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Alternative proof and interpretations for a recent state-dependent importance sampling scheme

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Abstract Recently, a state-dependent change of measure for simulating overflows in the two-node tandem queue was proposed by Dupuis et al. (Ann. Appl. Probab. 17(4):1306–1346, 2007), together with a proof of its asymptotic optimality. In the present paper, we present an alternative, shorter and simpler proof. As a side result, we obtain interpretations for several of the quantities involved in the change of measure in terms of likelihood ratios.

Keywords Importance sampling · Asymptotic optimality · Tandem queue

Mathematics Subject Classification (2000) 60K25 · 65C05

1 Introduction

Since the late 1980s, there has been an interest in the estimation of probabilities of rare overflow events in queueing networks using simulation, one of the main application areas being the performance analysis of telecommunication systems. In order to estimate such small probabilities efficiently, a technique known as importance sampling is often applied, where the model is simulated under an alternative probability measure under which the rare event becomes less rare. Conclusions about the probability of interest can be drawn by weighing the observations by the so-called likelihood ratio. The challenge then is to choose a good alternative measure for the simulation. One possible criterion is to choose a measure that is *asymptotically optimal* (or *asymptotically efficient*), which means that the required simulation time increases less than exponentially fast as the probability becomes small. Initial attempts used changes of measure that do not vary with the model's state; e.g., the arrival and service rates are replaced by other values, but these values are kept constant [9]. It turns out that already for a relatively simple queueing network problem, namely overflow of the total population of two queues in tandem, such a change of measure is not asymptotically optimal; see [3, 6]. In several publications [4, 10], state-dependent changes of measure were proposed for this two-node tandem queue and experimentally found to be asymptotically optimal; however, for none of them a rigid mathematical optimality proof is available.

In [5], Dupuis, Sezer and Wang introduce a state-dependent change of measure for several models, including the two-node tandem network. Their change of measure is based on game theory, which is used to derive an equation for the optimal change of measure, and the construction of an approximate solution to this equation. Their main and unique result is a proof that the change of measure associated to this approximate solution is asymptotically optimal.

Unfortunately, both the construction of the change of measure in [5], and especially the proof for its optimality, are rather lengthy and technical. In the present paper, we present a simpler proof of the asymptotic optimality of their change of measure. Furthermore, we use observations from our proof to provide alternative (i.e., non-game-theoretic)

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interpretations for some of the quantities and conditions used in the construction of the change of measure. Both of these contributions may be helpful to better understand the change of measure, and to extend these types of results to other models.

This paper is structured as follows. In Sect. 2 we present the two-node tandem model, fixing some notation and giving the associated simulation problem. In Sect. 3 we review the change of measure as proposed in [5]. Section 4 contains the main result of the paper, namely an alternative and shorter proof for the asymptotic optimality of this change of measure. Last but not least, we discuss our findings in Sect. 5, including interpretations for some of the functions involved in the change of measure, and the way in which our proof can be generalized to other models.

2 Model and preliminaries

In this section we introduce the model and some background on importance sampling; for more detailed explanations we refer to [5], whose notation we also use. Consider a tandem system of two $M/M/1$ queues, with arrival rate λ and service rates μ_1 and μ_2. The joint queue length process constitutes a continuous time Markov process, but since we are interested in the probability p_n that the total number of customers reaches n before 0 (starting at an empty system), we may as well consider the embedded discrete time Markov chain. This process, representing the state immediately after the jth transition epoch, will be denoted by $Z_j = (Z_{1,j}, Z_{2,j})$.

As in [5] we define vectors v_i, $i = 0, 1, 2$, in the directions that the process Z_j can jump, and let $\Theta[v_i]$ be the corresponding probabilities. Since we assume without loss of generality that $\lambda + \mu_1 + \mu_2 = 1$, we have $v_0 = (1, 0)$, $v_1 = (-1, 1)$ and $v_2 = (0, -1)$ with $\Theta[v_0] = \lambda$, $\Theta[v_1] = \mu_1$ and $\Theta[v_2] = \mu_2$. However, note that when queue k is empty, a transition v_k is impossible, $k = 1, 2$. To cope with this, the process Z_j is slightly modified, by introducing extra self-loop transitions with probability $\Theta[v_k]$ for states in $\{(n_1, n_2) : n_k = 0\}$, $k = 1, 2$.

As in [5], it will be convenient to work with the scaled process

$$X_j \equiv \frac{1}{n} Z_j,$$

which has the advantage that it suffices to consider the same set of states for any n. We define the interior of this set $D = \{(x_1, x_2) : x_i > 0, x_1 + x_2 < 1\}$, the so-called exit boundary $\partial_e = \{(x_1, x_2) : x_i \geq 0, x_1 + x_2 = 1\}$, the other boundaries $\partial_1 = \{(0, x_2) : 0 < x_2 < 1\}$ and $\partial_2 = \{(x_1, 0) : 0 < x_1 < 1\}$, and finally the entire relevant part of the state space $\bar{D} = D \cup \partial_e \cup \partial_1 \cup \partial_2$. Note that when $X_j \in \partial_k$, then $X_{j+1} = X_j$ with probability $\Theta[v_k] = \mu_k$, due to our modification.

