varepsilon \geqslant 0$.

In combination with the above notation the next two lemmas play key roles in the proofs of Theorems 1 and 2.

Hypothesis 3. There is an $L \geqslant 0$ with the following property. Given any $s \in (0, \infty)$ for which $c(s) < 0$ there exists an $s' \in (0, s)$ and a $\kappa \in [0, 8)$ such that

$$\int_{s'}^{s} \max\{c(w), 0\}\, da(w) = 0$$

and

$$b(r) \leqslant b(s) + L(s - r) + \left| \kappa \int_r^s c(w)\, da(w) \right|^{1/2} \qquad \text{for all} \quad r \in (s', s).$$

LEMMA 6. *Suppose that Hypothesis* 3 *holds. Then if* $0 < M(\lambda, \varepsilon) < \infty$ *for some* $\lambda \geqslant L$ *and* $\varepsilon \geqslant 0$ *necessarily* $c(M(\lambda, \varepsilon)) \geqslant 0$.

Proof. For convenience we drop λ and ε from the notation of M. Suppose contrary to the assertion of the lemma that $c(M) < 0$. Then setting $a^*(s) := a(M) - a(M-s)$, $b^* := b(M-s) - b(M)$, $c^*(s) := -c(s)$ and $\lambda^* := -\lambda$, recalling (2.9) it can be verified that $\theta^*(s) := \theta(M-s; \lambda, \varepsilon)$ is a solution of Eq. (1.37) on $[0, M)$ with a^*, b^*, c^* and λ^* in lieu of a, b, c and λ. Furthermore, by Hypothesis 3 there exists a $\delta \in (0, M]$ and a $\kappa \in [0, 8)$ such that

$$\int_0^\delta \min\{c^*(r), 0\}\, da^*(r) = 0$$

and

$$\lambda^* s + b^*(s) \leqslant \left| \kappa \int_0^s c^*(r)\, da^*(r) \right|^{1/2} \qquad \text{for all} \quad s \in [0, \delta).$$

by Lemma 3(ii) though this is only possible if $\theta^* \equiv 0$ on $[0, \delta_0)$ for some $\delta_0 \in (0, \delta)$. Whence $\theta(s; \lambda, \varepsilon) \equiv 0$ on $(M - \delta_0, M]$, and therefore

$$\int_0^s 1/\theta(r; \lambda, \varepsilon)\, da(r) = \infty \qquad \text{for all} \quad s \in (M - \delta_0, M).$$

However, this contradicts the definition of M. ∎

Hypothesis 4. For each function c_k from Hypothesis 2, Hypothesis 3 holds with c_k in place of c with the selfsame value of L. Furthermore, given any $s \in (0, \infty)$ and decreasing sequence of positive functions $\{\psi_k\}_k^\infty \subseteq C([0, s])$ which converges to a function $\psi \in C([0, s])$ in the limit $k \to \infty$ and for which

$$\lim_{k \to \infty} \int_0^s 1/\psi_k(w)\, da(w) < \infty \tag{2.10}$$

and

$$\lim_{k \to \infty} \int_{r_1}^{r_2} c_k(w)/\psi_k(w)\, da(w) \quad \text{exists and is finite} \tag{2.11}$$

for any $0 \leqslant r_1 < r_2 \leqslant s$, there holds

$$\int_{r_1}^{r_2} |c(w)/\psi(w)|\, da(w) < \infty \tag{2.12}$$

and

$$\int_{r_1}^{r_2} c(w)/\psi(w)\,da(w) = \lim_{k\to\infty} \int_{r_1}^{r_2} c_k(w)/\psi_k(w)\,da(w) \tag{2.13}$$

for all $0 < r_1 < r_2 < s$.

LEMMA 7. *Suppose that Hypothesis* 4 *holds. For each* $k \geqslant 1$ *and* $\varepsilon > 0$ *let* $\theta_k(s; \lambda, \varepsilon)$ *denote the maximal solution of* (2.1) *with* c_k *in place of* c *and let* $M_k(\lambda, \varepsilon)$ *denote the corresponding analogue of* $M(\lambda, \varepsilon)$ *defined by* (2.8). *Set*

$$N(\lambda) := \inf\{M_k(\lambda, \varepsilon) : k \geqslant 1 \text{ and } \varepsilon > 0\}. \tag{2.14}$$

Then if $M(\lambda, 0) = 0$ *and* $N(\lambda) > 0$, *necessarily*

$$\int_0^{\mu} 1/\theta_k(s; \lambda, \varepsilon)\,da(s) \uparrow \infty \qquad \text{as} \quad k \uparrow \infty \qquad \text{and} \qquad \varepsilon \downarrow 0$$

for all $\mu \in (0, N(\lambda))$.

Proof. We shall prove this lemma by showing that if $N(\lambda) > 0$ and there exists a $\mu \in (0, N(\lambda))$ such that

$$\Xi := \sup\left\{\int_0^{\mu} 1/\theta_k(s; \lambda, \varepsilon)\,da(s) : k \geqslant 1 \text{ and } \varepsilon > 0\right\} < \infty$$

then $M(\lambda, 0) \geqslant \mu$. For this purpose we observe to begin with that under the aforesaid circumstances $M_k(\lambda, \varepsilon) > \mu$, subsequently

$$\begin{aligned} \theta_k(r_2; \lambda, \varepsilon) = \theta_k(r_1; \lambda, \varepsilon) &+ \lambda r_2 + b(r_2) - \lambda r_1 - b(r_1) \\ &- \int_{r_1}^{r_2} c_k(w)/\theta_k(w; \lambda, \varepsilon)\,da(w) \end{aligned} \tag{2.15}$$

for any $0 \leqslant r_1 < r_2 \leqslant \mu$, and

$$\int_0^{\mu} 1/\theta_k(r; \lambda, \varepsilon)\,da(r) < \Xi \tag{2.16}$$

for every $k \geqslant 1$ and $\varepsilon > 0$. Furthermore, by Lemma 5 we have

$$\theta_{k_1}(s; \lambda, \varepsilon_1) \leqslant \theta_{k_2}(s; \lambda, \varepsilon_2) \qquad \text{for all} \quad s \in [0, \mu]$$

and $k_1 \geqslant k_2 \geqslant 1$ and $0 < \varepsilon_1 < \varepsilon_2$. Thus we can define the nonnegative function

$$\theta^*(s) := \lim_{k\to\infty \quad \varepsilon\to 0} \theta_k(s; \lambda, \varepsilon)$$

on $[0, \mu]$. Moreover, adapting the proof of a lemma in [75] it is possible to show that θ^* is continuous on $[0, \mu]$ with $\theta^*(0)=0$ (cf. Lemma 1(iii)). Whence, combining (2.15) and (2.16) with Hypothesis 4 there holds

$$\int_{r_1}^{r_2} |c(w)/\theta^*(w)| \, da(w) < \infty \qquad \text{for all} \quad 0 < r_1 < r_2 < \mu,$$

and we may let $k \uparrow \infty$ and $\varepsilon \downarrow 0$ in (2.15) to deduce

$$\theta^*(r_2) = \theta^*(r_1) + \lambda r_2 + b(r_2) - \lambda r_1 - b(r_1) - \int_{r_1}^{r_2} c(w)/\theta^*(w) \, da(w)$$

for any $0 < r_1 < r_2 < \mu$. Letting $r_1 \downarrow 0$ it follows that θ^* solves (1.37) on $[0, \mu]$. Thus $\tilde{M}(\lambda, 0) \geqslant \mu$ and $\theta(s; \lambda, 0) \geqslant \theta^*(s)$ for all $s \in [0, \mu)$. Applying the Monotone Convergence Theorem to (2.16) subsequently yields

$$\int_0^{\mu} 1/\theta(s; \lambda, 0) \, da(r) \leqslant \Xi.$$

Thus indeed $M(\lambda, 0) \geqslant \mu$ ∎

2.3. *Incidental Results*

In this subsection we state and prove a number of specific results concerning (1.37) under the constraint (1.38). These are introduced for the later discussion of applications of Theorems 1 and 2.

LEMMA 8. *Consider Eq.* (1.37) *with two different sets of coefficients* $a^{(i)}$, $b^{(i)}$ *and* $c^{(i)}$ *satisfying Hypothesis* 1 *and parameter* λ_i *for* $i = 1, 2$. *Let* σ *denote the Lebesgue–Stieltjes measure associated with Lebesgue–Stieltjes integration with respect to* $a^{(2)}$ *on* $(0, \infty)$.

(a) *Suppose that there exists a* $0 < l < \infty$ *and a constant* β *such that* $a^{(2)}(s) = a^{(1)}(s)$ *on* $[0, l)$, $b^{(2)}(s) - b^{(1)}(s) + \beta s$ *is nondecreasing on* $[0, l)$, $c^{(2)}(s) \leqslant c^{(1)}(s)$ *almost everywhere with respect to* σ *on* $(0, l)$, *and that*

$$c^{(2)}(s)/[1 + |c^{(1)}(s)|] \qquad \textit{is essentially bounded} \tag{2.17}$$

with respect to σ *in every compact subset of* $(0, l)$.

(b) *Suppose that there exists a* $0 < l < \infty$ *and constants* β *and* γ *such that* $a^{(2)}(s) - a^{(1)}(s)$ *is nonincreasing on* $[0, l)$, $b^{(2)}(s) + \beta s \geqslant b^{(1)}(s)$ *for all* $s \in [0, l)$, *and* $\max\{0, c^{(2)}(s)\} \leqslant c^{(1)}(s) + \gamma s$ *almost everywhere with respect to* σ *on* $(0, l)$.

