Path-fan Ramsey numbers

A.N.M. Salman¹, H.J. Broersma²

Faculty of Electrical Engineering, Mathematics and Computer Science, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

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Abstract

For two given graphs $F$ and $H$, the Ramsey number $R(F, H)$ is the smallest positive integer $p$ such that for every graph $G$ on $p$ vertices the following holds: either $G$ contains $F$ as a subgraph or the complement of $G$ contains $H$ as a subgraph. In this paper, we study the Ramsey numbers $R(P_n, F_m)$, where $P_n$ is a path on $n$ vertices and $F_m$ is the graph obtained from $m$ disjoint triangles by identifying precisely one vertex of every triangle ($F_m$ is the join of $K_1$ and $mK_2$). We determine the exact values of $R(P_n, F_m)$ for the following values of $n$ and $m$: $1 \leq n \leq 5$ and $m \geq 2$; $n \geq 6$ and $2 \leq m \leq (n+1)/2$; $6 \leq n \leq 7$ and $m \geq n-1$; $n \geq 8$ and $n-1 \leq m \leq n$ or $((q \cdot n - 2q + 1)/2 \leq m \leq (q \cdot n - q + 2)/2$ with $3 \leq q \leq n-5)$ or $m \geq (n-3)^2/2$; odd $n \geq 9$ and $((q \cdot n - 3q + 1)/2 \leq m \leq (q \cdot n - 2q)/2$ with $3 \leq q \leq (n-3)/2$ or $((q \cdot n - q - n + 4)/2 \leq m \leq (q \cdot n - 2q)/2$ with $(n-1)/2 \leq q \leq n-5$). Moreover, we give nontrivial lower bounds and upper bounds for $R(P_n, F_m)$ for the other values of $m$ and $n$.

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1. Introduction

Throughout this paper, all graphs are finite and simple. Let $G$ be such a graph. We write $V(G)$ or $V$ for the vertex set of $G$ and $E(G)$ or $E$ for the edge set of $G$. The graph $\overline{G}$ is the complement of $G$, i.e., the graph obtained from the complete graph on $|V(G)|$ vertices by deleting the edges of $G$. The graph $H = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$ (implying that the edges of $H$ have all their end vertices in $V'$).

If $e = \{u, v\} \in E$ (in short, $e = uv$), then $u$ is called adjacent to $v$, and $u$ and $v$ are called neighbors. For $x \in V$, define $N(x) = \{y \in V \mid xy \in E\}$ and $N[x] = N(x) \cup \{x\}$. If $S \subseteq V(G)$, $S \neq V(G)$, then $G - S$ denotes the subgraph of $G$ induced by $V(G) \setminus S$. If $|S| = 1$, then we also use $G - z$ for $S = \{z\}$ instead of $G - \{z\}$. If $e \in E(G)$, then $G - e = (V(G), E(G) \setminus \{e\})$.

We denote by $P_n$, $C_n$ and $K_n$ the path, the cycle and the complete graph on $n$ vertices, respectively. A fan $F_m$ is a graph on $2m + 1$ vertices obtained from $m$ disjoint triangles ($K_{3m}$) by identifying precisely one vertex of every triangle.
(\(F_m\) is the join of \(K_1\) and \(mK_2\)). The vertex corresponding to \(K_1\) is called the hub of the fan. For illustration, consider \(F_5\) in Fig. 1.

Given two graphs \(F\) and \(H\), the Ramsey number \(R(F, H)\) is defined as the smallest positive integer \(p\) such that every graph \(G\) on \(p\) vertices satisfies the following condition: \(G\) contains \(F\) as a subgraph or \(\overline{G}\) contains \(H\) as a subgraph.

In 1967, Geréncser and Gyárfás [4] determined all Ramsey numbers for paths versus paths. After that, Ramsey numbers \(R(P_n, H)\) for paths versus other graphs \(H\) have been investigated in several papers, for example: Parsons [6] when \(H\) is a complete graph; Faudree et al. [2] when \(H\) is a cycle; Parsons [7] when \(H\) is a star; Häggkvist [5] when \(H\) is a complete bipartite graph; Faudree et al. [3] when \(H\) is a tree; Surahmat and Baskoro [9], Chen et al. [1] and Salman and Broersma [8] when \(H\) is a wheel. We study Ramsey numbers for paths versus fans.