We may now introduce τ_n, the first time that $Z_{1,j} + Z_{2,j}$ hits n, staying away from 0, as follows in terms of the scaled process X_j:

$$\tau_n = \inf\{t > 0 : X_t \in \partial_e, X_j \neq (0, 0) \text{ for } j = 1, \dots, t-1\}.$$

Notice that τ_n is a defective random variable, where $\tau_n = \infty$ will denote the event in which X_j hits $(0, 0)$ before ∂_e (i.e. the event in which $Z_{1,j} + Z_{2,j}$ visits 0 before n). We are interested in the probability p_n that the total number of customers reaches n before 0, starting at an empty system, which we can write as

$$p_n = \mathbb{P}[\tau_n < \infty \mid X_0 = (0, 0)].$$

Asymptotically, as n grows large, it is known that p_n decays exponentially fast, at some rate

$$\gamma = -\lim_{n\to\infty} n^{-1} \log p_n. \tag{1}$$

Since reversing the order of service rates has no influence on p_n, we will from now on assume $\mu_2 \leq \mu_1$, in which case we know $\gamma = -\log(\lambda/\mu_2)$, see [6].

Now suppose we estimate p_n by simulation, and let $I(A)$ be the indicator function of the event $\tau_n < \infty$ for a path $A = (X_j, j = 0, \dots, \tau)$ in any simulation run. If we perform simulations under the normal measure, starting at $X_0 = (0, 0)$, we clearly have $p_n = \mathbb{E}[I(A)]$. However, in order to speed up the simulation using importance sampling, we simulate under a (state-dependent) alternative measure $\mathbb{Q}$ which attributes a probability $\bar{\Theta}[v_i|x]$ to a transition in direction v_i if the current state of the process X_j is x. In this case the probability p_n can be found as

$$p_n = \mathbb{E}^{\mathbb{Q}}[L(A)I(A)],$$

where $\mathbb{E}^{\mathbb{Q}}$ denotes expectation under the new measure $\mathbb{Q}$, and $L(A)$ is the likelihood ratio of the path under consideration, i.e.,

$$L(A) = \frac{\mathbb{P}(A)}{\mathbb{Q}(A)} = \prod_{j=0}^{\tau_n - 1} \frac{\Theta[Y_j]}{\bar{\Theta}[Y_j|X_j]}, \tag{2}$$

where $Y_j = n(X_{j+1} - X_j)$, unless $X_{j+1} = X_j$, in which case $Y_j = v_k$ if $X_j \in \partial_k$.

In order to prove asymptotic optimality for the measure $\mathbb{Q}$, we need to show that

$$\lim_{n\to\infty} \frac{\log \mathbb{E}^{\mathbb{Q}}[L^2(A)I(A)]}{\log p_n} \geq 2,$$

where the expectation is again taken under the new measure $\mathbb{Q}$. This limit on the second moment ensures that the estimator's relative error grows subexponentially in n, which by definition is asymptotic optimality (cf. [9]). Using the above, this simplifies to

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}[L(A)I(A)] \leq -2\gamma, \tag{3}$$

where this expectation is taken under the normal measure $\mathbb{P}$. In order to prove (3), it is important to bound the likelihood ratio from above for the particular change of measure used. The precise form of this new measure, i.e., the form of the functions $\bar{\Theta}[v_i|x]$, is the subject of the next section.

3 Change of measure from Dupuis et al.

For the purpose of this paper, it suffices to describe what the change of measure proposed in [5] looks like without going into details about its derivation.

A central role in the change of measure from [5] is played by the function $W(x)$, defined for all $x \in \bar{D}$, which comes about as an approximate solution to a set of equations derived using game theory.

The function $W(x)$ is constructed in three steps. First, three affine functions $\bar{W}_k^\delta(x)$ are constructed, parameterized by some δ, as follows:

$$\bar{W}_k^\delta(x) = \langle r_k, x\rangle + 2\gamma - k\delta, \quad k = 1, 2, 3, \tag{4}$$

where $\langle\cdot\,,\cdot\rangle$ denotes the inner product, and the vectors r_i are given by

$$r_1 = 2\gamma(-1, -1);$$
$$r_2 = 2\gamma(-1, 0);$$
$$r_3 = (0, 0).$$

These affine functions have the property of satisfying a condition derived using game theory, namely that $\mathbb{H}(D\bar{W}_k^\delta) \geq 0$ (with equality for $k = 1$), where $D\bar{W}_k^\delta = r_k$ is the gradient of $\bar{W}_k^\delta(x)$ and $\mathbb{H}$ denotes a function known as the Hamiltonian. The precise definition and meaning of $\mathbb{H}$ are not important here and can be found in [5] but we note that its form may be found easily from (8) below.

Next, the minimum of these three affine functions is taken, producing a piecewise affine function $\bar{W}^\delta = \bar{W}_1^\delta \wedge \bar{W}_2^\delta \wedge \bar{W}_3^\delta$. Notice that we may decompose the set $\bar{D}$ into three regions, depending on which of the three functions $\bar{W}_k^\delta$ attains the minimum. With each of these regions, a constant (i.e. not state-dependent) change of measure can be associated, determined by the corresponding vector r_k as specified below. In fact, the constant change of measure associated with r_1 is precisely the state-independent one proposed by [9], in which the service rate of the bottleneck queue (which is μ_2 in our case) is interchanged with the arrival rate λ. A sketch of the function $\bar{W}^\delta$ is provided in Fig. 1; see