Then in both cases (a) *and* (b) *if* (1.37) *with* $i = 1$ *has a solution which satisfies* (1.38) *the same can be said of* (1.37) *with* $i = 2$ *for* $\lambda_2 > \lambda_1 + \beta$.

Proof. Case (a) follows from Lemma 5. The only detail which needs particular attention is the validation of (2.7). The hypothesis (2.17) takes care of this. We concentrate therefore on the proof of case (b). For this case we set

$$\alpha := (\lambda_2 - \lambda_1 - \beta)/2 > 0$$

and use superscripts within brackets to distinguish between the various quantities associated with the solution of (1.37) for $i=1$ and for $i=2$. For any $\varepsilon > 0$ and $s \in [0, \min\{\tilde{M}^{(2)}(\lambda_2, \varepsilon), \tilde{M}^{(1)}(\lambda_1, 0), l\})$ such that $\theta^{(2)}(r; \lambda_2, \varepsilon) > \theta^{(1)}(r; \lambda_1, 0)$ for all $r \in [0, s)$ we compute

$$\begin{aligned}
&\int_0^s c^{(2)}(r)/\theta^{(2)}(r; \lambda_2, \varepsilon)\, da^{(2)}(r) \\
&\qquad \leqslant \int_0^s [c^{(1)}(r) + \gamma r]/\theta^{(2)}(r; \lambda_2, \varepsilon)\, da^{(2)}(r) \\
&\qquad \leqslant \int_0^s [c^{(1)}(r) + \gamma r]/\theta^{(1)}(r; \lambda_1, 0)\, da^{(1)}(r) \\
&\qquad \leqslant \int_0^s c^{(1)}(r)/\theta^{(1)}(r; \lambda_1, 0)\, da^{(1)}(r) + \gamma s \int_0^s 1/\theta^{(1)}(r; \lambda_1, 0)\, da^{(1)}(r).
\end{aligned}$$

So that

$$\begin{aligned}
\theta^{(2)}(s; \lambda_2, \varepsilon) &= \varepsilon + \lambda_2 s + b^{(2)}(s) - \int_0^s c^{(2)}(r)/\theta^{(2)}(r; \lambda_2, \varepsilon)\, da^{(2)}(r) \\
&\geqslant \varepsilon + \lambda_2 s + b^{(1)}(s) - \beta s - \int_0^s c^{(1)}(r)/\theta^{(1)}(r; \lambda_1, 0)\, da^{(1)}(r) \\
&\quad - \gamma s \int_0^s 1/\theta^{(1)}(r; \lambda_1, 0)\, da^{(1)}(r) \\
&= \varepsilon + \left[2\alpha - \gamma \int_0^s 1/\theta^{(1)}(r; \lambda_1, 0)\, da^{(1)}(r) \right] s + \theta^{(1)}(s; \lambda_1, 0).
\end{aligned}$$

It follows that if we choose $\delta \in (0, \min\{\tilde{M}^{(1)}(\lambda, 0)), l\})$ so small that

$$\gamma \int_0^\delta 1/\theta^{(1)}(r; \lambda_1, 0)\, da^{(1)}(r) < \alpha,$$

there holds $M^{(2)}(\lambda, \varepsilon) \geqslant \delta$ and

$$\theta^{(2)}(s; \lambda_2, \varepsilon) \geqslant \theta^{(1)}(s; \lambda_1, 0) + \alpha s > 0 \qquad \text{for all} \quad s \in (0, \delta)$$

and $\varepsilon > 0$. Subsequently, applying Lemma 1, Eq. (1.37) with $i=2$ has a solution $\theta^{(2)}(s; \lambda_2, 0)$ on an interval $[0, \tilde{M}^{(2)}(\lambda_2, 0))$ with $\tilde{M}^{(2)}(\lambda_2, 0) \geqslant \delta$. Moreover

$$\int_0^\delta 1/\theta^{(2)}(r; \lambda_2, 0)\, da^{(2)}(r) \leqslant \int_0^\delta 1/\theta^{(1)}(r; \lambda_1, 0)\, da^{(1)}(r) < \infty. \quad \blacksquare$$

LEMMA 9. *Suppose there exists a* $0 < l < \infty$ *such that* (2.6) *holds and with* $Q(s)$ *defined by* (2.5) *that* $b(s) + KQ(s) + \rho s$ *is nondecreasing on* $[0, l)$ *for some constants* $K \geqslant 0$ *and* ρ. *Then for any* $\lambda > \max\{0, \rho + K\}$ *there holds* $\tilde{M}(\lambda, 0) \geqslant l$ *and there exist positive constants* K_1 *and* K_2 *which depend only on* λ, K *and* ρ *such that*

$$K_1 \max\{b(s), Q(s), s\} \leqslant \theta(s; \lambda, 0) \leqslant K_2 \max\{b(s), Q(s), s\} \tag{2.18}$$

for all $s \in [0, l)$.

Proof. It can be easily verified that the function $\theta^{(1)}(s) := K^* Q(s)$ with $K^* := 4/(K + \sqrt{K^2 + 8})$ fulfills Eq. (1.37) on $[0, l)$ when $\lambda = 0$ and b is replaced by the function $b^{(1)}(s) := -KQ(s)$. Subsequently, applying Lemma 5 we have $\tilde{M}(\lambda, 0) \geqslant l$ and

$$\theta(s; \lambda, 0) \geqslant \theta^{(1)}(s) = K^* Q(s) \tag{2.19}$$

for all $s \in [0, l)$ for each $\lambda \geqslant \rho$.

Now if $\lambda \geqslant \rho$, from (1.37) and (2.6) we deduce $\theta(s; \lambda, 0) \geqslant \lambda s + b(s)$ for any $s \in [0, l)$. Whence, if also $\lambda \geqslant 0$ we have

$$\theta(s; \lambda, 0) \geqslant b(s) \tag{2.20}$$

whilst $\theta(s; \lambda, 0) \geqslant \lambda s - KQ(s) - \rho s$ for all $s \in [0, l)$. So that

$$\theta(s; \lambda, 0) \geqslant (\lambda - \rho - K)\, s \qquad \text{if} \quad Q(s) \leqslant s \tag{2.21}$$

for any $s \in [0, l)$. Combining (2.19), (2.20) and (2.21) leads to the left-hand inequality in (2.18) for any $\lambda > \max\{0, \rho + K\}$ with $K_1 = \min\{K^*, 1, \lambda - \rho - K\}$. On the other hand, substituting (2.19) directly into the integral term of (1.37) for $\lambda \equiv \rho$ gives

$$\theta(s; \lambda, 0) \leqslant \lambda s + b(s) + 2Q(s)/K^*$$

for any $s \in [0, l)$. This provides the right-hand inequality in (2.18). $\blacksquare$

LEMMA 10. *Suppose that there exists a* $0 < l < \infty$ *such that* $b(s) \leqslant 0$ *for all* $s \in (0, l)$,

$$\int_0^l |c(s)/b(s)|\, da(s) < \infty,$$

and

$$b(s) + K \int_0^s c(r)/b(r)\, da(r) + \rho s \qquad \textit{is nondecreasing on } [0, l)$$

for some constants $K > 0$ *and* ρ. *Let* $K^* := 2/(K + \sqrt{K^2 + 4K})$. *Then for any* $\lambda \geqslant \rho(1 + K^*)$ *there holds* $\tilde{M}(\lambda, 0) \geqslant l$ *and* $\theta(s; \lambda, 0) \geqslant -K^* b(s)$ *for all* $s \in [0, l)$.

Proof. When $\lambda = 0$ and $b(s)$ in Eq. (1.37) is replaced by

$$b^{(1)}(s) := -K^* b(s) - \frac{1}{K^*} \int_0^s \frac{c(r)}{b(r)}\, da(r)$$

the resulting equation admits the solution $\theta^{(1)}(s) := -K^* b(s)$ on $[0, l)$. This yields the stated result via Lemma 5. ∎

LEMMA 11. *Suppose that there exists a* $0 < l < \infty$ *such that* a *and* b *are absolutely continuous on* $[0, l)$, $(ca')(s) \leqslant 0$ *for almost all* $s \in (0, l)$, *and* ca'/b' *is continuous and nonnegative on* $[0, l)$.

(a) *Suppose furthermore that* $-(ca'/b')(s) - Kb(s) + \rho s$ *is nondecreasing on* $[0, l)$ *for some constants* $K > 0$ *and* ρ. *Let*

$$K^* := 2/(1 + \sqrt{1 + 4K}). \tag{2.22}$$

Then for any $\lambda \geqslant \rho K^*$ *there holds* $\tilde{M}(\lambda, 0) \geqslant l$ *and* $\theta(s; \lambda, 0) \geqslant K^*(ca'/b')(s)$ *for all* $s \in [0, l)$.

(b) *Suppose furthermore that* $-(ca'/b')(s) - Kb(s) + \rho s$ *is nonincreasing on* $[0, l)$ *for some constants* $K > 0$ *and* ρ. *Let* K^* *be given by* (2.22). *Then for any* $\lambda \leqslant \rho K^*$ *there holds* $\theta(s; \lambda, 0) \leqslant K^*(ca'/b')(s)$ *for all* $s \in [0, \min\{\tilde{M}(\lambda, 0), l\})$.