2. Main results

In this paper we determine the Ramsey numbers \(R(P_n, F_m)\) for the following values of \(n\) and \(m\): \(1 \leq n \leq 5\) and \(m \geq 2\); \(n \geq 6\) and \(2 \leq m \leq (n + 1)/2\); \(6 \leq n \leq 7\) and \(m \geq n - 1\); \(n \geq 8\) and \(n - 1 \leq m \leq n\) or \((q \cdot n - 2q + 1)/2 \leq m \leq (q \cdot n - q + 2)/2\) with \(3 \leq q \leq n - 5\) or \(m \geq (n - 3)^2/2\); odd \(n \geq 9\) and \((q \cdot n - 3q + 1)/2 \leq m \leq (q \cdot n - 2q)/2\) with \(3 \leq q \leq (n - 3)/2\) or \((q \cdot n - q - n + 4)/2 \leq m \leq (q \cdot n - 2q)/2\) with \((n - 1)/2 \leq q \leq n - 5\). We will present the Ramsey numbers for ‘small’ paths versus fans in Proposition 1, the Ramsey numbers for paths versus ‘small’ fans in Theorem 3, and the Ramsey numbers for paths versus ‘large’ fans in the corollaries based on Lemmas 4 and 6. Moreover, we give nontrivial lower bounds and upper bounds for \(R(P_n, F_m)\) for (odd \(n \geq 11\) and \((q \cdot n - q + 4)/2 \leq m \leq (q \cdot n - 3q + n - 3)/2\) with \(2 \leq q \leq (n - 7)/2\)) or (even \(n \geq 8\) and \((q \cdot n - q + 3)/2 \leq m \leq (q \cdot n - 2q + n - 2)/2\) with \(2 \leq q \leq n - 5\) or \((n \geq 6\) and \((n + 2)/2 \leq m \leq n - 2)\) in Corollaries 8, 9 and Theorem 10.

Proposition 1. Let \(m \geq 2\). Then

\[
R(P_n, F_m) = \begin{cases} 
1 & \text{for } n = 1, \\
2m + 1 & \text{for } n = 2 \text{ or } 3.
\end{cases}
\]

Proof. The cases for which \(n = 1\) or \(2\) are (almost) trivial and left to the reader. We only give the proof in case \(n = 3\): the graph consisting of \(m\) disjoint copies of \(K_2\) shows that \(R(P_3, F_m) > 2m\). Now suppose \(G\) is a graph on \(2m + 1\) vertices, and assume \(G\) contains no \(P_3\). We will show that \(\overline{G}\) contains an \(F_m\). Since \(|V(G)|\) is odd and \(G\) contains no \(P_3\), there is a vertex \(z \in V(G)\) with \(|N(z)| = 0\). Since \(G - z\) contains no \(P_3\), the vertices of \(V(G)\setminus\{z\}\) have degree at least \(2m - 2\) in \(\overline{G} - z\). This implies there exists a cycle \(C_{2m}\) in \(\overline{G} - z\). Hence \(\overline{G}\) contains an \(F_m\) (even a wheel on \(2m + 1\) vertices).

The next lemma plays a key role in the proofs for the remaining cases.

Lemma 2. Let \(n \geq 3\) and \(G\) be a graph on at least \(n\) vertices containing no \(P_n\). Let the paths \(P^1, P^2, \ldots, P^k\) in \(G\) be chosen in the following way: \(\bigcup_{j=1}^{k} V(P^j) = V(G)\), \(P^1\) is a longest path in \(G\), and, if \(k > 1\), \(P^{i+1}\) is a longest path in \(G - \bigcup_{j=1}^{i} V(P^j)\) for \(1 \leq i \leq k - 1\). Denote by \(\ell_j\) the numbers of vertices on the path \(P^j\). Let \(z\) be an end vertex of \(P^k\).
Then:

(i) \(\ell_1 \geq \ell_2 \geq \cdots \geq \ell_k\);
(ii) If \(\ell_k \geq \lfloor n/2 \rfloor\), then \(N(z) \subset V(P_k)\);
(iii) If \(\ell_k < \lfloor n/2 \rfloor\), then \(|N(z)| \leq \lfloor n/2 \rfloor - 1\).