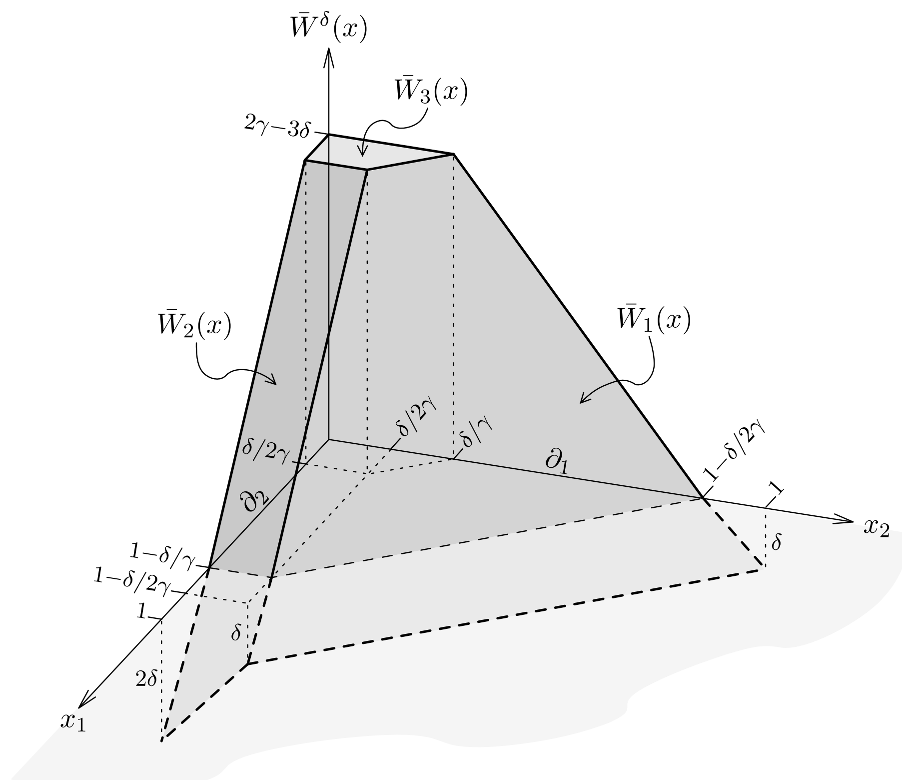

Fig. 1 Unmollified, piecewise affine function $\bar{W}^\delta(x)$

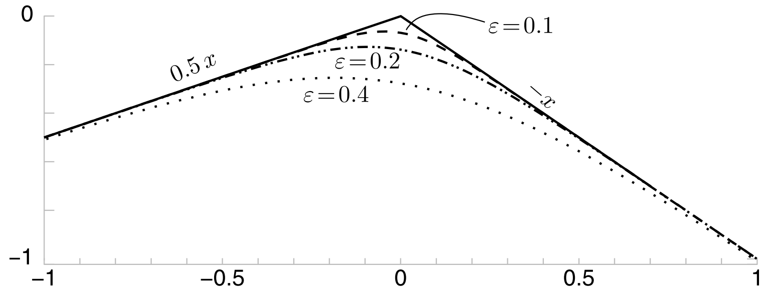

Fig. 2 Mollification example in one dimension: $W^\epsilon(x) = -\epsilon \log \sum_{k=1}^{2} e^{-\bar{W}_k(x)/\epsilon}$ versus x, with $\bar{W}_1(x) = x/2$ and $\bar{W}_2(x) = -x$

also Fig. 4 from [5]. Notice that the widths of the regions corresponding to r_2 and r_3 scale with the parameter δ.

Finally, a mollification procedure is applied, to make the resulting function W smooth along the boundaries of the three subsets of $\bar{D}$, and hence to make the transition from one type of measure (say determined by r_2) to another (say determined by r_1) not too sudden, as the path of the process X_j traverses $\bar{D}$. The specific mollification in [5], parameterized by ϵ, is given by

$$W^{\epsilon,\delta}(x) = -\epsilon \log \sum_{k=1}^{3} e^{-\bar{W}_k^\delta(x)/\epsilon} \tag{5}$$

and illustrated in Fig. 2. Note that as $\epsilon \to 0$, the function $W^{\epsilon,\delta}(x)$ simply converges to $\bar{W}^\delta(x)$. Moreover, the value of ϵ determines the 'smoothness' of $W^{\epsilon,\delta}(x)$ along the boundaries mentioned above. In the rest of the paper we will write $W(x)$ instead of $W^{\epsilon,\delta}(x)$ for brevity, since the parameters ϵ and δ are taken fixed, except in the very last step of the proof.

The state-dependent change of measure in each state x is strongly related to the *gradient* of $W(x)$ in x, which we will denote as $DW(x)$. In fact we can write this gradient as a state-dependent weighted average of the vectors r_k:

$$\begin{aligned} DW(x) &= \sum_{k=1}^{3} \rho_k(x) r_k \\ &\text{with } \rho_k(x) = \frac{e^{-\bar{W}_k^\delta(x)/\epsilon}}{\sum_j e^{-\bar{W}_j^\delta(x)/\epsilon}}. \end{aligned} \tag{6}$$

Proposition 3.2 in [5] associates to each vector p a change of measure as follows (with some minor abuse of notation for $\bar{\Theta}$):

$$\bar{\Theta}(p)[v_i] = N(p)\Theta[v_i]e^{-\langle p,v_i\rangle/2}, \quad i = 0, 1, 2, \tag{7}$$

with normalization constant

$$N(p) = \left[\sum_{i=0}^{2} \Theta[v_i]e^{-\langle p,v_i\rangle/2}\right]^{-1} = e^{\mathbb{H}(p)/2}. \tag{8}$$