Proof. We note that when $\lambda = 0$ and b is replaced by the function $b^{(1)}(s) := b(s)/K^* + K^*(ca'/b')(s)$ Eq. (1.37) admits the solution $\theta^{(1)}(s) := K^*(ca'/b')(s)$. Moreover, by Lemma 4 this function is the only admissible solution of this integral equation. Lemma 5 subsequently once more provides the desired result. ∎

LEMMA 12. *Suppose that there exists a* $0<l<\infty$ *such that* $b\equiv 0$ *on* $[0, l)$ *and* (2.4) *holds. With* $Q(s)$ *given by* (2.5) *set*

$$\sigma=\limsup_{s\downarrow 0} Q(s)/s.$$

(a) *If* $\lambda<\sqrt{2}\,\sigma$ *then* $\tilde{M}(\lambda, 0)=0$.

(b) *If* $\lambda>\sqrt{8}\,\sigma$ *then* $\tilde{M}(\lambda, 0)>0$ *and there exists a* $\delta\in(0, \tilde{M}(\lambda, 0))$ *such that* $\lambda s/2\leqslant\theta(s; \lambda, 0)\leqslant\lambda s$ *for all* $s\in[0, \delta)$.

Proof. Part (a) represents an application of Lemma (iii). Part (b) follows from an application of Lemma 3(i). ∎

LEMMA 13. *Suppose that* $a(s)=s^m$, $b(s)=b_0 s^n$ *and* $c(s)=c_0 s^p$ *for some constants* $m>0$, $n>0$, $p>-m$, b_0 *and* c_0. *Then there exists a* λ^* *such that* $\tilde{M}(\lambda^*, 0)>0$ *if and only if the constants satisfy one of the following eight combinations. Moreover, with the value of q stated in each case, there exists a* $\lambda^{**}\geqslant\lambda^*$ *such that for any* $\lambda>\lambda^{**}$ *there holds*

$$\theta(s; \lambda, 0)\sim\theta_0 s^q \qquad \text{as} \quad s\downarrow 0$$

for some $\theta_0>0$.

(i) $c_0<0$, $n\geqslant 1$ *or* $b_0=0$; *with* $q=\min\{(m+p)/2, 1\}$.

(ii) $c_0<0$, $n<1$, $b_0<0$; *with* $q=\max\{m+p-n, (m+p)/2\}$.

(iii) $c_0<0$, $n<1$, $b_0>0$; *with* $q=\min\{n, (m+p)/2\}$.

(iv) $c_0=0$, $n\geqslant 1$ *or* $b_0=0$; *with* $q=1$.

(v) $c_0=0$, $n<1$, $b_0>0$; *with* $q=n$.

(vi) $c_0>0$, $n\geqslant 1$ *or* $b_0=0$, $m+p\geqslant 2$; *with* $q=1$.

(vii) $c_0>0$, $n<1$, $0<b_0<2\sqrt{mc_0/n}$, $m+p>2n$; *with* $q=n$.

(viii) $c_0>0$, $n<1$, $b_0\geqslant 2\sqrt{mc_0/n}$, $m+p\geqslant 2n$; *with* $q=n$.

For the proof of this and the next lemma, see [75, 78].

LEMMA 14. *Suppose that* $a(s)=s\,|\ln s|^{-m}$, $b(s)=b_0 s\,|\ln s|^{-n}$ *and* $c(s)=c_0 s\,|\ln s|^{-p}$ *for some constants* m, n, p, b_0 *and* c_0. *Then there exists a* λ^* *such that* $\tilde{M}(\lambda^*, 0)>0$ *if and only if the constants satisfy one of the following eight combinations. Moreover, with the value of q stated in each case, there exists a* $\lambda^{**}\geqslant\lambda^*$ *such that for any* $\lambda>\lambda^*$ *there holds*

$$\theta(s; \lambda, 0)\sim\theta_0 s\,|\ln s|^{-q} \qquad \text{as} \quad s\downarrow 0$$

for some $\theta_0 > 0$.

(i) $c_0 < 0$, $n \geqslant 0$ *or* $b_0 = 0$; *with* $q = \min\{(m+p)/2, 0\}$.

(ii) $c_0 < 0$, $n < 0$, $b_0 < 0$; *with* $q = \max\{m+p-n, (m+p)/2\}$.

(iii) $c_0 < 0$, $n < 0$, $b_0 > 0$; *with* $q = \min\{n, (m+p)/2\}$.

(iv) $c_0 = 0$, $n \geqslant 0$ *or* $b_0 = 0$; *with* $q = 0$.

(v) $c_0 = 0$, $n < 0$, $b_0 > 0$; *with* $q = n$.

(vi) $c_0 > 0$, $n \geqslant 0$ *or* $b_0 = 0$, $m + p \geqslant 0$; *with* $q = 0$.

(vii) $c_0 > 0$, $n < 0$, $0 < b_0 \leqslant 2\sqrt{c_0}$, $m + p > 2n$; *with* $q = n$.

(viii) $c_0 > 0$, $n < 0$, $b_0 > 2\sqrt{c_0}$, $m + p \geqslant 2n$; *with* $q = n$.

3. Proof and Discussion of Main Results

3.1. *Travelling Waves*

The first goal of this section is to show that solutions of the nonlinear integral equation (1.37) really lead to travelling-wave type solutions of (1.1). We had done this formally in Subsection 1.4. However, there we avoided all the difficulties associated with the dearth of regularity in the coefficients of (1.1) and of the definition of a generalized solution.

We maintain the notation used in the previous section and consider the maximal solution $\theta(s; \lambda, \varepsilon)$ of Eq. (2.1) on $[0, M(\lambda, \varepsilon))$ with $M(\lambda, \varepsilon)$ defined by (2.8) for every λ and $\varepsilon \geqslant 0$. We let

$$\Delta(\lambda, \varepsilon) := \int_0^{M(\lambda, \varepsilon)} 1/\theta(r; \lambda, \varepsilon)\, da(r).$$

Next we define the function f by

$$f(\xi) = M(\lambda, \varepsilon) \qquad \text{for} \quad \xi \leqslant -\Delta(\lambda, \varepsilon)$$

$$\int_0^{f(\xi)} 1/\theta(r; \lambda, \varepsilon)\, da(r) = -\xi \qquad \text{for} \quad -\Delta(\lambda, \varepsilon) < \xi < 0 \tag{3.1}$$

and

$$f(\xi) = 0 \qquad \text{for} \quad \xi \geqslant 0.$$

Finally, for fixed $(x_0, t_0) \in \mathbb{R}^2$ we define the function $v(x, t; x_0, t_0, \lambda, \varepsilon)$ by

$$v(x, t; x_0, t_0, \lambda, \varepsilon) = f(x - x_0 - \lambda t + \lambda t_0) \tag{3.2}$$

and set

$$\Omega(x_0, t_0, \lambda, \varepsilon) = \{(x, t) \in \mathbb{R}^2 \colon -\Delta(\lambda, \varepsilon) < x - x_0 - \lambda t + \lambda t_0 < \infty\}.$$

Ignoring the implicit dependence on the parameters x_0, t_0, λ and ε we can now formulate the following.

LEMMA 15. *Suppose that* $M > 0$ *and let* D *denote a domain of the form* (1.3), (1.4) *such that* $\bar{D} \subseteq \Omega$.

(i) *If* $\varepsilon = 0$ *then* v *is a generalized solution of Eq.* (1.1) *in* D.

(ii) *If* $\varepsilon > 0$ *then* v *is a generalized subsolution of Eq.* (1.1) *in* D. *Moreover if* $\eta_1 < x_0 + \lambda t - \lambda t_0 < \eta_2$ *for some* $t \in [\tau_1, \tau_2]$ *then* v *is a generalized strict subsolution of* (1.1) *in* D.

Proof. We introduce the notation

$$\beta_\iota(t) = x_0 + \lambda t - \lambda t_0 - \int_0^\iota 1/\theta(r)\, da(r)$$

for any $t \in \mathbb{R}$ and

$$\Omega_\iota := \{(x, t) \in \Omega \colon x < \beta_\iota(t)\}$$

for any $\iota \in [0, M)$. We let A denote the inverse of a on $[0, a(M))$ and define the map Ψ on $[0, a(M))$ by

$$\Psi(z) = \int_0^{A(z)} 1/\theta(r)\, da(r) = \int_0^z 1/\theta(A(y))\, dy.$$

Note that for any $(x, t) \in \Omega$ and $\iota \in [0, M)$ there holds

$$v(x, t) > \iota \qquad \text{if and only if} \quad (x, t) \in \Omega_\iota$$

and for any $(x, t) \in \Omega_0$ we have

$$\Psi(a(v(x, t))) = \beta_0(t) - x.$$

Plainly v is nonnegative, continuous and strictly bounded above by M in $\bar{D}$. We assert that for any bounded rectangle $R := (x_1, x_2) \times (t_1, t_2] \subseteq D$ and nonnegative function $\phi \in C^{2,1}(\bar{R})$ satisfying (1.5) there holds:

$$c(v) \in L^1(R \cap \Omega_\iota) \qquad \text{for all} \quad \iota > 0, \tag{3.3}$$

$$\iint_R c(v)\, \phi\, dx\, dt := \lim_{\iota \downarrow 0} \iint_{R \cap \Omega_\iota} c(v)\, \phi\, dx\, dt \qquad \text{exists} \tag{3.4}$$

and satisfies

$$\iint_R [a(v)\,\phi_{xx} - b(v)\,\phi_x + c(v)\,\phi + v\phi_t]\,dx\,dt$$

$$= \int_{x_1}^{x_2} [v(x, t_2)\,\phi(x, t_2) - v(x, t_1)\,\phi(x, t_1)]\,dx$$

$$+ \int_{t_1}^{t_2} [a(v(x_2, t))\,\phi_x(x_2, t) - a(v(x_1, t))\,\phi_x(x_1, t)]\,dt$$

$$+ \varepsilon \int_{I_0} \phi(\beta_0(t), t)\,dt, \tag{3.5}$$

where

$$I_\iota := \{t \in [t_1, t_2] : \beta_\iota(t) \in (x_1, x_2)\}.$$

Proving this suffices to prove the lemma.

