ON THE TYPE AND THE STANDARDFORM OF INDUCED UA REPRESENTATIONS

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ABSTRACT. We derive to which of the three types an irreducible UA representation which is obtained with the procedure of generalised induction belongs. We analyse the question whether or not the irreducible induced UA representations are on standardform. The results hold as well for induced PUA representations.

1. INTRODUCTION

It is well known from Wigner (1959) that the irreducible unitary-antiunitary (UA) representations of a group are naturally divided into three types and that they can be chosen from their equivalence classes in such a way that they have a special form, called standardform. It is very important if one deals with irreducible UA representations that one knows to which type they belong and that they are on standardform; for instance the lemma of Schur and the orthogonality relations for matrix elements, generalised for irreducible UA representations, are different for each of the three types and can be formulated suitably only for irreducible UA representations which are on standardform (see for instance [1])

A powerful method for the construction of a complete set of inequivalent irreducible UA representations of a group is the procedure of generalised induction, given by Shaw and Lever [2]. The aim of this paper is to derive to which of the types an irreducible UA representation which is obtained with this procedure belongs and to analyse the question under which conditions the irreducible induced UA representations are on standardform.

Although the most general form of representations occurring in physical applications are projective (P)UA representations rather than UA representations we restrict ourselves here to UA representations in order to avoid tedious calculations with factor systems. It is, however, a straightforward exercise to show that all the results of this paper hold as well for induced PUA representations.

2. PRELIMINARIES

Let G be a finite group and G_0 a subgroup of G of index 2. A UA representation of G with

respect to G_0 is a mapping D from G into the unitary matrices of some dimension n such that

$$D(g)D^g(g') = D(gg'), \quad \forall g, g' \in G \quad (2.1)$$

where D^g is defined by

$$D^g = \begin{cases} D & \text{if } g \in G_0 \\ D^* & \text{if } g \notin G_0 \end{cases}. \quad (2.2)$$

In the following we shall drop the phrase 'with respect to G_0 '.

Two UA representations D_1 and D_2 of G are equivalent if there exists a unitary matrix U such that

$$D_1(g)U^g = UD_2(g) \quad \forall g \in G. \quad (2.3)$$

A UA representation D is reducible if it is equivalent with a UA representation D' which has the form

$$D'(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix} \quad \forall g \in G. \quad (2.4)$$

According to Wigner [3] the irreducible UA representations are divided into three types: the restriction $D \downarrow G_0$ of an irreducible UA representation D of G to G_0 , which is a unitary representation of G_0 , is irreducible (Type I), is reducible into two equivalent irreducible components (Type II), or is reducible into two inequivalent irreducible components (Type III).

Let a be a fixed element of $G \setminus G_0$. An irreducible UA representation of G of Type II is equivalent with a UA representation which has the form

$$D(g) = \begin{pmatrix} \Delta(g) & 0 \\ 0 & \Delta(g) \end{pmatrix} \quad \forall g \in G_0; \quad D(a) = \begin{pmatrix} 0 & U \\ -U & 0 \end{pmatrix}. \quad (2.5)$$

An irreducible UA representation of G of Type III is equivalent with a UA representation which has the form

$$D(g) = \begin{pmatrix} \Delta(g) & 0 \\ 0 & \bar{\Delta}(g) \end{pmatrix} \quad \forall g \in G_0; \quad D(a) = \begin{pmatrix} 0 & \Delta(a^2) \\ 1 & 0 \end{pmatrix} \quad (2.6)$$

where $\bar{\Delta}(g)$ is given by $\Delta(g) = \Delta^*(a^{-1}ga)$.

An irreducible UA representation is said to be on standard form if it is of Type I, if it is of Type II and satisfies eqn. (2.5) or if it is of Type III and satisfies eqn. (2.6). So each irreducible UA representation is equivalent with an irreducible UA representation which is on standard form.