**Proof.** (i) Obviously follows from the choice of the paths. From this choice we can also deduce that for any integer \(x\) with \(1 \leq x < k\), the number of neighbors of \(z\) in \(V(P^x)\) is

\[
\left\{ \begin{array}{ll}
\ell_x + 1 - 2\ell_k & \text{if } \ell_x \geq 2\ell_k + 1, \\
0 & \text{if } \ell_x < 2\ell_k + 1.
\end{array} \right.
\]

This can be checked easily: first order the neighbors of \(z\) on \(P^x\) according to the order of their appearance on \(P^x\) in a fixed orientation. Then observe that between any two successive neighbors of \(z\) on \(P^x\), there is at least one nonneighbor of \(z\), while before the first and after the last neighbor of \(z\) on \(P^x\), there are at least \(\ell_k\) nonneighbors of \(z\).

(ii) Assume \(\ell_k \geq \lfloor n/2 \rfloor\). Then \(2\ell_k + 1 \geq n > \ell_1\). So by the above observation, we conclude that there is no neighbor of \(z\) in \(V(G) \setminus V(P^k)\).

(iii) Now assume \(\ell_k < \lfloor n/2 \rfloor\). If \(z\) has no neighbors in \(V(G) \setminus V(P^k)\), we are done. If \(z\) has some neighbors in \(V(G) \setminus V(P^k)\), similar counting arguments as above yield the desired result: denote by \(h_1, \ldots, h_t\) the numbers of vertices on the paths \(P^1, \ldots, P^k\) that contain a neighbor of \(z\), chosen in such a way that \(h_t \geq \cdots \geq h_1\), and denote by \(d_1, \ldots, d_t\) the numbers of neighbors of \(z\) on the corresponding paths. Then, arguing as above, we obtain \(h_1 = \ell_k \geq d_1 + 1\) and \(h_2 \geq 2h_1 + 2d_2 - 1\). Similarly, observing that \(z\) connects any two of the considered paths, and using the same elementary counting techniques, we get (if \(t \geq 3\)) \(h_j \geq 2(h_j-1 - 1/2 + 2) + 2d_j - 1 = h_j - 1 + 2d_j + 2\) for \(3 \leq j \leq t\). This implies (for \(t \geq 2\)) \(h_j \geq 2(d_1 + \cdots + d_t) + 2(t - 2) + 1 \geq 2|N(z)| + 1\). Since \(h_t \leq n - 1\) and \(|N(z)|\) are integers, this yields the desired result. \(\square\)

**Theorem 3.** Let \(n \geq 4\) and \(2 \leq m \leq \lfloor (n+1)/2 \rfloor\). Then \(R(P_n, F_m) = 2n - 1\).

**Proof.** The graph \(2K_{n-1}\) shows that \(R(P_n, F_m) > 2n - 2\). Let \(G\) be a graph on \(2n - 1\) vertices and assume \(G\) contains no \(P_n\). We are going to show that \(\overline{G}\) contains an \(F_m\). Choose the paths \(P^1, \ldots, P^k\) and the vertex \(z\) as in Lemma 2. Since \(|V(G)| = 2n - 1\) and \(G\) does not contain a \(P_k\), \(k \geq 3\) and \(\ell_k \leq (2n - 1)/3\). If \(\ell_k < \lfloor n/2 \rfloor\), then by Lemma 2(ii) we obtain \(|N(z)| \leq \ell_k + 1 \leq (2n - 1)/3 - 1\). Hence, \(|N(z)| \leq (2n - 1)/3\). We are going to show that there is an \(F_m\) in \(\overline{G}\) with \(z\) as a hub. We distinguish the following two cases.

Case 1: \(|N(z)| \leq \lfloor n/2 \rfloor - 1\).

Then \(|V(G) \setminus N[z]| \geq (2n - 1) - |n/2| \geq n + m - 1\). We can apply the result from [2] that \(R(P_n, C_{2m}) = n + m - 1\) for \(2 \leq m \leq \lfloor (n+1)/2 \rfloor\). This implies that \(\overline{G} - N[z]\) contains a \(C_{2m}\). So, there is an \(F_m\) in \(\overline{G}\) with \(z\) as a hub (there is even a wheel on \(2m + 1\) vertices).

Case 2: \(|N(z)| \geq \lfloor n/2 \rfloor\).