By standard Lebesgue integration theory Ψ is absolutely continuous on $(0, a(M))$ and $\Psi'(z) = 1/\theta(A(z))$ for almost all $z \in (0, a(M))$. Subsequently

$$\int_{x^-}^{x^+} |c(v(x, t))|\,dx = -\int_{a(v(x^-, t))}^{a(v(x^+, t))} |c(v(\beta_0(t) - \Psi(y), t))|\,\Psi'(y)\,dy$$

$$= \int_{a(v(x^+, t))}^{a(v(x^-, t))} |c(A(y))|/\theta(A(y))\,dy$$

$$= \int_{v(x^+, t)}^{v(x^-, t)} |c(r)/\theta(r)|\,da(r) \tag{3.6}$$

for any $\beta_M(t) < x^- < x^+ < \beta_0(t)$ and $t \in \mathbb{R}$. Hence

$$c(v(x, t)) \in L^1_{\mathrm{loc}}(\beta_M(t), \beta_0(t)) \qquad \text{for all} \quad t \in \mathbb{R} \tag{3.7}$$

and defining

$$\nu := \sup\{v(x, t) : (x, t) \in R\}$$

we have the estimate

$$\iint_{R \cap \Omega_\iota} |c(v(x, t))|\,dx\,dt \leqslant \int_{t_1}^{t_2} \int_{\beta_\nu(t)}^{\beta_\iota(t)} |c(v(x, t))|\,dx\,dt$$

$$= (t_2 - t_1) \int_\iota^\nu |c(r)/\theta(r)|\,da(r).$$

This yields (3.3).

To prove (3.4) we note that for any $\beta_M(t) < x^- < x^+ < \beta_0(t)$ and $t \in \mathbb{R}$ setting

$$\underline{\theta} = \inf\{\theta(v(x, t)) : x^- \leqslant x \leqslant x^+\}$$

and

$$\bar{\theta} = \sup\{\theta(v(x, t)) : x^- \leqslant x \leqslant x^+\}$$

there holds

$$\begin{aligned} -\underline{\theta}(x^+ - x^-) &= \underline{\theta} \int_{v(x^-, t)}^{v(x^+, t)} 1/\theta(r)\, da(r) \\ &\leqslant a(v(x^+, t)) - a(v(x^-, t)) \\ &\leqslant \bar{\theta} \int_{v(x^-, t)}^{v(x^+, t)} 1/\theta(r)\, da(r) \\ &= -\bar{\theta}(x^+ - x^-) \end{aligned}$$

by (3.1), (3.2). Hence $(a(v))_x$ exists in Ω_0,

$$(a(v))_x\,(x, t) = -\theta(v(x, t)) \qquad \text{for all} \quad (x, t) \in \Omega_0 \tag{3.8}$$

and $(a(v))_x$ is nonnegative and continuous in Ω_0. Similarly, one can show that $a(v)$ is continuously differentiable with respect to t in Ω_0 and

$$(a(v))_t\,(x, t) = \lambda\theta(v(x, t)) \qquad \text{for all} \quad (x, t) \in \Omega_0. \tag{3.9}$$

Because θ solves (2.1) on $[0, M)$ it follows from (3.8) that

$$\begin{aligned} &((a(v))_x + b(v) + \lambda v)(x^+, t) - ((a(v))_x + b(v) + \lambda v)(x^-, t) \\ &\quad = \int_{v(x^-, t)}^{v(x^+, t)} c(r)/\theta(r)\, da(r) \end{aligned}$$

for any $\beta_M(t) < x^- < x^+ < \beta_0(t)$ and $t \in \mathbb{R}$. However, recalling (3.7) and repeating the argument in (3.6) without the absolute value signs,

$$\int_{v(x^-, t)}^{v(x^+, t)} c(r)/\theta(r)\, da(r) = -\int_{x^-}^{x^+} c(v(x, t))\, dx$$

for any such x^-, x^+ and t. Thus, moreover, the function $(a(v))_x + b(v) + \lambda v$ is absolutely continuous with respect to x on $(\beta_M(t), \beta_0(t))$ and

$$((a(v))_x + b(v) + \lambda v)_x\,(x, t) + c(v(x, t)) = 0 \tag{3.10}$$

for almost all $x \in (\beta_M(t), \beta_0(t))$ for every $t \in \mathbb{R}$. Multiplying (3.10) by ϕ and integrating by parts over $R \cap \Omega_\iota$ for $\iota > 0$ we compute

$$\iint_{R \cap \Omega_\iota} [a(v)\,\phi_{xx} - b(v)\,\phi_x - \lambda v \phi_x + c(v)\,\phi]\,dx\,dt$$

$$= \int_{t_1}^{t_2} [a(v(x_2, t))\,\phi_x(x_2, t) - a(v(x_1, t))\,\phi_x(x_1, t)]\,dt$$

$$+ \int_{I_\iota} \{[\theta(\iota) - b(\iota) - \lambda\iota]\,\phi(\beta_\iota(t), t) + a(\iota)\,\phi_x(\beta_\iota(t), t))$$

$$- a(v(x_2, t))\,\phi_x(x_2, t)\}\,dt.$$

So, letting $\iota \downarrow 0$ we derive (3.4). In fact, noting that $v(x, t) \equiv 0$ on $R \backslash \Omega_0$, in the limit $\iota \downarrow 0$ we obtain

$$\iint_{R} [a(v)\,\phi_{xx} - b(v)\,\phi_x - \lambda v \phi_x + c(v)\,\phi]\,dx\,dt$$

$$= \int_{t_1}^{t_2} [a(v(x_2, t))\,\phi_x(x_2, t) - a(v(x_1, t))\,\phi_x(x_1, t)]\,dt$$

$$+ \varepsilon \int_{I_0} \phi(\beta_0(t), t)\,dt. \tag{3.11}$$

As a result of the above analysis, (3.5) is the only component of our central assertion still awaiting proof. Moreover, in the light of (3.11), to confirm (3.5) we merely have to show that

$$\iint_R [\lambda v \phi_x + v \phi_t]\,dx\,dt = \int_{x_1}^{x_2} [v(x, t_2)\,\phi(x, t_2) - v(x, t_1)\,\phi(x, t_1)]\,dx. \tag{3.12}$$

To verify this, we refer to (3.8) and (3.9) and note that consequently

$$(a(v))_t\,(x, t) = -\lambda (a(v))_x\,(x, t) \tag{3.13}$$

at any point $(x, t) \in \Omega_0$. Trivially though (3.13) is true for all $(x, t) \in \Omega \backslash \bar{\Omega}_0$. Hence for any continuously-differentiable function $F\colon [0, a(v)] \to [0, \infty)$ we have

$$-\lambda (F(a(v)))_x\,(x, t) = (F(a(v)))_t\,(x, t) \tag{3.14}$$

for all $(x, t) \in \bar{R}$ such that $x \neq \beta_0(t)$. Multiplying (3.14) by ϕ and integrating by parts yields

$$\iint_R [\lambda F(a(v))\, \phi_x + F(a(v))\, \phi_t]\, dx\, dt$$

$$= \int_{x1}^{x} [F(a(v(x, t_2)))\, \phi(x, t_2) - F(a(v(x, t_1)))\, \phi(x, t_1)]\, dx. \quad (3.15)$$

Consequently, if we replace F in (3.15) by a sequence of functions which approximate A the inverse of a in the limit, and take this limit, we obtain (3.12).

This completes the proof of the assertion (3.5) and therewith the proof of the lemma. ∎

Lemma 16. *Suppose that* $0 < M < \infty$ *and* $c(M) \geqslant 0$ *and let* D *denote a domain of the form* (1.3), (1.4). *Then* v *is a generalized subsolution of Eq.* (1.1) *in* D. *Moreover if* $\varepsilon > 0$ *and* $\eta_1 < x_0 + \lambda t - \lambda t_0 < \eta_2$ *for some* $t \in [\tau_1, \tau_2]$ *then* v *is a generalized strict subsolution of* (1.1) *in* D.

Proof. We continue from the proof of Lemma 15. Repeating the analysis in the proof of this lemma we find that (1.6) and (1.7) hold and compute

$$\iint_R [a(v)\, \phi_{xx} - b(v)\, \phi_x + c(v)\, \phi + v\phi_t]\, dx\, dt$$

$$= \int_{x_1}^{x_2} [v(x, t_2)\, \phi(x, t_2) - v(x, t_1)\, \phi(x, t_1)]\, dx$$

$$+ \int_{t_1}^{t_2} [a(v(x_2, t))\, \phi_x(x_2, t) - a(v(x_1, t))\, \phi_x(x_1, t)]\, dt$$

$$+ \varepsilon \int_{I_0} \phi(\beta_0(t), t)\, dt + \iint_{R \setminus \Omega} c(M)\, \phi\, dx\, dt$$

for any bounded rectangle $R := (x_1, x_2) \times (t_1, t_2] \subseteq D$ and any nonnegative function $\phi \in C^{2,1}(\bar{R})$ satisfying (1.5). This gives the result. ∎

Lemma 15 shows that solutions of (1.37) lead to generalized solutions of (1.1) of travelling-wave type. Moreover, by Lemmas 15 and 16, for any $\varepsilon > 0$ and $\lambda \geqslant L$ the solution of (2.1) generates a generalized strict subsolution of (1.1).