Now let H be an invariant subgroup of G which belongs to G_0 , let K be the quotient group G/H and let K_0 be the quotient group G_0/H . The elements of H will be denoted by $a, b, c, \dots$ and elements of K by $\alpha, \beta, \gamma, \dots$. Take for each coset $\alpha \in K$ a representative $r(\alpha)$, and let $r(e) = e$, if e and e are the unit elements of K and G , respectively. Then each element of G can uniquely be written as (a, α) if we define

$$(a, \alpha) = ar(\alpha). \quad (2.7)$$

We suppose that the irreducible unitary representations of H are known. From these a complete set of inequivalent irreducible UA representations of G is obtained with the procedure of generalised induction [2], which will be briefly reviewed here.

Let Δ be an irreducible unitary representation of H . For each $\alpha \in K$ define the irreducible unitary representation Δ_α of H by

$$\Delta_\alpha(a) = \Delta^\alpha(r^{-1}(\alpha)ar(\alpha)), \quad (2.8)$$

where Δ^α is a shorthand symbol for $\Delta^{(e, \alpha)}$.

The elements $\alpha \in K$ for which Δ and Δ_α are equivalent form a subgroup $\bar{K}$ of K . The little group $L(G)$ of Δ is defined by

$$L(G) = \{(a, \alpha) | a \in H, \alpha \in \bar{K}\} \quad (2.9)$$

$L(G)$ is a subgroup of G which contains H but is not necessarily an invariant subgroup. The set $\{\Delta_\alpha | \alpha \in K\}$ is called the G -orbit of Δ .

For any $\alpha \in \bar{K}$ there exists a unitary operator $U(\alpha)$ such that

$$\Delta(a) = U(\alpha)\Delta_\alpha(a)U^{-1}(\alpha) \quad \forall a \in H. \quad (2.10)$$

Define $V(a, \alpha)$ by

$$V(a, \alpha) = \Delta(a)U(\alpha) \quad (2.11)$$

for each $(a, \alpha) \in L(G)$. Then it can be shown that

$$V(a, \alpha)V^\alpha(b, \beta) = \mu(\alpha, \beta)V((a, \alpha)(b, \beta)) \quad (2.12)$$

for some mapping $\mu: \bar{K} \times \bar{K} \rightarrow U(1)$, which means that V is a $PU(A)$ representation of $L(G)$ (i.e. a PUA representation if $L(G) \not\subseteq G_0$ and a PU representation if $L(G) \subseteq G_0$) whose factor system μ is a factor system of K_0 .

Now let E be an irreducible $PU(A)$ representation of $\bar{K}$ with factor system μ^* :

$$E(\alpha)E^\alpha(\beta) = \mu^*(\alpha, \beta)E(\alpha\beta) \quad (2.13)$$

then, with the definition

$$\mathcal{D}(a, \alpha) = V(a, \alpha) \otimes E(\alpha) \quad \forall a \in H, \forall \alpha \in \bar{K} \quad (2.14)$$

$\mathcal{D}$ is an irreducible $U(A)$ representation of $L(G)$.

Let $g_1 = e, g_2, \dots, g_p$ be left-coset-representations of G with respect to $L(G)$. Then $G = \sum_{i=1}^p g_i L(G)$.

For each $U(A)$ rep $\mathcal{D}$ of $L(G)$ the induced UA rep $\mathcal{D} \uparrow G$ is defined by

$$(\mathcal{D} \uparrow G)_{j_l, k_s}(g) = \mathcal{D}_{l's}^{g_j}(g_j^{-1} g g_k) \delta(g_j^{-1} g g_k, L(G)) \quad (2.15)$$

where $\delta(g, L(G))$ is defined by

$$\delta(g, L(G)) = \begin{cases} 1 & \text{if } g \in L(G) \\ 0 & \text{if } g \notin L(G). \end{cases}$$

A complete set of inequivalent irreducible UA representations is obtained in the following way:

- (i) Take one irreducible unitary representation Δ of H from each G -orbit.
- (ii) For each Δ , obtain the $U(A)$ representations of $L(G)$ given by eqn. (2.15) where E runs through a complete set of inequivalent irreducible $PU(A)$ representations of $\bar{K}$ with factor system μ^* .
- (iii) Induce these $U(A)$ representations to G .

Note that the irreducible unitary representations of G_0 can be obtained from the irreducible unitary representations of H in exactly the same way: one only has to replace G by G_0 and K by K_0 in the method described above. In fact, we wrote $L(G)$ and G -orbit instead of just L and orbit in order to be able to distinguish between $L(G)$ and $L(G_0)$, G -orbit and G_0 -orbit.