By Lemma 2(ii), we find \(N(z) \subset V(P^k)\). Hence, \(\ell_k \geq \lfloor n/2 \rfloor + 1\). Since \(|V(G)| = 2n - 1\), \(k = 3\). Take the first \(m\) vertices of \(P^1\) (in some fixed orientation) and name them \(u_1, \ldots, u_m\), starting at an end vertex. Also take the first \(m\) vertices of \(P^2\) (in some fixed orientation) and name them \(v_1, \ldots, v_m\), starting at an end vertex. Since \(P^1\) is chosen as a longest path in \(G\), it is obvious that \(u_i v_i \notin E(G)\) (\(i = 1, \ldots, m\)). So there is an \(F_m\) in \(\overline{G}\) with \(z\) as a hub. \(\square\)

The following lemma provides upper bounds that yield several exact Ramsey numbers in the sequel.

**Lemma 4.** If \(n \geq 4\) and \(m \geq n - 1\), then

\[
R(P_n, F_m) \leq \begin{cases} 
2m + n - 1 & \text{for } 2m = 1 \mod (n - 1), \\
2m + n - 2 & \text{for other values of } m.
\end{cases}
\]
Proof. Let $G$ be a graph that contains no $P_n$ and has order $|V(G)| = \begin{cases} 2m + n - 1 & \text{for } 2m = 1 \text{ mod } (n-1), \\ 2m + n - 2 & \text{for other values of } m. \end{cases}$ (2)

Choose the paths $P^1, \ldots, P^k$ and the vertex $z$ in $G$ as in Lemma 2. Because of (2), not all $P^i$ can have $n - 1$ vertices, so $\ell_k \leq n - 2$. By similar arguments as in the proof of Theorem 3, this implies $|N(z)| \leq n - 3$. We will use the following result that has been proved in [2]: $R(P^i, C_k) = s + \lfloor t/2 \rfloor - 1$ for $s \geq \lfloor (3t + 1)/2 \rfloor$. We distinguish the following cases.

Case 1: $|N(z)| \leq \lfloor n/2 \rfloor - 2$ or $n$ is odd and $|N(z)| = \lfloor n/2 \rfloor - 1$.

Since $|V(G) \setminus N[z]| \geq 2m + \lfloor n/2 \rfloor - 1$, we find that $\overline{G - N[z]}$ contains a $C_{2m}$. So, there is an $F_m$ in $\overline{G}$ with $z$ as a hub.

Case 2: $n$ is even and $|N(z)| = n/2 - 1$.

Since $|V(G) \setminus N[z]| \geq (2m + n - 2) - n/2 = 2m + n/2 - 2$, we find that $\overline{G - N[z]}$ contains a $C_{2m-1}$; denote its vertices by $v_1, v_2, v_3, \ldots, v_{2m-1}$ in the order of appearance on the cycle with a fixed orientation. There are $n/2 - 1$ vertices in $U = V(G) \setminus (V(C_{2m-1}) \cup N[z])$, say $u_1, u_2, \ldots, u_{n/2-1}$. If some vertex $v_i$ ($i = 1, \ldots, 2m - 1$) is not neighbor of some vertex $u_j$ ($j = 1, \ldots, n/2 - 1$), w.l.o.g. assume $v_{2m-1}u_1 \not\in E(G)$. Then $\overline{G}$ contains an $F_m$ with $z$ as a hub and additional edges $v_1v_2, v_2v_3, \ldots, v_{2m-3}v_{2m-2}, v_{2m-1}u_1$. Now let us assume each of the $v_i$ is adjacent to all $u_j$ in $G$. For every choice of a subset of $n/2$ vertices from $V(C_{2m-1})$, there is a path on $n - 1$ vertices in $G$ alternating between the vertices of this subset and the vertices of $U$, starting and terminating in two arbitrary vertices from the subset. Since $G$ contains no $P_n$, there are no edges $v_iv_j \in E(G)$ ($i, j \in \{1, \ldots, 2m - 1\}$). This implies $V(C_{2m-1}) \cup \{z\}$ induces a $K_{2m}$ in $\overline{G}$. Since $G$ contains no $P_n$, no $v_i$ is adjacent to a vertex of $N(z)$. This implies $\overline{G}$ contains a $K_{2m+1 - z}$ for any vertex $w \in N(z)$, and hence $\overline{G}$ contains an $F_m$ with one of the $v_i$ as a hub.