The vector p may depend on the current state x, and can be interpreted as $DW(x)$. In fact, we can distinguish two ways in which a change of measure can be obtained from a given function $W(x)$ (see also Sect. 3.8.6 of [5]):

- For each state x, calculate the gradient $DW(x)$ and use $p = DW(x)$ in (7) to compute the new transition probabilities for the state x:

$$\begin{aligned} \bar{\Theta}[v_i|x] &= \bar{\Theta}(DW(x))[v_i] \\ &= \Theta[v_i]e^{-\langle DW(x),v_i\rangle/2}e^{\mathbb{H}(DW(x))/2}. \end{aligned} \tag{9}$$

- For each state x, use (6) to calculate the weighing factors $\rho_k(x)$ for each of the components $\bar{W}_k^\delta(x)$, and then define $\bar{\Theta}[v_i]$ as the accordingly weighted average of the $\bar{\Theta}(r_k)[v_i]$, $k = 1, 2, 3$, which are calculated using $p = D\bar{W}_k^\delta = r_k$ in (7); this results in (cf. (3.16) in [5])

$$\begin{aligned} \bar{\Theta}[v_i|x] &= \sum_k \rho_k(x)\bar{\Theta}(r_k)[v_i] \\ &= \sum_k \rho_k(x)\Theta[v_i]e^{-\langle r_k,v_i\rangle/2}e^{\mathbb{H}(r_k)/2}. \end{aligned} \tag{10}$$

In [5], the change of measure in (10) is used because of some practical advantages. In the next section we will build our proof firstly on (9), because the interpretation is easier for this change of measure; after that, we will show that essentially the same arguments also hold for change of measure (10).

Finally, we mention that the behavior of ϵ and δ as functions of n is crucial for desirable behavior of the change of measure(s). We will assume the following.

Assumption 1 The positive numbers ϵ and δ depend on n in such a way that the following four conditions are met:

$$\lim_{n\to\infty} \epsilon = 0, \tag{11}$$

$$\lim_{n\to\infty} \delta = 0, \tag{12}$$

$$\lim_{n\to\infty} n\epsilon = \infty, \tag{13}$$

$$\lim_{n\to\infty} \frac{\epsilon}{\delta} = 0. \tag{14}$$

Note that these conditions are the same as those in [5].

Remark 1 The specific function $W(x)$ that was found in [5] using game theory, and that satisfies all the above, essentially leads to a variant of the well-known, state-independent measure from [9] in which λ and μ_2 are interchanged (here corresponding to r_1), but the measure is modified such that visits to the horizontal boundary ∂_2 are no longer harmful for the likelihood ratio. This is done here by mollifying it with another measure (corresponding to r_2 here), the influence of which is only noticeable in a region close to ∂_2.

4 Asymptotic optimality

In this section we present our proof that both changes of measure (9) and (10) are asymptotically optimal, starting with (9). In order to prove (3) we start with the following lemma, which presents a decomposition of the likelihood $L(A)$ of a path A in terms of the Hamiltonian $\mathbb{H}$ and the (gradient of the) function $W(x)$. In the lemma, and in fact in most of the arguments below, we fix n, and hence ϵ and δ; only in the proof of the main results we will let $n\to\infty$.

Lemma 1 *The likelihood $L(A)$ of any path $A=(X_j, j=0,\dots,\tau)$ under change of measure (9) satisfies*

$$\begin{aligned}\log L(A) &= \frac{n}{2}\sum_{j=0}^{\tau-1}\langle DW(X_j), X_{j+1}-X_j\rangle\\ &\quad+\sum_{k=1}^{2}\frac{1}{2}\sum_{j=0}^{\tau-1}\langle DW(X_j), v_k\rangle 1\{X_j=X_{j+1}\in\partial_k\}\\ &\quad-\frac{1}{2}\sum_{j=0}^{\tau-1}\mathbb{H}(DW(X_j)).\end{aligned}\tag{15}$$

Proof From (9) we see that if $X_j\neq X_{j+1}$, the log likelihood ratio of the jth step is given by:

$$\begin{aligned}&\log\frac{\Theta[n(X_{j+1}-X_j)]}{\bar\Theta(DW(X_j))[n(X_{j+1}-X_j)]}\\ &\quad=\frac{n}{2}\langle DW(X_j), X_{j+1}-X_j\rangle-\frac{1}{2}\mathbb{H}(DW(X_j)).\end{aligned}$$

If on the other hand $X_j=X_{j+1}$ and $X_j\in\partial_k$, the log likelihood ratio of the jth step is given by

$$\begin{aligned}&\log\frac{\Theta[v_k]}{\bar\Theta(DW(X_j))[v_k]}\\ &\quad=\frac{1}{2}\langle DW(X_j), v_k\rangle-\frac{1}{2}\mathbb{H}(DW(X_j)).\end{aligned}$$

Combining these results completes the proof. □

Note that the lemma holds sample-path wise, for any path of length τ (including possible self-loop transitions at the boundaries). Most terms in (15) will turn out to vanish in the limit as $n\to\infty$; only the first term will give a real contribution. The following lemma shows that this term is in fact close to $\frac{n}{2}(W(X_\tau)-W(X_0))$, by giving an upper bound on the difference.

Lemma 2 *For any path $(X_j, j=0,\dots,\tau)$ under change of measure (9), the first term in (15) satisfies*

$$\begin{aligned}&\Bigg|\frac{n}{2}\sum_{j=0}^{\tau-1}\langle DW(X_j), X_{j+1}-X_j\rangle\\ &\quad-\frac{n}{2}(W(X_\tau)-W(X_0))\Bigg|\le\frac{17\gamma^2}{n\epsilon}\tau\end{aligned}\tag{16}$$

for sufficiently large $n\epsilon$.