3. THE TYPE AND THE STANDARD FORM OF INDUCED UA REPRESENTATIONS

Let Δ be an irreducible unitary representation of H , $L(G)$ its little group, $\mathcal{D}$ the irreducible $U(A)$ representation of $L(G)$ given by eqn. (2.14) where E is an irreducible $PU(A)$ representation of $\bar{K}$ with factor system μ^* , and $\mathcal{D} \uparrow G$ the irreducible UA representation of G given by eqn. (2.15). The first question we will answer in this section is: what is the type of $\mathcal{D} \uparrow G$? The second question is: can we manage that $\mathcal{D} \uparrow G$ is on standardform?

To answer these questions we have to distinguish two different cases: $L(G)$ belongs to G_0 or $L(G)$ does not belong to G_0 . Let us first consider the case that $L(G)$ belongs to G_0 . Then $L(G)$ equals $L(G_0)$. The G -orbit of Δ contains two G_0 -orbits of Δ : the G_0 -orbit of Δ and the G_0 -orbit of $\bar{\Delta}$. The restriction of $\mathcal{D} \uparrow G$ to H (notation: $(\mathcal{D} \uparrow G) \downarrow H$) contains as irreducible components only elements of the G -orbit of Δ ; each element of this orbit occurs an equal number of times. But since $((\mathcal{D} \uparrow G) \downarrow G_0) \downarrow H = (\mathcal{D} \uparrow G) \downarrow H$ the restriction of $(\mathcal{D} \uparrow G) \downarrow G_0$ to H contains as irreducible components the elements of two different G_0 -orbits. This means that $(\mathcal{D} \uparrow G) \downarrow G_0$ must contain at least two inequivalent irre-

ducible components; so $\mathcal{D} \uparrow G$ must be of Type III.

Let $g_1 = e, g_2, \dots, g_p$ be left-coset-representatives of G_0 with respect to $L(G)$. Then we may choose the left-coset-representatives of G with respect to $L(G)$ to be $g_1 = e, g_2, \dots, g_p, a, ag_2, \dots, ag_p$. Define

$$g_{jm} (j = 1, \dots, p; m = 1, 2) \text{ by } g_{j1} = g_j \text{ and } g_{j2} = ag_j.$$

Then $G = \sum_{j=1}^p \sum_{m=1}^2 g_{jm} L(G)$. Equation (2.15) can now be written as

$$(\mathcal{D} \uparrow G)_{jmt, kns}(g) = \mathcal{D}_{ts}^{g_{jm}}(g_{jm}^{-1} g g_{kn}) \delta(g_{jm}^{-1} g g_{kn}, L(G)). \quad (3.1)$$

For $g \in G_0$ the blocks $(\mathcal{D} \uparrow G)_{j1t, k2s}(g)$ and $(\mathcal{D} \uparrow G)_{j2t, k1s}(g)$ are equal to zero;

$$(\mathcal{D} \uparrow G)_{j1t, k1s}(g) = \mathcal{D}_{ts}(g_{j1}^{-1} g g_{k1}) \delta(g_{j1}^{-1} g g_{k1}, L(G_0)) = (\mathcal{D} \uparrow G_0)_{jt, ks}(g) \quad (3.2)$$

and

$$(\mathcal{D} \uparrow G)_{j2t, k2s}(g) = \mathcal{D}_{ts}^*(g_{j2}^{-1} g g_{k2}) \delta(g_{j2}^{-1} g g_{k2}, L(G_0)) = (\mathcal{D} \uparrow G_0)_{jt, ks}^*(a^{-1} g a). \quad (3.3)$$