Case 3: Suppose that there is no choice for $P^k$ and $z$ such that one of the former cases applies. Then $|N(w)| \geq \lfloor n/2 \rfloor$ for any end vertex $w$ of a path on $\ell_k$ vertices in $G - \bigcup_{j=1}^{k-1} V(P^j)$. This implies all neighbors of such $w$ are in $V(P^k)$ and $\ell_k \geq \lfloor n/2 \rfloor$. So for the two end vertices $z_1$ and $z_2$ of $P^k$ we have that $|N(z_i) \cap V(P^k)| \geq \lfloor n/2 \rfloor \geq \ell_k/2$. Let $P^k : z_1 = v_1v_2 \ldots v_{\ell_k} = z_2$. Then by standard arguments in Hamiltonian graph theory, we can find an index $i \in \{2, \ldots, \ell_k - 1\}$ such that $z_1v_{i+1}$ and $z_2v_i$ are edges of $G$. It is clear that we can find a cycle on $\ell_k$ vertices in $G$. This implies that any vertex of $V(P^k)$ could serve as $w$. By the assumption of this last case, we conclude that there are no edges in $G$ between $V(P^k)$ and the other vertices. This also implies that all vertices of $P^k$ have degree at least $2m$ in $\overline{G}$.

We now turn to $P^{k-1}$ and consider one of its end vertices $w$. Since $\ell_{k-1} \geq \ell_k \geq \lfloor n/2 \rfloor$, similar arguments as in the proof of Lemma 2 show that all neighbors of $w$ are on $P^{k-1}$. If $|N(w)| < \lfloor n/2 \rfloor$, we get an $F_m$ in $\overline{G}$ as in Case 1 or Case 2. So we may assume $|N(w)| \cap V(P^{k-1}) \geq \lfloor n/2 \rfloor \geq \ell_{k-1}/2$ for both end vertices $w_1$ and $w_2$ of $P^{k-1}$. By similar arguments as before we obtain a cycle on $\ell_{k-1}$ vertices in $G$. This implies that any vertex of $V(P^{k-1})$ could serve as $w$. By the assumption of the last case, we conclude that there are no edges in $G$ between $V(P^{k-1})$ and the other vertices. This also implies that all vertices of $P^{k-1}$ have degree at least $2m - 1$ in $\overline{G}$. (Note that $P^{k-1}$ can have $n - 1$ vertices, whereas $\ell_k \leq n - 2$.)

Repeating the above arguments for $P^{k-2}, \ldots, P^1$ we eventually conclude that all vertices of $G$ have degree at least $2m - 1$ in $\overline{G}$. Now let $H = \overline{G} - V(P^1)$. Then all vertices in $V(H)$ have degree at least $2m - 1 - \ell_k \geq m + (n - 1) - 1 - \ell_k \geq \frac{1}{2}(2m + 2n - 4 - \ell_k - (n - 2)) = \frac{1}{2}(2m + n - 2 - \ell_k) \geq \frac{1}{2}(|V(H)| - 1)$. This implies there exists a Hamilton path in $H$. Since $|V(H)| \geq 2m$ and $z$ is a neighbor of all vertices in $H$, it is clear that $\overline{G}$ contains an $F_m$ with $z$ as a hub. This completes the proof of Lemma 4.

Corollary 5. If $(4 \leq n \leq 7$ and $m \geq n - 1$) or $(n \geq 8$ and $n - 1 \leq m \leq n$ or $(q \cdot n - 2q + 1)/2 \leq m \leq (q \cdot n - q + 2)/2$ for $3 \leq q \leq n - 5$) or $(m \geq (n - 3)^2/2)$, then

$$R(P_n, P_m) = \begin{cases} 2m + n - 1 & \text{for } 2m = 1 \text{ mod } (n-1), \\ 2m + n - 2 & \text{for other values of } m. \end{cases}$$

Proof. Let $r$ denote the remainder of $2m$ divided by $n - 1$, so $2m = p(n - 1) + r$ for some $0 \leq r \leq n - 2$. Then for $(4 \leq n \leq 7$ and $m \geq n - 1$) or $(n \geq 8$ and $n - 1 \leq m \leq n$ or $(q \cdot n - 2q + 1)/2 \leq m \leq (q \cdot n - q + 2)/2$ for $3 \leq q \leq n - 5$)
or \( m \geq (n - 3)^2/2 \), the graphs

\[
\begin{align*}
(p - 1)K_{r-1} & \cup 2K_{n-2} & \text{for } r = 0, \\
(p + 1)K_{n-1} & \text{for } r = 1 \text{ or } 2, \\
(p + r + 1 - n)K_{n-1} & \cup (n + 1 - r)K_{n-2} & \text{for other values of } r
\end{align*}
\]

show that

\[
R(P_n, F_m) > \begin{cases} 2m + n - 2 & \text{for } m = 1 \text{ mod}(n - 1), \\
2m + n - 3 & \text{for other values of } m.
\end{cases}
\]