Proof The mean-value theorem says that $W(x+y)-W(x)=\langle DW(x+\eta y), y\rangle$ for some η such that $0\le\eta\le 1$ (where η may depend on x, y, ϵ, and δ). Since we also know, cf. (6), that

$$DW(x+\eta y)=\sum_k\rho_k(x+\eta y)r_k\tag{17}$$

with

$$\rho_k(x+\eta y)=\frac{R_k(x)e^{-\langle r_k,y\rangle\eta/\epsilon}}{\sum_j R_j(x)e^{-\langle r_j,y\rangle\eta/\epsilon}}$$

and $R_k(x)=e^{-\bar W_k^\delta(x)/\epsilon}$, we can write

$$\begin{aligned}&W(x+y)-W(x)-\langle DW(x),y\rangle\\ &\quad=\frac{\sum_k R_k(x)e^{-\langle r_k,y\rangle\eta/\epsilon}\langle r_k,y\rangle}{\sum_k R_k(x)e^{-\langle r_k,y\rangle\eta/\epsilon}}-\frac{\sum_k R_k(x)\langle r_k,y\rangle}{\sum_k R_k(x)}\\ &\quad=\frac{\sum_k R_k(x)\langle r_k,y\rangle\big(e^{-\langle r_k,y\rangle\eta/\epsilon}\frac{\sum_i R_i(x)}{\sum_i R_i(x)e^{-\langle r_i,y\rangle\eta/\epsilon}}-1\big)}{\sum_k R_k(x)}.\end{aligned}$$

Thus, conveniently replacing k by the values of k that maximize or minimize relevant terms, we find

$$\begin{aligned}&\big|W(x+y)-W(x)-\langle DW(x),y\rangle\big|\\ &\quad\le|\langle DW(x),y\rangle|\\ &\qquad\times\max_k\left|e^{-\langle r_k,y\rangle\eta/\epsilon}\frac{\sum_i R_i(x)}{\sum_i R_i(x)e^{-\langle r_i,y\rangle\eta/\epsilon}}-1\right|\\ &\quad\le|\langle DW(x),y\rangle|\\ &\qquad\times\max\left(1-\frac{\min_k e^{-\langle r_k,y\rangle\eta/\epsilon}}{\max_k e^{-\langle r_k,y\rangle\eta/\epsilon}},\frac{\max_k e^{-\langle r_k,y\rangle\eta/\epsilon}}{\min_k e^{-\langle r_k,y\rangle\eta/\epsilon}}-1\right).\end{aligned}$$

In view of (6) and the definitions of r_k and v_i, we have $|DW(x)|\le\max_k|r_k|\le 2\gamma\sqrt{2}$ and $|X_{j+1}-X_j|\le\sqrt{2}/n$.

Thus, substituting X_j for x and $X_{j+1}-X_j$ for y, and using that $\langle u,v\rangle \le |u|\,|v|$, we obtain

$$\begin{aligned}&\left|W(X_{j+1})-W(X_j)-\langle DW(X_j),X_{j+1}-X_j\rangle\right|\\&\quad\le \frac{4\gamma}{n}\left(\frac{e^{4\gamma\eta/n\epsilon}}{e^{-4\gamma\eta/n\epsilon}}-1\right)\\&\quad=\frac{4\gamma}{n}\left(\frac{8\gamma\eta}{n\epsilon}+\mathcal{O}\left(\frac{8\gamma\eta}{n\epsilon}\right)^2\right)\le\frac{33\gamma^2}{n^2\epsilon}\end{aligned}$$

for sufficiently large $n\epsilon$. Finally, we conclude:

$$\begin{aligned}&\left|\sum_{j=0}^{\tau-1}\langle DW(X_j)\,,\ X_{j+1}-X_j\rangle-\big(W(X_\tau)-W(X_0)\big)\right|\\&\quad\le\sum_{j=0}^{\tau-1}\frac{33\gamma^2}{\epsilon n^2}=\frac{33\gamma^2\tau}{\epsilon n^2}.\end{aligned}$$

□

In the above lemma, the bound on the right hand side is proportional to the path length τ. In the final asymptotic optimality proof, this will lead to terms like $\mathbb{E}e^{\theta_n\tau_n}$, that need to grow at most subexponentially in n, when $\theta_n\to 0$. The following lemma shows this to be true in several steps that are interesting in their own right. Key to the result is that we consider the time-reversed process, which is also a tandem queue, with μ_1 and μ_2 interchanged. For this process we first consider the busy period σ (in discrete time), which we define here as the first entrance time of the process into state $(0,0)$, starting at $(1,0)$. Again we include possible self-loop transitions.

Lemma 3 (i) *For sufficiently small $\theta>0$ we have $\mathbb{E}e^{\theta\sigma}<\infty$, where σ is the busy period in a two-node tandem queue.*

(ii) *For any sequence $\theta_n\ge 0$ such that $\lim_{n\to\infty}\theta_n=0$ we have $\lim_{n\to\infty}\mathbb{E}e^{\theta_n\sigma}=1$.*

(iii) *For any sequence $\theta_n\ge 0$ such that $\lim_{n\to\infty}\theta_n=0$ we have $\lim_{n\to\infty}(1/2)\log\mathbb{E}e^{\theta_n\tau_n}=0$, where τ_n is the length of a path in a two-node tandem queue from any state (n_1,n_2) with $n_1+n_2=n$ to state $(0,0)$.*

(iv) *For any sequence $\theta_n\ge 0$ such that $\lim_{n\to\infty}\theta_n=0$ we have*

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}\left[e^{\theta_n\tau_n}|\tau_n<\infty\right]=0,$$

where τ_n is the path length of a successful path from state $(0,0)$ to some state (n_1,n_2) with $n_1+n_2=n$, without visiting $(0,0)$.