The blocks $(\mathcal{D} \uparrow G)_{j1t, k1s}(a)$ and $(\mathcal{D} \uparrow G)_{j2t, k2s}(a)$ are equal to zero;

$$(\mathcal{D} \uparrow G)_{j1t, k2s}(a) = \mathcal{D}_{ts}(g_{j1}^{-1} a g_{k2}) \delta(g_{j1}^{-1} a g_{k2}, L(G_0)) = \mathcal{D}_{ts}(g_{j1}^{-1} a^2 g_{k1}) \delta(g_{j1}^{-1} a^2 g_{k1}, L(G_0)) \quad (3.4)$$

and

$$(\mathcal{D} \uparrow G)_{j2t, k1s}(a) = \mathcal{D}_{ts}^*(g_{j2}^{-1} a g_{k1}) \delta(g_{j2}^{-1} a g_{k1}, L(G_0)) = \mathcal{D}_{ts}^*(g_j^{-1} g_k) \delta(g_j^{-1} g_k, L(G_0)) = \delta_{jk} \delta_{ts}. \quad (3.5)$$

So with this choice of left-coset-representatives we obtained the following blockform for $\mathcal{D} \uparrow G$:

$$\begin{aligned} (\mathcal{D} \uparrow G)(g) &= \begin{pmatrix} (\mathcal{D} \uparrow G_0)(g) & 0 \\ 0 & (\mathcal{D} \uparrow G_0)^*(a^{-1} g a) \end{pmatrix} \quad \forall g \in G_0; \\ (\mathcal{D} \uparrow G)(a) &= \begin{pmatrix} 0 & (\mathcal{D} \uparrow G_0)(a^2) \\ \mathbb{I} & 0 \end{pmatrix} \end{aligned} \quad (3.6)$$

which is exactly the standardform for an irreducible UA representation of Type III. It is obvious that with an arbitrary choice of the left-coset-representatives $\mathcal{D} \uparrow G$ is not on standardform in general.

Now let us consider the case that $L(G)$ does not belong to G_0 . Then $L(G_0)$ is equal to

$L(G) \cap G_0$ and is a subgroup of $L(G)$ of index 2. The G -orbit of Δ is equal to the G -orbit of Δ . Let $\bar{K}_0$ be the quotient group $L(G_0)/H$. $\bar{K}_0$ is a subgroup of $\bar{K}$ of index 2. Let α_0 be an element of $\bar{K} \setminus \bar{K}_0$. The element $a \in G \setminus G_0$ we choose to be $a = (e, \alpha_0)$. The left-coset-representatives $g_1 = e, g_2, \dots, g_p$ of G with respect to $L(G)$ are chosen to belong to G_0 , which means that they are also left-coset-representatives of G_0 with respect to $L(G_0)$. This means that the relation

$$(\mathcal{D} \uparrow G) \downarrow G_0 = (\mathcal{D} \downarrow L(G_0)) \uparrow G_0 \quad (3.7)$$

is valid with this choice of left-coset-representatives. $\mathcal{D}$ is given by eqn. (2.14) where E is an irreducible PUA representation of $\bar{K}$ with respect to $\bar{K}_0$ with factor system μ^* .

The type and the standardform of an irreducible PUA representation is defined in the same way as the type and the standardform of an irreducible UA representation, the only difference being the appearance of the factor system in eqn. (2.6) which reads

$$D(g) = \begin{pmatrix} \Delta(g) & 0 \\ 0 & \bar{\Delta}(g) \end{pmatrix} \quad \forall g \in G_0; \quad D(a) = \begin{pmatrix} 0 & \sigma(a, a)\Delta(a^2) \\ \mathbb{I} & 0 \end{pmatrix} \quad (3.8)$$

where $\bar{\Delta}(g)$ is given by $\bar{\Delta}(g) = \sigma^*(a, a^{-1}ga)\sigma(g, a)\Delta^*(a^{-1}ga)$ and σ is the factor system of D .

$E \downarrow \bar{K}_0$ is irreducible, consists of two equivalent irreducible components or consists of two inequivalent irreducible components if E is of Type I, II or III, respectively. From eqn. (2.14) it then follows that also $\mathcal{D} \downarrow L(G_0)$ is irreducible, consists of two equivalent irreducible components or consists of two inequivalent components respectively. But from eqn. (3.7) it then follows that the same holds for $(\mathcal{D} \uparrow G) \downarrow G_0$. This shows that $\mathcal{D} \uparrow G$ is of Type I, II or III if E is of Type I, II or III, respectively.