Lemma 4 completes the proof. □

**Lemma 6.** If \( n \) is odd, \( n \geq 9 \) and \( (q \cdot n - q + 3)/2 \leq m \leq (q \cdot n - 2q + n - 2)/2 \) with \( 2 \leq q \leq 2(n/2) - 5 \), then \( R(P_n, F_m) \leq 2m + n - 3 \).

**Proof.** The proof is modelled along the lines of the proof of Lemma 4. Let \( G \) be a graph on \( 2m + n - 3 \) vertices, and assume \( G \) contains no \( P_n \). We will show that \( \overline{G} \) contains an \( F_m \). Choose the paths \( P^1, \ldots, P^k \) and the vertex \( z \) in \( G \) as in Lemma 2. Since \( |V(G)| = 2m + n - 3 \) with \( n \geq 9 \) and \( (q \cdot n - q + 3)/2 \leq m \leq (q \cdot n - 2q + n - 2)/2 \) with \( 2 \leq q \leq 2(n/2) - 5 \), \( k \geq q + 2 \), and therefore not all \( P^j \) can have more than \( n - 3 \) vertices. So \( \ell_j \leq n - 3 \). By similar arguments as in the proof of Theorem 3, this implies \( |N(z)| \leq n - 4 \). We will use the following result that has been proved in [2]:

\[
R(P_l, C_3) = s + \lfloor t/2 \rfloor - 1 \text{ for } s \geq \lfloor (3t + 1)/2 \rfloor.
\]

We distinguish the following cases.

**Case 1:** \( |N(z)| \leq \lfloor n/2 \rfloor - 2 \).

Since \( |V(G)\setminus N[z]| \geq 2m + \lfloor n/2 \rfloor - 1 \), we find that \( \overline{G} - N[z] \) contains a \( C_{2m} \). So, there is an \( F_m \) in \( \overline{G} \) with \( z \) as a hub.

**Case 2:** \( |N(z)| = \lfloor n/2 \rfloor - 1 \).

Since \( |V(G)\setminus N[z]| = (2m + n - 3) - \lfloor n/2 \rfloor = 2m + \lfloor n/2 \rfloor - 2 \), we find that \( \overline{G} - N[z] \) contains a \( C_{2m-1} \); denote its vertices by \( v_1, v_2, v_3, \ldots, v_{2m-1} \) in the order of appearance on the cycle with a fixed orientation. There are \( \lfloor n/2 \rfloor - 1 \) vertices in \( U = V(G) \setminus (V(C_{2m-1}) \cup N[z]) \), say \( u_1, u_2, \ldots, u_{\lfloor n/2 \rfloor - 1} \). If some vertex \( v_i \), \( i = 1, \ldots, 2m - 1 \), is a neighbor of some vertex \( u_j \), \( j = 1, \ldots, \lfloor n/2 \rfloor - 1 \), w.l.o.g., assume \( v_{2m-1}u_1 \notin E(G) \). Then \( \overline{G} \) contains an \( F_m \) with \( z \) as a hub and additional edges \( v_1v_2, v_2v_3, \ldots, v_{2m-3}v_{2m-2}, v_{2m-1}u_1 \). Now let us assume each of the \( u_j \) is adjacent to all \( u_j \) in \( G \). For every choice of a subset of \( \lfloor n/2 \rfloor \) vertices from \( V(C_{2m-1}) \), there is a path on \( n - 2 \) vertices in \( G \) alternating between the vertices of this subset and the vertices of \( U \), starting and terminating in arbitrary vertices from the subset. Let \( z_1 \in N(z) \). Since \( G \) contains no \( P_n \), there are no edges \( v_i z \in E(G) \) and \( v_i z_1 \in E(G) \) for \( i = 1, \ldots, 2m - 1 \) and there is only (at most) one edge \( v_i v_j \in E(G) \) (for some \( i, j = 1, \ldots, 2m - 1 \)). Suppose \( v_1v_2 \in E(G) \). This implies \( \overline{G} \) contains an \( F_m \) with hub \( v_{2m-1} \) and additional edges \( v_1z, v_2z_1, v_3v_4, \ldots, v_{2m-3}v_{2m-2}, v_{2m-1}u_1 \). The other cases are similar.