3.2. *The Proof of Theorem* 1

If (1.37) has a continuous nonnegative solution whose reciprocal is integrable with respect to a in the sense of Lebesgue–Stieltjes in a right neighbourhood of zero for some parameter λ, then in terms of the theory developed in the previous section, $M(\lambda, 0) > 0$ for some λ. Theorem 1 is subsequently a corollary of the next result.

LEMMA 17. *Suppose that $M(\lambda, 0) > 0$ for some λ. Then given any $0 < \delta < \Delta(\lambda, 0)$ there exists a $\tau \in (0, T]$ such that $\zeta(t) \leqslant \zeta(0) + \delta + \lambda t$ for all $t \in [0, \tau]$.*

Proof. Consider the function $v(x, t; \zeta(0) + \delta, 0, \lambda, 0)$ defined in the previous subsection. Note that $v(\zeta(0), 0; \zeta(0) + \delta, 0, \lambda, 0) > 0$ whilst $u(\zeta(0), 0) = 0$. Hence, utilizing the continuity of u and v, we can find a $\tau \in (0, T]$ so small that when

$$D := (\zeta(0), \infty) \times (0, \tau]$$

there holds $\bar{D} \subseteq \Omega(\zeta(0) + \delta, 0, \lambda, 0)$ and

$$v(\zeta(0), t; \zeta(0) + \delta, 0, \lambda, 0) \geqslant u(\zeta(0), t) \qquad \text{for all} \quad t \in [0, \tau].$$

Besides

$$v(x, 0; \zeta(0) + \delta, 0, \lambda, 0) \geqslant 0 = u(x, 0) \qquad \text{for all} \quad x \in [\zeta(0), \infty).$$

Now though, by Lemma 15, v is a generalized solution of Eq. (1.1) in D. Invoking the comparison principle, Hypothesis 2, subsequently tells us that

$$v(x, t; \zeta(0) + \delta, 0, \lambda, 0) \geqslant u(x, t) \qquad \text{for all} \quad (x, t) \in \bar{D}.$$

This yields $u(x, t) = 0$ for all $x \geqslant \zeta(0) + \delta + \lambda t$ and $t \in [0, \tau]$. ∎

3.3. *The Proof of Theorem* 2

Suppose that $P[0]$ is not empty. Then by the continuity of u there exists a $\tau \in (0, T]$ such that $P[t]$ is not empty for $t \in [0, \tau]$. Let $t_1 \in (0, \tau]$ and $x_0 \in P[t_1]$ be arbitrary. Subsequently choose a μ and a $t_0 \in [0, t_1)$ such that

$$u(x_0, t) \geqslant \mu > 0 \qquad \text{for all} \quad t \in [t_0, t_1].$$

Assuming Hypotheses 1–4 and defining

$$D := (x_0, \infty) \times (t_0, t_1]$$

we shall now show that if $M(\lambda, 0) = 0$ for all λ there holds

$$u(x, t) > 0 \qquad \text{for all} \quad (x, t) \in \bar{D} \qquad \text{with} \quad x < x_0 + \lambda(t - t_0) \tag{3.16}$$

for every fixed $\lambda \geqslant L$. Because $\lambda \geqslant L$ and also x_0 and t_1 were arbitrary, showing this suffices to prove the theorem.

We use the fact that a solution of (2.1) for any $\varepsilon > 0$ yields a generalized strict subsolution of (1.1). For each $k \geqslant 1$ and $\varepsilon > 0$ we let $\theta_k(s; \lambda, \varepsilon)$ denote the maximal solution of (2.1) with c_k in place of c, and define $M_k(\lambda, \varepsilon)$, $v_k(x, t; x_0, t_0, \lambda, \varepsilon)$ and $\Omega_k(x_0, t_0, \lambda, \varepsilon)$ as the corresponding analogues of $M(\lambda, \varepsilon)$, $v(x, t; x_0, t_0, \lambda, \varepsilon)$ and $\Omega(x_0, t_0, \lambda, \varepsilon)$. Finally, we define $N(\lambda)$ by (2.14). We distinguish two cases dependent on the magnitude of $N(\lambda)$.

(a) The case $N(\lambda) > 0$.

Without loss of generality, we may assume that $\mu < N(\lambda)$. By Lemma 7, we can pick a $k \geqslant 1$ so large and an $\varepsilon > 0$ so small that

$$\int_0^{\mu} 1/\theta_k(r; \lambda, \varepsilon)\, da(r) > \lambda(t_1 - t_0).$$

There then holds $\Omega_k(x_0, t_0, \lambda, \varepsilon) \supseteq \bar{D}$ and

$$v_k(x_0, t; x_0, t_0, \lambda, \varepsilon) < \mu \leqslant u(x_0, t) \qquad \text{for all} \quad t \in [t_0, t_1].$$

Whilst,

$$v_k(x, t_0; x_0, t_0, \lambda, \varepsilon) = 0 \leqslant u(x, t_0) \qquad \text{for all} \quad x \in [x_0, \infty). \tag{3.17}$$

So, since $v_k(x, t; x_0, t_0, \lambda, \varepsilon)$ is a generalized strict subsolution of Eq. (1.1) with c_k in lieu of c in D by Lemma 15, we deduce

$$v_k(x, t; x_0, t_0, \lambda, \varepsilon) \leqslant u(x, t) \qquad \text{for all} \quad (x, t) \in \bar{D}$$

via the comparison principle of Hypothesis 2. This gives (3.16).

(b) The case $N(\lambda) = 0$.

In this case we may pick a $k \geqslant 1$ so large and an $\varepsilon > 0$ so small that $M_k(\lambda, \varepsilon) \leqslant \mu$. Although now we do not necessarily have $\Omega_k(x_0, t_0, \lambda, \varepsilon) \supseteq \bar{D}$, here Lemma 16 implies that $v_k(x, t; x_0, t_0, \lambda, \varepsilon)$ is a generalized strict subsolution of Eq. (1.1) with c_k in lieu of c in D. Furthermore,

$$v_k(x_0, t; x_0, t_0, \lambda, \varepsilon) \leqslant M_k(\lambda, \varepsilon) \leqslant u(x_0, t) \qquad \text{for all} \quad t \in [t_0, t_1]$$

whilst (3.17) still holds. Subsequently, applying the comparison principle as before, we obtain (3.16) again.

3.4. *Discussion*

It may be noted that we do not actually need Hypotheses 1–4 in full to prove Theorems 1 and 2. Specifically, all that is required for the proof of Theorem 1 is Hypothesis 1 and part (i) of Hypothesis 2. Whilst for the proof of Theorem 2, Hypothesis 1, part (ii) of Hypothesis 2, Hypothesis 3 and Hypothesis 4 suffice. Moreover, all that we really require from Hypothesis 2 is that it allows us to compare u, the given generalized solution of Eq. (1.1), with the constructed generalized solutions and strict subsolutions of (1.1) of travelling-wave type. In other words, we could have formulated any restricted version of Hypothesis 2 as long as it admitted the application of our particular comparison functions.

Hypothesis 3 is incidentally also a stronger restriction on the coefficients in Eq. (1.1) than we require. The formulation given above is more convenient for checking than the general hypothesis though. At this juncture, we just indicate that neither Hypothesis 3 nor Hypothesis 4 is a particularly severe restriction on the admissiblity of the coefficients in (1.1). Taking each member of the sequence c_k in Hypothesis 2 part (ii) identical to c for instance, we note that if $c \equiv 0$ then Hypotheses 3 and 4 are satisfied vacuously. Whilst if $b \equiv 0$ and c does not change sign on $(0, \infty)$ then Hypotheses 3 and 4 are also automatically satisfied. Furthermore, if c is upper semi-continuous from the left and given any $s \in (0, \infty)$ there exists an $\alpha \in (0, 1]$ such that

$$\liminf_{r \uparrow s} \frac{a(s)-a(r)}{(s-r)^{2\alpha}} = \infty > \limsup_{r \uparrow s} \frac{b(s)-b(r)}{(s-r)^{\alpha}}$$

then Hypothesis 3 holds. Moreover if c and the sequence $\{c_k\}_{k=1}^{\infty}$ from Hypothesis 2 part (ii) are uniformly bounded above or below in every compact subset of $(0, \infty)$ then Hypothesis 4 also holds. Thus, for instance, Hypotheses 3 and 4 are valid if $a, b \in C([0, \infty)) \cap C^1(0, \infty)$, $c \in C(0, \infty)$, $c(s) \leqslant 0$ for all $s > 0$ or $c(s) \geqslant 0$ for all $s > 0$, and (1.12) holds. Suffice then to summarize that Hypotheses 3 and 4 are fulfilled by the coefficients of the equations covered in the earlier work on finite speed of propagation which was reviewed in Subsection 1.2.

In the light of results in [75] Hypothesis 3 could have been replaced with the following weaker assumption.