Suppose that E is of Type II and is on standardform. Then we may write

$$E(\alpha) = \begin{pmatrix} F(\alpha) & 0 \\ 0 & F(\alpha) \end{pmatrix} \quad \forall \alpha \in K_0; \quad E(\alpha_0) = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \quad (3.9)$$

or, using double indices

$$E(\alpha)_{mp, nq} = F(\alpha)_{mn} \delta_{pq} \quad \forall \alpha \in K_0; \quad E(\alpha_0)_{mp, nq} = A_{mn}(\delta_{1p} \delta_{2q} - \delta_{1q} \delta_{2p}) \quad (3.10)$$

for $p, q = 1, 2; m, n = 1, \dots, \frac{1}{2} \dim E$ and where F is an irreducible PU representation of $\bar{K}_0$ with factor system μ^* .

Equation (2.14) can be written as

$$\mathcal{D}(a, \alpha)_{kmp, \ell nq} = V(a, \alpha)_{k\ell} F(\alpha)_{mp, nq} \quad \forall a \in H, \quad \forall \alpha \in \bar{K}. \quad (3.11)$$

Without loss of generality we may assume that the left-coset-representatives $g_1 = e, g_2, \dots, g_p$ are given by $(e, \alpha_1), (e, \alpha_2), \dots, (e, \alpha_p)$ where $\alpha_1 = e, \alpha_2, \dots, \alpha_p$ are left-coset-representatives of K_0 with respect to $\bar{K}_0$. Then eqn. (2.15) can be written as:

$$(\mathcal{D} \uparrow G)_{ikmp, j\ell nq}(a, \alpha) = \mathcal{D}_{kmp, \ell nq}((e, \alpha_1)^{-1}(a, \alpha)(e, \alpha_i))\delta(\alpha_i^{-1}\alpha\alpha_j, \bar{K}). \quad (3.12)$$

For $\alpha \in \bar{K}_0$ this becomes, with eqns. (3.10) and (3.11)

$$(\mathcal{D} \uparrow G)_{ikmp, j\ell nq}(a, \alpha) = V_{k\ell}((e, \alpha_1)^{-1}(a, \alpha)(e, \alpha_j))F(\alpha_i^{-1}\alpha\alpha_j)_{mn}\delta_{pq}\delta(\alpha_i^{-1}\alpha\alpha_j, \bar{K}_0). \quad (3.13)$$

So if we write $\mathcal{D} \uparrow G$ in blockform with respect to the indices p and q we obtain

$$(\mathcal{D} \uparrow G)(a, \alpha) = \begin{pmatrix} (\mathcal{D}' \uparrow G_0)(a, \alpha) & 0 \\ 0 & (\mathcal{D}' \uparrow G_0)(a, \alpha) \end{pmatrix} \quad \forall a \in H, \quad \forall \alpha \in \bar{K}_0, \quad (3.14)$$

where $\mathcal{D}'$ is the irreducible unitary representation of $L(G_0)$ given by

$$\mathcal{D}'(a, \alpha) = (V \downarrow L(G_0))(a, \alpha) \otimes F(\alpha) \quad \forall a \in H, \quad \forall \alpha \in \bar{K}_0. \quad (3.15)$$

From the eqns. (3.11) and (3.12) it follows

$$(\mathcal{D} \uparrow G)_{ikmp, j\ell nq}(e, \alpha_0) = V_{k\ell}((e, \alpha_i)^{-1}(e, \alpha_0)(e, \alpha_j))E_{mp, nq}(\alpha_i^{-1}\alpha_0\alpha_j)\delta(\alpha_i^{-1}\alpha\alpha_j, \bar{K}). \quad (3.16)$$

Since

$$E(\alpha_i^{-1}\alpha_0\alpha_j) = \mu(\alpha_i^{-1}\alpha_0\alpha_j\alpha_0^{-1}, \alpha_0) \begin{pmatrix} 0 & F(\alpha_i^{-1}\alpha_0\alpha_j\alpha_0^{-1})A \\ -F(\alpha_i^{-1}\alpha_0\alpha_j\alpha_0^{-1})A & 0 \end{pmatrix} \quad (3.17)$$

it follows from eqn. (3.16) that the two diagonal blocks of $(\mathcal{D} \uparrow G)(e, \alpha_0)$ are equal to zero and that the two off-diagonal blocks only differ from each other by a minus sign. This shows that $\mathcal{D} \uparrow G$ is on standardform if we write the matrices in blockform with respect to the indices p and q . This result is independent of the choice of the left-coset-representatives.