**Case 3:** Suppose that there is no choice for \( P^k \) and \( z \) such that one of the former cases applies. Then \( |N(w)| \geq \lfloor n/2 \rfloor \) for any end vertex \( w \) of a path on \( \ell k \) vertices in \( G - \bigcup_{j=1}^{k-1} V(P^j) \). This implies all neighbors of such \( w \) are in \( V(P^k) \) and \( \ell k \geq \lfloor n/2 \rfloor + 1 \). So for the two end vertices \( z_1 \) and \( z_2 \) of \( P^k \) we have that \( |N(z_i) \cap V(P^k)| \geq \lfloor n/2 \rfloor \geq \ell k/2 \). By similar arguments as in the proof of Case 3 of Lemma 4 we obtain a cycle on \( \ell k \) vertices in \( G \). This implies that any vertex of \( V(P^k) \) could serve as \( w \). By the assumption of this last case, we conclude that there are no edges in \( G \) between \( V(P^k) \) and the other vertices. This also implies that all vertices of \( P^k \) have degree at least \( 2m - 2 \) in \( \overline{G} \). (Note that \( P^k \) can have \( n - 1 \) vertices, whereas \( \ell k \leq n - 3 \).)

Repeating the above arguments for \( P^{k-1}, \ldots, P^1 \) we eventually conclude that all vertices of \( G \) have degree at least \( 2m - 2 \) in \( \overline{G} \). Now let \( H = \overline{G} - V(P^k) \). Then all vertices in \( V(H) \) have degree at least \( 2m - 2 - \ell k \geq m + (n + 1) - 2 - \ell k \geq 1/2(2m + 2n - 2 - \ell k - (n - 3)) = 1/2(2m + n + 1 - \ell k) = 1/2(|V(H)| + 4) \). This implies there exists a Hamilton
cycle in \( H \). Since \( |V(H)| \geq 2m \) and \( z \) is a neighbor of all vertices in \( H \), it is clear that \( \overline{G} \) contains an \( F_m \) with \( z \) as a hub. This completes the proof of Lemma 6. \( \square \)

**Corollary 7.** If \( n \) is odd, \( n \geq 9 \) and either \((q \cdot n - 3q + 1)/2 \leq m \leq (q \cdot n - 2q)/2 \) with \( 3 \leq q \leq (n - 3)/2 \) or \((q \cdot n - q - n + 4)/2 \leq m \leq (q \cdot n - 2q)/2 \) with \( (n - 1)/2 \leq q \leq n - 5 \), then \( R(P_n, F_m) = 2m + n - 3 \).

**Proof.** For odd \( n \geq 9 \) and \( m = (q \cdot n - 2q - j)/2 \) with either \((3 \leq q \leq (n - 3)/2 \) and \( 0 \leq j \leq q - 1 \) or \((n - 1)/2 \leq q \leq n - 5 \) and \( 0 \leq j \leq n - q - 4 \), the graph \((q - j - 1)K_{n - 2} \cup (j + 2)K_{n - 3} \) shows that \( R(P_n, F_m) > 2m + n - 4 \). Lemma 6 completes the proof. \( \square \)

**Corollary 8.** If \( n \) is odd, \( n \geq 11 \) and \((q \cdot n - q + 4)/2 \leq m \leq (q \cdot n - 3q + n - 3)/2 \) with \( 2 \leq q \leq (n - 7)/2 \), then

\[
2m + n - 3 \geq R(P_n, F_m) \geq \max \left\{ \left\lceil \frac{2m}{n - 1} \right\rceil (n - 1) + n, \ 2m + \left\lceil \frac{2m - 1}{2m/(n - 1)} \right\rceil \right\}.
\]

**Proof.** Let \( t = \lceil 2m/(n - 1) \rceil \) and \( s \) denote the remainder of \( 2m - 1 \) divided by \( t \). Then for \( m \) and \( n \) satisfying \( (2m/(n - 1))(n - 1) + n \geq 2m + \lceil (2m - 1)/t \rceil \), the graph \( tK_{n - 1} \) shows that \( R(P_n, F_m) > (2m/(n - 1))(n - 1) + n - 1 \).