Proof (i) First, consider the corresponding continuous time Markov chain (CTMC), and define the random variable T as the busy period in this process, i.e., T is the first entrance time into state $(0,0)$, starting in state $(1,0)$. By Theorem 1 of [2], the relaxation time of this process is finite, which implies that the process is exponentially ergodic as defined in [1]. It follows by Lemma 6.3 in Chap. 6 of [1] that some $\vartheta>0$ exists such that $\mathbb{E}e^{\vartheta T}<\infty$.

To find the corresponding discrete-time result, let X_i, $i=1,2,\dots$, be the sojourn time in the CTMC between the ith and $(i+1)$st transition of the embedded discrete time Markov chain after leaving state $(0,0)$ (interpreting the first transition as the one at which the process leaves $(0,0)$). Due to the self loops in the embedded process, which correspond to virtual transitions in the CTMC of type v_k when queue k is empty, we have that the X_i are i.i.d. and exponentially distributed with rate $\lambda+\mu_1+\mu_2=1$. Because we have $T=\sum_{i=1}^{\sigma}X_i$, it now follows that

$$\mathbb{E}e^{\vartheta T}=\mathbb{E}\big[\big(\mathbb{E}e^{\vartheta X_1}\big)^\sigma\big]=\mathbb{E}\left[\left(\frac{1}{1-\vartheta}\right)^\sigma\right]=\mathbb{E}e^{-\log(1-\vartheta)\sigma}.$$

Since this exists for some $\vartheta\in]0,1[$, this completes the proof of part (i) by choosing $\theta\le-\log(1-\vartheta)$.

(ii) This follows immediately from part (i) by dominated convergence.

(iii) The path length τ_n can be written as $S_n+S_{n-1}+\cdots+S_1$, where S_i is the length of a path starting in a state (n_1,n_2) with $n_1+n_2=i$ until its first visit to a state (m_1,m_2) with $m_1+m_2=i-1$. We claim that S_i must be stochastically smaller than the busy period σ of the tandem system (in discrete time). To show this we consider two (discrete time) processes on the same probability space: Z_j starting in some state $Z_0=(n_1,n_2)$ with $n_1+n_2=i$, and $\bar{Z}_j$ starting in $\bar{Z}_0=(1,0)$. We claim that for any $j\ge 0$

$$\begin{aligned}&0\le(Z_{1,j}+Z_{2,j})-(\bar{Z}_{1,j}+\bar{Z}_{2,j})\le n_1+n_2-1\quad\text{and}\\&Z_{2,j}\ge\bar{Z}_{2,j}.\end{aligned}\tag{18}$$

Clearly, (18) is true for $j=0$. Furthermore, for each of the three transition types, one easily verifies that if (18) is true before the jth transition, it is also true after that transition, so by induction (18) is true for all $j\ge 0$. It follows that if $\bar{Z}_j=(0,0)$ for some j, then $Z_{1,j}+Z_{2,j}\le n_1+n_2-1$. Thus, regardless of its initial state (n_1,n_2), the process Z_j will always reach some state (m_1,m_2) with $m_1+m_2=i-1$ *at or before* the time the process $\bar{Z}_j$ reaches $(0,0)$ for the first time.

Introducing i.i.d. copies σ_i of σ, it now follows that

$$\begin{aligned}\frac{1}{n}\log\mathbb{E}e^{\theta_n\tau_n}&\le\frac{1}{n}\log\mathbb{E}e^{\theta_n\sum_{i=1}^{n}\sigma_i}\\&=\frac{1}{n}\log\left(\mathbb{E}e^{\theta_n\sigma}\right)^n=\log\mathbb{E}e^{\theta_n\sigma}.\end{aligned}$$

Using part (ii) of this lemma one sees that as $\theta_n\to 0$, this exists and goes to zero. On the other hand, $\frac{1}{n}\log\mathbb{E}e^{\theta_n\tau_n}\ge\frac{1}{n}\log\mathbb{E}e^0=0$, which completes the proof of part (iii).

(iv) Consider the time-reversed Markov chain for this system, see e.g. Theorem 1.12 in [8]. It can be easily verified that this is again a two-node tandem queue, but with the first and second queues interchanged, i.e. with n_1 replaced by n_2 and μ_1 by μ_2. As a consequence, the conditional path length of interest is the same as the length of a path in the reversed process towards state (0, 0), starting from any state (n_1, n_2) with $n_1 + n_2 = n$, given that it does not visit any such states in between. Since such a path is always shorter than *any* path from (n_1, n_2) with $n_1 + n_2 = n$ to state (0, 0), we can apply part (iii) of the lemma to the time-reversed system, which gives the desired result. □

We are now ready to prove the main theorems.