Now suppose that E is of Type III and is on standardform. Then we may write

$$E(\alpha) = \begin{pmatrix} F(\alpha) & 0 \\ 0 & \bar{F}(\alpha) \end{pmatrix} \quad \forall \alpha \in \bar{K}_0, \quad (3.18)$$

where $\bar{F}(\alpha)$ is given by

$$\bar{F}(\alpha) = \mu(\alpha_0, \alpha_0^{-1}\alpha\alpha_0)\mu^*(\alpha, \alpha_0)F^*(\alpha_0^{-1}\alpha\alpha_0). \quad (3.19)$$

Also for this case eqns. (3.11) and (3.12) are valid.

If we write $\mathcal{D} \uparrow G$ in blockform with respect to the indices p and q we obtain

$$(\mathcal{D} \uparrow G)(a, \alpha) = \begin{pmatrix} (\mathcal{D}' \uparrow G_0)(a, \alpha) & 0 \\ 0 & (\mathcal{D}' \uparrow G_0)(a, \alpha) \end{pmatrix}, \quad (3.20)$$

where $\mathcal{D}'$ is given by eqn. (3.15) and $\mathcal{D}'$ is given by

$$\tilde{\mathcal{D}}'(a, \alpha) = (V \downarrow L(G_0))(a, \alpha) \otimes \tilde{F}(\alpha) \quad \forall a \in H, \quad \forall \alpha \in \bar{K}_0. \quad (3.21)$$

Since in general $(\mathcal{D}' \uparrow G_0)(a, \alpha)$ is not equal to $(\mathcal{D}' \uparrow G_0)^*((e, \alpha_0)^{-1}(e, \alpha_0))$ we find that $\mathcal{D} \uparrow G$ is not on standardform in general. This difficulty may be overcome by finding the unitary transformation which brings $\mathcal{D} \uparrow G$ on standardform. However, if we realize that $\mathcal{D} \uparrow G$ is of Type III and that the standardform of an irreducible UA representation (eqn. (2.6)) is known if one of the irreducible components of its restriction to the unitary subgroup is known we may forget about the induced representation $\mathcal{D} \uparrow G$ and consider instead the irreducible UA representation D of G given by

$$D(a, \alpha) = \begin{pmatrix} (\mathcal{D}' \uparrow G_0)(a, \alpha) & 0 \\ 0 & (\mathcal{D}' \uparrow G_0)^*((e, \alpha_0)^{-1}(a, \alpha)(e, \alpha_0)) \end{pmatrix} \quad \forall a \in H, \quad \forall \alpha \in K_0 \quad (3.22a)$$

and

$$D(e, \alpha_0) = \begin{pmatrix} 0 & (\mathcal{D}' \uparrow G_0)((e, \alpha_0)^2) \\ 1 & 0 \end{pmatrix} \quad (3.22b)$$

which is on standardform.

However, we must keep in mind that D and $\mathcal{D} \uparrow G$ are equivalent but not equal in general.

4. SUMMARY OF THE RESULTS

We distinguished two cases: $L(G)$ belongs to G_0 and $L(G)$ does not belong to G_0 . If $L(G)$ belongs to G_0 then $\mathcal{D} \uparrow G$ is of Type III and, with a proper choice of left-coset-representatives and a proper order of the indices, is on standardform. If $L(G)$ does not belong to G_0 the type of $\mathcal{D} \uparrow G$ is the same as the type of the irreducible PUA representation E of K which is used to obtain $\mathcal{D}$ (with eqn. (2.14)). If $\mathcal{D} \uparrow G$ and E are of Type II then $\mathcal{D} \uparrow G$ is on standardform (with a proper order of the indices). If $\mathcal{D} \uparrow G$ and E are of Type III then $\mathcal{D} \uparrow G$ is not on standardform in general, but in this case an equivalent UA representation which is on standardform can be written down at once (eqn. (3.22)).

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