For other values of \( m \) and \( n \), the graph \( sK_{\lceil (2m - 1)/t \rceil} \cup ((t - s + 1)K_{\lceil (2m - 1)/t \rceil} \) shows that \( R(P_n, F_m) > 2m - 1 + \lceil (2m - 1)/(2m/(n - 1)) \rceil \).

The upper bound comes from Lemma 6. \( \square \)

**Corollary 9.** If \( n \) is even, \( n \geq 8 \) and \((q \cdot n - q + 3)/2 \leq m \leq (q \cdot n - 2q + n - 2)/2 \) with \( 2 \leq q \leq n - 5 \), then

\[
2m + n - 2 \geq R(P_n, F_m) \geq \max \left\{ \left\lceil \frac{2m}{n - 1} \right\rceil (n - 1) + n, \ 2m + \left\lceil \frac{2m - 1}{2m/(n - 1)} \right\rceil \right\}.
\]

**Proof.** Let \( t = \lceil 2m/(n - 1) \rceil \) and \( s \) denote the remainder of \( 2m - 1 \) divided by \( t \). Then for \( m \) and \( n \) satisfying \( 2m/(n - 1))(n - 1) + n \geq 2m + \lceil (2m - 1)/t \rceil \), the graph \( tK_{n - 1} \) shows that \( R(P_n, F_m) > 2m/(n - 1)(n - 1) + n - 1 \).

For other values of \( m \) and \( n \), the graph \( sK_{\lceil (2m - 1)/t \rceil} \cup ((t - s + 1)K_{\lceil (2m - 1)/t \rceil} \) shows that \( R(P_n, F_m) > 2m - 1 + \lceil (2m - 1)/(2m/(n - 1)) \rceil \).

The upper bound comes from Lemma 4. \( \square \)

**Theorem 10.** If \( n \geq 6 \) and \((n + 2)/2 \leq m \leq n - 2 \), then

\[
2m + \left\lceil \frac{3n}{2} \right\rceil - 2 \geq R(P_n, F_m) \geq \begin{cases} 2n - 1 & \text{for } n + 2 \leq m \leq n + \lfloor n/3 \rfloor, \\ 3m - 1 & \text{for } \frac{2}{n + \lfloor n/3 \rfloor} < m \leq n - 2. \end{cases}
\]

**Proof.** For \( n \geq 6 \) and \((n + 2)/2 \leq m \leq (n + \lfloor n/3 \rfloor)/2 \), the graph \( 2K_{n - 1} \) shows that \( R(P_n, F_m) > 2n - 2 \). For \( n \geq 6 \) and \((n + \lfloor n/3 \rfloor)/2 < m \leq n - 2 \), the graph \( K_m \cup 2K_{m - 1} \) shows that \( R(P_n, F_m) > 3m - 2 \).

Let \( G \) be a graph on \( 2m + \lfloor 3n/2 \rfloor - 2 \) vertices, and assume \( G \) contains no \( P_n \). Choose the paths \( P^1, \ldots, P^k \) and the vertex \( z \) in \( G \) as in Lemma 2. By Lemma 2, \( |N(z)| \leq n - 2 \). Hence, \( |V(G)\backslash N[z]| \geq 2m + \lfloor n/2 \rfloor - 1 \). We can apply the result from [2] that \( R(P_n, C_{2m}) = 2m + \lfloor n/2 \rfloor - 1 \) for \( 2 \leq n \leq 2m \). This implies that \( G - N[z] \) contains a \( C_{2m} \). So, there is an \( F_m \) in \( \overline{G} \) with \( z \) as a hub (there is even a wheel on \( 2m + 1 \) vertices). \( \square \)

### 3. Conclusion

In this paper we determined the exact Ramsey numbers for paths versus fans of varying orders. The numbers are indicated in Table 1. We used different capitals to distinguish the results in the previous section that led to these numbers. The shaded elements indicate open cases. For these cases we established nontrivial lower bounds and upper bounds for \( R(P_n, F_m) \). We learned from one of the anonymous referees that Yunqing Zhang et al. established similar results in a recent paper. We are not aware of the present status of this paper.
Table 1
The Ramsey numbers for paths versus fans

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| Proposition 1 | D | 2m + 1 |
| Theorem 3     | I | 2n - 1 |
| Corollary 5   | L | 2m + n - 1 |
| Corollary 5   | N | 2m + n - 2 |
| Corollary 7   | P | 2m + n - 3 |

References