Hypothesis 3′. There is an $L \geqslant 0$ with the following property. Given any $s \in (0, \infty)$ for which $c(s) < 0$ letting

$$s_0 := \inf \left\{ r \in [0, s] : \int_r^s \max\{c(w), 0\}\, da(w) = 0 \right\}$$

and

$$\theta_0(r) := Lr + b(r) - Ls - b(s) \qquad \text{for} \quad r \in [s_0, s],$$

$$s_{j+1} := \inf \left\{ r \in [s_j, s] : \theta_j(w) \geqslant 0 \text{ for all } w \in [r, s] \right.$$

$$\left. \text{and } \int_r^s |c(w)/\theta_j(w)|\, da(w) < \infty \right\}$$

and

$$\theta_{j+1}(r) := Lr + b(r) - Ls - b(s) + \int_r^s c(w)/\theta_j(w)\, da(w)$$

for $r \in [s_{j+1}, s]$ for every $j \geqslant 0$,

$$s_\infty := \sup\{s_j : 0 \leqslant j < \infty\}$$

and

$$\theta_\infty(r) := \inf\{\theta_j(r) : 0 \leqslant j < \infty\} \qquad \text{for} \quad r \in [s_\infty, s],$$

there holds

$$s_\infty > s_0 \qquad \text{or} \qquad \int_{s_\infty}^s 1/\theta_\infty(r)\, da(r) = \infty.$$

Our final remark on the proofs of Theorems 1 and 2 pertains to each of the Hypotheses 1, 3 and 4. This is that it is not necessary that the given properties of the coefficients a, b and c hold on the whole of $[0, \infty)$. Since all the arguments used in this paper are local, it is sufficient that the stated properties be valid in some right neighourhood of zero.

4. Applications

4.1. *Equations with Reduced Forms*

Here we consider the consequences of Theorems 1 and 2 when Eq. (1.1) has a number of special forms which arise in various areas of practical interest [4, 5, 7, 8, 19, 25, 28–30, 33, 37, 38, 53, 56, 109–111, 113, 114]. For the full Eq. (1.1), Theorems 1 and 2 state that there is finite speed of propagation if and only if (1.37) has a solution whose reciprocal is integrable with respect to a in the sense of Lebesgue–Stieltjes in a right neighbourhood of zero. This criterion can be reformulated in more explicit terms for the special forms of interest which we consider below.

Theorem 3. *The equation*

$$u_t = (a(u))_{xx} \tag{4.1}$$

displays finite speed of propagation if and only if

$$\int_0^\delta 1/s\, da(s) < \infty \qquad \textit{for some} \quad \delta \in (0, \infty). \tag{4.2}$$

THEOREM 4. *The equation*

$$u_t = (a(u))_{xx} + (b(u))_x \tag{4.3}$$

displays finite speed of propagation if and only if

$$\max\{-b(s), 0\} = \mathcal{O}(s) \qquad \textit{as} \quad s \downarrow 0 \tag{4.4}$$

and

$$\int_0^\delta 1/\max\{b(s), s\}\, da(s) < \infty \qquad \textit{for some} \quad \delta \in (0, \infty). \tag{4.5}$$

Proof of Theorems 3 *and* 4. When $c \equiv 0$, Eq. (1.37) reduces to the simple identity $\theta = \lambda s + b(s)$. Subsequently, setting

$$\sigma := \limsup_{s \downarrow 0} -b(s)/s,$$

it is easy to see that (1.37) has no nonnegative solution for any $\lambda < \sigma$. Whilst, (1.37) does have such a solution for every $\lambda > \sigma$. In the latter case moreover, letting $[0, \tilde{M}(\lambda, 0))$ denote the maximal interval of existence of this solution, for every $\lambda \geqslant \max\{0, \sigma + 2\}$ and $s \in [0, \tilde{M}(\sigma + 1))$ we can estimate

$$\begin{aligned}(\lambda + 1) \max\{b(s), s\} \geqslant \theta(s) &= (\sigma + 1)\, s + b(s) + \lambda s - (\sigma + 1)\, s \\ &\geqslant \max\{(\sigma + 1)\, s + b(s), \lambda s\} - (\sigma + 1)\, s \\ &= \max\{b(s), (\lambda - \sigma - 1)\, s\} \\ &\geqslant \max\{b(s), s\}.\end{aligned}$$

It follows that there is a λ such that (1.37) admits a nonnegative solution in a right neighbourhood of zero if and only if $\sigma < \infty$, and moreover that if λ is sufficiently large this solution satisfies (1.38) if and only if (4.5) holds. ∎

THEOREM 5. *Suppose that $c(s) \leqslant 0$ for all $s \in (0, \infty)$. Then the equation*

$$u_t = (a(u))_{xx} + c(u) \tag{4.6}$$

admits finite speed of propagation if and if

$$\int_0^\delta 1/\max\left\{\left|\int_0^s c(r)\, da(r)\right|^{1/2}, s\right\} da(s) < \infty \qquad \textit{for some} \quad \delta \in (0, \infty). \tag{4.7}$$

THEOREM 6. *Suppose that* $c(s) \geqslant 0$ *for all* $s \in (0, \infty)$. *Then the Eq.* (4.6) *admits finite speed of propagation if and only if*

$$\int_0^s c(r)\, da(r) = \mathcal{O}(s^2) \qquad \text{as} \quad s \downarrow 0 \tag{4.8}$$

and (4.2) *holds*

Proof of Theorems 5 *and* 6. Let $Q(s)$ be given by (2.5). Then if $Q \equiv 0$ in a right neighbourhood of zero, Eq. (1.37) is equivalent to the simple identity $\theta = \lambda s$. Subsequently Theorem 5 or 6 may be easily deduced from Theorems 1 and 2. On the other hand, if $Q(s) > 0$ for all $s > 0$, then Theorems 5 and 6 follow from Lemmas 9 and 12 respectively when these are combined with Theorems 1 and 2. ∎

4.2. *Relation to Earlier Work*

It is clear that Theorem 3 covers the earlier work [89, 115, 121, 122] on Eq. (4.1).

With regard to Theorem 4, (4.4) holds if and only if there is a $\lambda > 0$ and a $\delta \in (0, \infty)$ such that

$$\lambda s + b(s) > 0 \qquad \text{for all} \quad s \in (0, \delta). \tag{4.9}$$

Furthermore, in this event, retracing the proof of Theorem 4, it can be shown that $\max\{b(s), s\} \leqslant \lambda s + b(s) \leqslant (\lambda + 1) \max\{b(s), s\}$ for sufficiently large λ and small s. So (4.4) and (4.5) hold if and only if there is a λ which fulfils (4.9) and

$$\int_0^\delta 1/[\lambda s + b(s)]\, da(s) < \infty \qquad \text{for some} \quad \delta \in (0, \infty). \tag{4.10}$$

The conditions (4.9) and (4.10) were precisely those previously established in [71] as being necessary and sufficient for finite speed of propagation of Eq. (4.3) under additional smoothness assumptions on the coefficients. Thus, under the relevant circumstances, the present criterion and the earlier one are equivalent.

To see that Theorem 5 covers the previous results on Eq. (4.6) when $c(s) \leqslant 0$ for all $s \in (0, \infty)$ requires a little more analysis. It is not too difficult to check that the theorem generalizes the work of Kalashnikov [91], Chen [31] and Song [136]. For if $a \in C^1(0, \infty)$ and

$$c(s)a'(s) = \mathcal{O}(s) \qquad \text{as} \quad s \downarrow 0, \tag{4.11}$$

then the theorem implies that (4.2) is necessary and sufficient for finite speed of propagation. Whilst, if

$$s = \mathcal{O}(c(s)a'(s)) \qquad \text{as} \quad s \downarrow 0 \tag{4.12}$$

then it implies that

$$\int_0^\delta 1 \Bigg/ \left| \int_0^s c(r)\, da(r) \right|^{1/2} da(s) < \infty \qquad \text{for some} \quad \delta \in (0, \infty)$$

is the necessary and sufficient criterion. For comparison with Kersner's results [102] though it is more convenient to consider the next variant of the theorem.

THEOREM 7. *Suppose that c is nonincreasing on* $[0, \infty)$. *Then Eq.* (4.6) *admits finite speed of propagation if and only if*

$$\int_0^\delta 1/\max\{|c(s)a(s)|^{1/2}, s\}\, da(s) < \infty \qquad \textit{for some} \quad \delta \in (0, \infty). \tag{4.13}$$

Proof. In the light of Theorem 5 it is enough to show that (4.13) holds when (4.7) is true, and vice versa. Let A denote the inverse of a on the range of a. By the monotonicity of c,

$$\left| \int_0^s c(r)\, da(r) \right| \leqslant |c(s)| \int_0^s da(r) = |c(s)a(s)|$$

for any $s \in (0, \infty)$. So, (4.7) implies (4.13). On the other hand,

$$\begin{aligned} \left| \int_0^s c(r)\, da(r) \right| &\geqslant \left| \int_{A(a(s)/2)}^s c(r)\, da(r) \right| \\ &\geqslant \left| \int_{A(a(s)/2)}^s c(A(a(s)/2))\, da(r) \right| \\ &= |c(A(a(s)/2))\, a(s)/2| \end{aligned}$$

for any $s \in (0, \infty)$. Hence

$$\max\left\{ \left| \int_0^s c(r)\, da(r) \right|^{1/2}, s \right\} \geqslant \max\{ |c(A(a(s)/2))\, a(s)/2|^{1/2}, A(a(s)/2)\}.$$

Subsequently, applying the change of variables $s \to A(a(s)/2)$, we see that (4.13) also implies (4.7). ∎

COROLLARY. *Suppose that $a \in C^1(0, \infty)$, a is convex on $[0, \infty)$, c is nonincreasing on $[0, \infty)$, and $c(s)a'(s)/s$ is nondecreasing on $(0, \infty)$. Then Eq. (4.6) admits finite speed of propagation if and only if*

$$\int_0^\delta 1/\max\left\{\int_0^s |c(r)\, a'(r)/r|^{1/2}\, dr, s\right\} da(s) < \infty \qquad \textit{for some} \quad \delta \in (0, \infty).$$