Theorem 1 *Under Assumption 1, change of measure (9) is asymptotically optimal.*

Proof Fixing n, and hence ϵ and δ, we first provide some relevant bounds on the last two terms in Lemma 1. For the last term, note that the first claim of Lemma B.1 in [5] states that $\mathbb{H}(DW(x)) \geq 0$ for all $x \in D$, not allowing x to be on the boundaries ∂_1 or ∂_2. However, its proof does not use this restriction, so the claim holds also for boundary states, which implies

$$-\frac{1}{2}\sum_{j=0}^{\tau-1}\mathbb{H}(DW(X_j)) \leq 0. \tag{19}$$

For the second term we use the third claim of Lemma B.1 in [5] where we have for $x \in \partial_k$, $k = 1, 2$, that

$$\langle DW(x), v_k\rangle \leq 2\gamma e^{-\delta/\epsilon},$$

from which we immediately obtain the crude bound

$$\sum_{k=1}^{2}\frac{1}{2}\sum_{j=0}^{\tau-1}\langle DW(X_j), v_k\rangle 1\{X_j = X_{j+1} \in \partial_k\} \leq \gamma e^{-\delta/\epsilon}\tau. \tag{20}$$

Note that the first and third claims from Lemma B.1 in [5] are the only parts of the asymptotic optimality proof in [5] that we use, and that these claims follow immediately from the properties of the functions $\mathbb{H}$ and W.

As a result, we now have for any successful path $A = (X_j, j = 0, \ldots, \tau_n)$ by Lemmas 1 and 2 and the above, that

$$\log L(A) \leq \frac{n}{2}(W(X_{\tau_n}) - W(X_0)) + \left(\gamma e^{-\delta/\epsilon} + \frac{17\gamma^2}{\epsilon n}\right)\tau_n.$$

The value of $W(X_0)$ can be bounded directly from (5):

$$\begin{aligned} W(X_0) &= -\epsilon\log\big(e^{(-2\gamma+\delta)/\epsilon} + e^{(-2\gamma+2\delta)/\epsilon} \\ &\quad + e^{(-2\gamma+3\delta)/\epsilon}\big) \\ &\leq -\epsilon\log(e^{(-2\gamma+\delta)/\epsilon}) = 2\gamma - \delta \end{aligned}$$

and

$$W(X_0) \geq -\epsilon\log(3e^{(-2\gamma+3\delta)/\epsilon}) = 2\gamma - \epsilon\log(3) - 3\delta.$$

Using $\langle X_{\tau_n}, r_1\rangle = -2\gamma$, $-2\gamma \leq \langle X_{\tau_n}, r_2\rangle \leq 0$, and $\langle X_{\tau_n}, r_3\rangle = 0$, a similar calculation yields

$$-\epsilon\log(3) - 3\delta \leq W(X_{\tau_n}) \leq -\delta.$$

Hence $W(X_{\tau_n}) - W(X_0) \leq -2\gamma + 2a(n)$, where $a(n)$ is such that $\lim_{n\to\infty} a(n) = 0$, so that we arrive at

$$\log L(A) \leq -n\gamma + na(n) + b(n)\tau_n$$

with

$$b(n) = \gamma e^{-\delta/\epsilon} + \frac{17\gamma^2}{\epsilon n}.$$

Thus we find immediately for *any* path A,

$$\begin{aligned} &\frac{1}{n}\log\mathbb{E}[L(A)I(A)] \\ &\quad = \frac{1}{n}\log\big(\mathbb{E}[L(A)|I(A)=1]\,\mathbb{P}[I(A)=1]\big) \\ &\quad \leq \frac{1}{n}\log\big(\mathbb{E}[e^{-n\gamma+na(n)+b(n)\tau_n} \mid \tau_n < \infty]p_n\big) \\ &\quad = \frac{1}{n}\big(-n\gamma + na(n) + \log\mathbb{E}[e^{b(n)\tau_n} \mid \tau_n < \infty] + \log p_n\big) \\ &\quad = -\gamma + a(n) + \frac{1}{n}\log\mathbb{E}[e^{b(n)\tau_n} \mid \tau_n < \infty] + \frac{1}{n}\log p_n. \end{aligned}$$

Due to the constraints (13) and (14), $\lim_{n\to\infty} b(n) = 0$, so we can apply the last part of Lemma 3 and (1) to conclude that

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}[L(A)I(A)] \leq -2\gamma,$$

as needed. □

Theorem 2 *Under Assumption 1, change of measure (10) is asymptotically optimal.*

Proof The likelihood ratio for a transition v from any state x under change of measure (10) satisfies

$$\log \frac{\Theta[v]}{\sum_k \rho_k(x)\Theta[v]e^{-\langle r_k,v\rangle/2}e^{\mathbb{H}(r_k)/2}}$$
$$= -\log \sum_k \rho_k(x)e^{-\langle r_k,v\rangle/2}e^{\mathbb{H}(r_k)/2}$$
$$\le -\sum_k \rho_k(x)\log e^{-\langle r_k,v\rangle/2} = \langle DW(x),v\rangle/2, \tag{21}$$

where the inequality holds due to the concavity of the logarithm (note that $\sum_k \rho_k(x) = 1$), and the fact that the vectors r_k are such that $\mathbb{H}(r_k) \ge 0$. Summing the above over all steps of a sample path A, we get exactly the same expression as the first two terms in the right-hand side of (15). We may now copy the proof of Theorem 1, except that the term (19) is not present. Thus, the upper bound on $\mathbb{E}[L(A)I(A)]$ for change of measure (9), as found in the proof of Theorem 1, is also an upper bound on $\mathbb{E}[L(A)I(A)]$ for change of measure (10). Hence, the latter is also asymptotically optimal. □

5 Discussion

5.1 Interpretation

In our asymptotic optimality proof, we essentially split the likelihood ratio of any sample path into two components: a *dominant* term that depends *only* on the start and end points of the path, and *remaining* terms which depend on the specific shape of the path, but which are typically small compared to the dominant term. In the proof in Sect. 4, this separation is not completely explicit: the dominant term shows up in Lemma 2 as $W(X_\tau) - W(X_0)$.

The identification of the dominating term emphasizes the fact that the likelihood ratio of a successful sample path is largely independent of the exact shape of the path. In particular, it is largely independent of the presence of cycles. The importance of this for a good performance of the estimator has been discussed before, see e.g. [7].

The remainder terms consist of two components: (1) terms that are present even if the function $W(x)$ would be affine—or in other words if we consider $W(x)$ locally and set $\epsilon = 0$ so that it is replaced by one of its constituent functions $\bar{W}_k^\delta(x)$; (2) additional components that are due to the mollification. Each of these will be discussed in some detail below.

5.1.1 Terms for affine W

The terms in this category are of two types: terms of the form $\mathbb{H}(DW)$, and terms of the form $\langle DW, v_i\rangle$ for boundary states. They can be interpreted as the likelihood ratio of a cyclic path, since for a cyclic path the dominant term, depending only on the beginning and end state, is zero. For cyclic subpaths containing τ steps that are entirely in the interior, we simply have that their log likelihood equals $-\tau\mathbb{H}(DW)/2$, while visits to boundary states introduce extra terms $\langle DW, v_i\rangle$. Thus, the conditions $\mathbb{H}(DW) \ge 0$ and $\langle DW, -v_i\rangle \ge 0$ from [5] are equivalent to the likelihood ratio of cycles being at most 1.

Thus, we have the following interpretations in case of an affine function W:

- $\mathbb{H}(DW)$ determines the likelihood ratio of cyclic paths in the interior;
- Boundary conditions on $\langle DW, -v_i\rangle$ co-determine the likelihood ratio of a cycle containing a boundary state;
- If the above two are negligible, the difference in W between two states is the likelihood ratio of any path connecting those states.

5.1.2 Terms related to the mollification

When mollification is used to "glue together" the different affine functions $\bar{W}_k^\delta(x)$, each of the three terms above gets an extra component:

- Even if $\mathbb{H}(D\bar{W}_k^\delta) = 0$ for each of the composing $\bar{W}_k^\delta$'s, the mollified W may have $\mathbb{H}(DW) \ne 0$, and thus cycles in the interior may have a non-zero contribution to the log likelihood ratio. This contribution vanishes as $\epsilon \to 0$.
- $\langle DW, -v_i\rangle$ may become negative, as pointed out in Sect. 3.8.3 of [5]. However, the effect of this vanishes (as $\delta, \epsilon \to 0$) when δ is large compared to ϵ, see (20).
- Since W is not a purely affine function, the equality $\langle DW(X_j), X_{j+1} - X_j\rangle = W(X_{j+1}) - W(X_j)$ (which forms the basis of the dominant component discussed in the beginning of this section) is only approximately true; see also Lemma 2. This can also be related to cyclic subpaths. Consider for instance a three-step cycle $(X_j, X_{j+1}, X_{j+2}, X_{j+3})$ where $X_{j+3} = X_j$. Its log likelihood ratio contains a term

$$\langle DW(X_j), X_{j+1} - X_j\rangle + \langle DW(X_{j+1}), X_{j+2} - X_{j+1}\rangle + \langle DW(X_{j+2}), X_j - X_{j+2}\rangle.$$

  The error made in the approximation depends on the step sizes $X_{j+1} - X_j$ in relation to the rate at which the gradient DW changes. The former are proportional to $1/n$ due to the scaling used, while the latter is proportional to ϵ due to the mollification. Hence, ϵ should be large compared to $1/n$ to ensure that the contribution to the likelihood of cyclic subpaths is nearly zero.

The above three observations provide intuitive justification for conditions (11), (14), and (13), respectively. The remaining condition (12) ensures that $W(X_\tau)$ does not vary too much over all possible final states X_τ (cf. Fig. 1), and thus that the dominant term in the likelihood ratio has little variance.

5.2 Generalization

Our asymptotic optimality proof in Sect. 4 is specific to the two-node tandem queue. However, we believe that the approach can be used in many other cases.

Already in [5], the game-theory-based method is applied to several other examples of (Jackson) networks. In each of those cases, the final change of measure is related to a mollified piecewise-affine function $W(x)$, in the same way as in the two-node tandem case. In particular, the decomposition as in Lemma 1, can be extended to these cases immediately. One situation that needs more attention, is that in which the boundary condition $\langle DW, v_i\rangle = 0$ is replaced by one in terms of a so-called boundary Hamiltonian. Lemma 2 can also be generalized easily to other measures. Lemma 3 for path lengths now uses results that are specific to the two-node tandem queue, so this lemma will need more work to generalize, depending on the model of interest.

We like to point out that the changes of measure to be used need not be directly based on (or determined by) the game-theoretic framework. The change of measure from [5] has a clear structure, being essentially the state-independent change of measure from [9], but gradually replaced by another measure near the 'harmful' boundary; see also Remark 1 at the end of Sect. 3. Similar constructions could be thought of in other models without invoking game theory and constructing proper subsolutions. Their asymptotic optimality might be proved using a likelihood-ratio calculation similar to the one given in the present paper.

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