Proof. By the convexity of a there holds

$$a(s) \leqslant sa'(s) \qquad \text{for all} \quad s \in (0, \infty).$$

Hence, since $c(s)$ is nonincreasing and $c(s)a'(s)/s$ is nondecreasing,

$$\left|\frac{c(s)a(s)}{s^2}\right| \leqslant \left|\frac{c(s)a'(s)}{s}\right| \leqslant \left|\frac{c(r)a'(r)}{r}\right| \leqslant \left|\frac{c(s)a'(r)}{r}\right| \leqslant \left|\frac{c(s)(a'(r))^2}{a(r)}\right|$$

for any $0 < r < s < \infty$. Taking the square root of the first, third and last expression and integrating with respect to r from 0 to s yields

$$|c(s)a(s)|^{1/2} \leqslant \int_0^s |c(r)a'(r)/r|^{1/2}\, dr \leqslant 2\, |c(s)a(s)|^{1/2}$$

for every $s \in (0, \infty)$. Whence, the corollary follows from Theorem 7. ∎

This corollary to Theorem 7 delivers the unequivocal generalization of Kersner's results [102] which we seek, for the hypotheses in this corollary were all utilized by Kersner. Moreover, (4.11) implies

$$\int_0^s |c(r)a'(r)/r|^{1/2}\, dr = \mathcal{O}(s) \qquad \text{as} \quad s \downarrow 0.$$

So when (4.11) holds the corollary states that (4.2) is necessary and sufficient for finite speed of propagation. Whilst, should (4.12) be the case correspondingly

$$s = \mathcal{O}\left(\int_0^s |c(r)a'(r)/r|^{1/2}\, dr\right) \qquad \text{as} \quad s \downarrow 0,$$

and in this case the corollary delivers

$$\int_0^\delta 1\bigg/\left(\int_0^s |c(r)a'(r)/r|^{1/2}\, dr\right) da(s) < \infty \qquad \text{for some} \quad \delta \in (0, \infty)$$

as the necessary and sufficient criterion for finite speed of propagation. These, disregarding the imposition of additional assumptions on the coefficients in Eq. (4.6), were Kersner's results [102].

Lastly, we compare Theorem 6 with the earlier work on (4.6) when $a \in C^1(0, \infty)$ and $c(s) \geqslant 0$ for all $s \in (0, \infty)$. For this combination Galaktionov [60] has indicated that the equation possesses finite speed of propagation when

$$\lim_{s \downarrow 0} \frac{1}{s} \int_0^s \frac{c(r)\, a'(r)}{r}\, dr \qquad \text{exists and is finite} \tag{4.14}$$

and (4.2) holds. Noting that

$$\int_0^s c(r)\, da(r) \leqslant s \int_0^s \frac{c(r)}{r}\, da(r) = s^2 \left(\frac{1}{s} \int_0^s \frac{c(r)\, a'(r)}{r}\, dr \right)$$

for any $s \in (0, \infty)$, (4.14) implies (4.8). So Galaktionov's deduction [60] is indeed compatible with our own.

The final theorem in this subsection provides a generalization of the results of Song [136] and serves to illustrate that Theorems 1 and 2 also cover his criteria for finite speed of propagation for the full Eq. (1.1) when c is negative.

THEOREM 8. *Suppose that a and b are absolutely continuous on $[0, l)$ with $a'(s) > 0$ and $c(s) < 0$ for almost all $s \in (0, l)$ for some $0 < l \leqslant \infty$.*

(a) *Suppose furthermore that there exists a constant $K > 0$ such that*

$$b(s) + K \left| \int_0^s c(r)\, da(r) \right|^{1/2} + Ks \qquad \textit{is nondecreasing on } [0, l).$$

Then Eq. (1.1) has finite speed of propagation if and only if

$$\int_0^\delta 1/\max \left\{ b(s), \left| \int_0^s c(r)\, da(r) \right|^{1/2}, s \right\} da(s) < \infty \qquad \textit{for some} \quad \delta \in (0, l).$$

(b) *Suppose furthermore that $b(s) \leqslant 0$ for all $s \in (0, l)$,*

$$\int_0^l c(s)/b(s)\, da(s) < \infty$$

and that there exists a constant $K > 0$ such that

$$b(s) + K \int_0^s c(r)/b(r)\, da(r) + Ks \qquad \textit{is nondecreasing on } [0, l).$$

Then Eq. (1.1) *has finite speed of propagation if*

$$\int_0^\delta |1/b(s)|\, da(s) < \infty \qquad \textit{for some} \quad \delta \in (0, l).$$

(c) *Suppose furthermore that* ca'/b' *is continuous and nonnegative on* $[0, l)$ *and there exist a constant* $K > 0$ *such that* $-(ca'/b')(s) - Kb(s) + Ks$ *is nondecreasing on* $[0, l)$. *Then Eq.* (1.1) *has finite speed of propagation if*

$$\int_0^\delta b'(s)/c(s)\, ds < \infty \qquad \textit{for some} \quad \delta \in (0, l). \tag{4.15}$$

(d) *Suppose furthermore that* ca'/b' *is continuous and nonnegative on* $[0, l)$ *and that there exists a constant* $K > 0$ *such that* $-(ca'/b')(s) - Kb(s) + \rho s$ *is nonincreasing in a rightneighbourhood of zero for every* $\rho > 0$. *Then Eq.* (1.1) *has finite speed of propagation only if* (4.15) *holds*.

Proof. This theorem is a consequence of Theorems 1 and 2 in the light of Lemmas 9–11. ∎

The reader may refer to [136] to check that the various hypotheses imposed by Song lead to one or other of the alternative cases in Theorem 8 and moreover that the ensuing conclusions are consistent with those of Song.

We feel obliged to point out that our results do differ in one essential way from some of the earlier work of finite speed of propagation for particular cases of Eq. (1.1) though. Cf. [71, 89, 121, 122]. Namely, the present conclusions are purely of a local nature. In general, we cannot and do not exclude the possibility that there is extinction of the solution in finite time [32, 51, 91, 93, 103], nor that the interface ζ may move off to infinity in finite time.

4.3. *An Alternative Definition of a Generalized Solution*

We remark that our analysis may be extended to cover a slightly different definition of a generalized solution and subsolution of Eq. (1.1). The difference is that instead of requiring (1.6) and (1.7) one requires the stronger condition

$$c(u) \in L^1(R)$$

for any bounded rectangle R contained in the closure of the domain of definition. Plainly any generalized solution or subsolution in the sense of this definition is also one in the sense previously used. The only price we have to pay for extension to this case is a slight reinforcement of Hypothesis 4.

For the alternative definition of a generalized solution and subsolution of Eq. (1.1), the last sentence of Hypothesis 4 should be modified to read as follows. Furthermore, given any $s \in (0, \infty)$ and decreasing sequence of positive functions $\{\psi_k\}_{k=1}^{\infty} \subseteq C([0, s])$ which converges to a function $\psi \in C([0, s])$ in the limit $k \to \infty$ and for which (2.10) and (2.11) apply, (2.12) holds for all $0 \leqslant r_1 < r_2 < s$ and (2.13) as before. Subsequently, altering the definition of $M(\lambda, \varepsilon)$ to

$$M(\lambda, \varepsilon) := \sup \left\{ s \in [0, \tilde{M}(\lambda, \varepsilon)) : \int_0^s [1 + |c(r)|]/\theta(r; \lambda, \varepsilon)\, da(r) < \infty \right\}$$

the theory contained in the previous sections can be carried through.

The stricter version of Hypothesis 4 can be seen to be satisfied if each member of the sequence $\{c_k\}_{k=1}^{\infty}$ in Hypothesis 2 and c do not change sign in a right neighbourhood of zero. Indeed, it is satisfied if these functions are uniformly essentially bounded on any compact subset of $[0, \infty)$ with respect to the measure associated with Lebesgue–Stieltjes integration with respect to a. Thus, even with the alternative definition of a generalized solution of (1.1), our results cover the earlier work on finite speed of propagation for the equation.

4.4. *Comparative Results*

Here we give more import to our results by showing how finite speed of propagation for one equation may be inferred from that for another.

THEOREM 9. *Consider Eq.* (1.1) *with two different sets of coefficients* $a^{(i)}$, $b^{(i)}$ *and* $c^{(i)}$ *which satisfy Hypotheses* 1, 3 *and* 4 *and are such that Hypotheses* 2 *holds for any generalized solution* u *of the Eq.* (1.1) *in* H *for which* $0 < \sup\{x \in (0, \infty) : u(x, 0) > 0\} < \infty$. *Let* σ *denote the Lebesgue–Stieltjes measure associated with Lebesgue–Stieltjes integration with respect to* a_2 *on* $(0, \infty)$.

(a) *Suppose that there exists a* $0 < l \leqslant \infty$ *and constant* β *such that*

$$a^{(2)}(s) = a^{(1)}(s) \qquad \text{on} \quad [0, l),$$

$$b^{(2)}(s) - b^{(1)}(s) + \beta s \qquad \text{is nondecreasing on } [0, l),$$

$$c^{(2)}(s) \leqslant c^{(1)}(s) \qquad \text{almost everywhere in } (0, l)$$

with respect to σ, *and that* $c^{(2)}(s)/[1 + |c^{(1)}(s)|]$ *is essentially bounded with respect to* σ *on every compact subset of* $(0, l)$.

(b) *Suppose that there exists a* $0 < l \leqslant \infty$ *and constants* β *and* γ *such that*

$$a^{(2)}(s) - a^{(1)}(s) \qquad \textit{is nonincreasing on } [0, l),$$

$$b^{(2)}(s) + \beta s \geqslant b^{(1)}(s) \qquad \textit{for all} \quad s \in [0, l),$$

and

$$\max\{c^{(2)}(s), 0\} \leqslant c^{(1)}(s) + \gamma s \qquad \textit{almost everywhere in } (0, l)$$

with respect to σ.

Then in both cases, if Eq. (1.1) *with* $i = 1$ *possesses finite speed of propagation the same can be said of the equation with* $i = 2$.

Proof. This theorem follows from Theorems 1 and 2 upon mobilizing Lemma 8. ▮

With the less roomy definition of a generalized solution of Eq. (1.1) discussed in the previous subsection, the only modification which has to be made to the above theorem is to replace the last assumption in part (a) by the assumption that $c^{(2)}(s)/[1 + |c^{(1)}(s)|]$ is essentially bounded with respect to σ on every compact subset of $[0, l)$.

An interesting corollary of Theorem 9 is that when the function c in the full Eq. (1.1) is Lipschitz continous at $s = 0$, Eq. (1.1) displays finite speed of propagation if and only if the Eq. (4.3) without the reaction terms does. Likewise, when the function b is Lipschitz continuous at $s = 0$, Eq. (1.1) displays finite speed of propagation if and only if the corresponding Eq. (4.6) without the convection term does. Whilst should b and c both the Lipschitz continuous at $s = 0$, then (1.1) displays finite speed of propagation if and only if the reduced Eq. (4.1) does. These last conclusions are borne out by the earlier work on finite speed of propagation [69, 71, 91, 102, 136].

4.5. *Specific Examples*

In this final subsection we discuss the implication of our results for four very specific equations.

The first example is Eq. (1.1) where the reaction term corresponds to a sink given by the Heaviside function

$$H(u) = \begin{cases} 1 & \text{for} \quad u > 0 \\ 0 & \text{for} \quad u = 0 \end{cases}$$

and the remaining coefficients a and b conform appropriately to Hypotheses 1 and 3. Hypotheses 4 holds automatically when one can take c_k identical to c in Hypotheses 2 for each $k \geqslant 1$. (In the light of the remarks in Subsection 3.4 it is actually enough that Hypotheses 1 and 3′ hold.)

When $a(s) \equiv s$ and $b(s) \equiv 0$ the Eq. (1.1) with $c(s) \equiv -H(s)$ has been derived as a model for the diffusion of oxygen in absorbing tissue [35, 38, 39, 50]. This particular equation can also be obtained under suitable conditions by a transformation of the classical Stefan problem. In this case the interface (1.10) corresponds with the usual free boundary in the Stefan problem [38, 49, 50, 131].

THEOREM 10. *The equation*

$$u_t = (a(u))_{xx} + (b(u))_x - H(u)$$

displays finite speed of propagation irrespective of the coefficients a and b.

Proof. As previously, for fixed λ and $\varepsilon > 0$ let $\theta(s; \lambda, \varepsilon)$ denote the maximal continuous nonnegative solution of (2.1) and $M(\lambda, \varepsilon)$ the variable defined by (2.8). Let L denote the value in Hypothesis 3 (or 3′). By Lemma 6 we have $\tilde{M}(\lambda, \varepsilon) \geqslant M(\lambda, \varepsilon) = \infty$ for all $\varepsilon > 0$ and $\lambda \geqslant L$. Subsequently

$$\int_0^s 1/\theta(r; \lambda, \varepsilon)\, da(r) = \varepsilon + \lambda s + b(s) - \theta(s; \lambda, \varepsilon)$$

for any $s \in (0, \infty)$ and $\lambda \geqslant L$. Hence defining $\tilde{N}(\lambda)$ by (2.2) and $\theta(s; \lambda, 0)$ by (2.3), we have $\tilde{N}(\lambda) = \infty$ and

$$\int_0^s 1/\theta(r; \lambda, 0)\, da(r) \leqslant \lambda s + b(s) \qquad \text{for any} \quad s \in (0, \infty) \tag{4.16}$$

and $\lambda \geqslant L$. Lemma 1 subsequently implies that $\tilde{M}(\lambda, 0) = \infty$ and by (4.16) there holds $M(\lambda, 0) > 0$ for any $\lambda \geqslant L$. Therefore Eq. (1.37) has a solution such that the conditions of Theorem 1 are met. ∎

Confronted with Theorem 10 one may naturally ask if the converse is true; that is to say, should the reaction term in (1.1) correspond to a source term described by the Heaviside function is it true that there is no finite speed of propagation whatever the coefficients *a* and *b*. The next theorem illustrates that this is not so.

THEOREM 11. *The equation*

$$u_t = (a(u))_{xx} + H(u) \tag{4.17}$$

displays finite speed of propagation if and only if

$$a(s) = \mathcal{O}(s^2) \qquad \textit{as} \quad s \downarrow 0. \tag{4.18}$$

Proof. By Theorem 6, Eq. (4.17) possesses finite speed of propagation if and only if (4.18) and (4.2) hold. To obtain the present result it subsequently suffices to show that (4.18) implies (4.2). Suppose therefore that (4.18) holds. Then by definition there exists an $\alpha > 0$ and a $0 < \delta < \infty$ such that

$$a(s) \leqslant \alpha s^2 \qquad \text{for all} \quad s \in (0, \delta).$$

Whence

$$\int_0^\delta 1/s \, da(s) \leqslant \int_0^\delta |\alpha/a(s)|^{1/2} \, da(s) = 2 \, |\alpha a(\delta)|^{1/2}.$$

So (4.18) does indeed infer (4.2). ∎

The following theorem also provided examples to illustrate that the converse to Theorem 10 is not true. This theorem concerns a special case of Eq. (1.1) which has been the subject of considerable study in recent years. See for instance [1, 3, 10, 11, 32, 34, 36, 40, 42, 47, 57, 59, 61–66, 68, 70, 73, 76, 79–82, 85–95, 97, 99, 100, 115, 117–119, 123, 125, 129, 133, 136, 138–142] and the references cited therein. The equation has been and still is of paramount interest as a tangible proptotype for the general Eq. (1.1).

THEOREM 12. *The equation*

$$u_t = (u^m)_{xx} + b_0(u^n)_x + \begin{cases} c_0 u^p & \text{for} \quad u > 0 \\ 0 & \text{for} \quad u = 0 \end{cases} \tag{4.19}$$

with real parameters $m > 0$, $n > 0$, $p > -m$, b_0 and c_0 admits finite speed of propagation if and only if one of the following hold.

(i) $c_0 < 0$, $n \geqslant 1$ *or* $b_{00} = 0$, *and* $m > \min\{p, 1\}$.

(ii) $c_0 < 0$, $n < 1$, $b_0 < 0$ *and* $p < \min\{m, n\}$.

(iii) $c_0 < 0$, $n < 1$, $b_0 > 0$ *and* $m > \min\{n, p\}$.

(iv) $c_0 = 0$, $n \geqslant 1$ *or* $b_0 = 0$, *and* $m > 1$.

(v) $c_0 = 0$, $n < 1$, $b_0 > 0$ *and* $m > n$.

(vi) $c_0 > 0$, $n \geqslant 1$ *or* $b_0 = 0$, $m > 1$ *and* $m + p \geqslant 2$.

(vii) $c_0 > 0$, $n < 1$, $0 < b_0 < 2\sqrt{mc_0/n}$, $m > n$ *and* $m + p > 2n$.

(viii) $c_0 > 0$, $n < 1$, $b_0 \geqslant 2\sqrt{mc_0/n}$, $m > n$ *and* $m + p \geqslant 2n$.

This theorem is a corollary of Theorems 1 and 2 and Lemma 13.

To close we summarize the character of finite speed of propagation for an equation which represents a weaker perturbation of the linear equation $u_t = u_{xx} + b_0 u_x + c_0 u$ than (4.19).

THEOREM 13. *The equation*

$$u_t = (u\,|\ln u|^{-m})_{xx} + b_0(u\,|\ln u|^{-n})_x + c_0 u\,|\ln u|^{-p}$$

with real parameters m, n, p, b_0 and c_0 admits finite speed of propagation if and only if one of the following hold.

(i) $c_0 < 0$, $n \geqslant 0$ *or* $b_0 = 0$, *and* $m > \min\{p+2, 1\}$.
(ii) $c_0 < 0$, $n < 0$, $b_0 < 0$ *and* $p + 2 < \min\{m, n+1\}$.
(iii) $c_0 < 0$, $n < 0$, $b_0 > 0$ *and* $m > \min\{n+1, p+2\}$.
(iv) $c_0 = 0$, $n \geqslant 0$, *or* $b_0 = 0$ *and* $m > 1$.
(v) $c_0 = 0$, $n < 0$, $b_0 > 0$ *and* $m > n+1$.
(vi) $c_0 > 0$, $n \geqslant 0$ *or* $b_0 = 0$, $m > 1$ *and* $m + p \geqslant 0$.
(vii) $c_0 > 0$, $n < 0$, $0 < b_0 \leqslant 2\sqrt{c_0}$, $m > n+1$ *and* $m + p > 2n$.
(viii) $c_0 > 0$, $n < 0$, $b_0 > 2\sqrt{c_0}$, $m > n+1$ *and* $m + p \geqslant 2n$.

This theorem follows from Theorems 1 and 2 and Lemma 14.

Note the similarity in structure between the results in Theorems 12 and 13. With one exception, if we replace n by $n+1$, p by $p+2$ and $\sqrt{mc_0/n}$ by $\sqrt{c_0}$ in the conclusions of Theorem 12 we obtain those of Theorem 13. The one exception is the marginal case $c_0 > 0$, $n < 1$, $b_0 = 2\sqrt{mc_0/n}$, $m > n$ and $m + p = 2n$.

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