THE QUEUE M|G|1 WITH MARKOV MODULATED ARRIVALS AND SERVICES*

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We study an M|G|1 queue in which both the arrival rate and the service time distribution depend on the state of an underlying finite-state Markov chain. The solution is obtained by a matrix factorization method. This leads to results for waiting times and queue lengths both at arrival epochs and in continuous time. A numerical algorithm for the calculation of several quantities of interest is described and some numerical examples are given.

1. Introduction. We study an M|G|1 queue with Markov modulated arrivals and services. Let $(X_t, t \geq 0)$ be a Markov chain on $\{1, 2, \dots, N\}$ with indecomposable generator Q with elements $Q_{ij}, i = 1, 2, \dots, N; j = 1, 2, \dots, N$; and $q_i = -Q_{ii}, i = 1, 2, \dots, N$. Denote its stationary distribution by $\pi = (\pi_1, \pi_2, \dots, \pi_N)^T$, i.e., $Q^T \pi = 0$ and $1\pi = 1$, where 1 is the N -dimensional row vector with all components equal to 1. Customers arrive at $T_1, T_2, \dots; T_1 = 0$. The service times are denoted by $S_1, S_2, \dots$. If $X_t = i$ customers arrive according to a Poisson process with intensity $\lambda_i > 0$ and an arriving customer has service time distribution G_i . Given $(X_t, t \geq 0)$ the $S_1, S_2, \dots$, are independent and also independent of the arrival process. Define $A_n = T_n - T_{n-1}, n = 2, 3, \dots$; and $Y_n = X_{T_n}, n = 1, 2, \dots$. Denote the waiting time of the n th customer by W_n and assume that the system is initially empty, i.e. $W_1 = 0$. Moreover denote for $i = 1, 2, \dots, N$;

$$B_i(\phi) = \int_0^\infty \exp(-\phi x) dG_i(x) \quad \text{and} \quad \beta_i = \int_0^\infty x dG_i(x) = -B'_i(0).$$

Throughout the paper we shall assume that there exists a $\delta > 0$ such that the $B_i(\phi)$ can be continued analytically to the region $\text{Re } \phi > -\delta$. The arrival process described above is called a Markov modulated arrival process. For this model we study the waiting times and queue lengths both at arrival epochs and in continuous time, generalizing known results for the ordinary M|G|1 queue. A more general arrival process, the so-called N-process, has been studied by Ramaswami [4]. He analyses the N|G|1 queue with a fixed service time distribution and obtains results for the busy period, the busy cycle, the steady-state queue length both at departure epochs and in continuous time, and for the virtual waiting time. Ramaswami claims that his results are computationally tractable, but he does not give any numerical examples. Burman and Smith [2] study the queue M|G|1 with Markov modulated arrivals. They give approximations for the expected steady state waiting time and queue length both in light and in heavy traffic. The special case in which the arrival process is a switched Poisson process, i.e. $(X_t, t \geq 0)$ is a 2-state Markov chain, has been analysed by van Hoorn and Seelen [5]. They obtain explicit numerical results for the steady-state queue length in continuous time.

Our model is a special case of the semi-Markov queue studied by Arjas [1]. The

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practical importance of the Markov modulated arrival process has been explained in [2], [4] and [5]. As we also allow the service time distribution to depend on the underlying Markov chain, the applicability of our model is even wider than that of the models discussed in the papers mentioned above.

The paper is organized as follows. In §2 we derive a system of Wiener–Hopf-type equations, and in §3 we give its solution using a factorization method. Results for the actual waiting time, the virtual waiting time and the queue length both at arrival epochs and in continuous time are given in §4. Finally in §5 a numerical algorithm is described and numerical examples are given. We shall use the following notations:

$1(A)$ is the indicator function of the event A ; $x^+ = \max(0, x)$; $x^- = \min(0, x)$; M^T is the transpose of the matrix M ; I is the $N \times N$ -identity matrix.

2. Derivation of a system of Wiener–Hopf-type equations. For $j = 1, 2, \dots, N$; $\operatorname{Re} \phi > 0$, $\operatorname{Re} \eta > 0$ and $|r| < 1$ we define

$$Z_j(r, \phi, \eta) = \sum_{n=1}^{\infty} r^n E(\exp(-\phi W_n - \eta T_n) 1(Y_n = j))$$

and we denote the N -dimensional column vector with components $Z_j(r, \phi, \eta)$ by $Z(r, \phi, \eta)$. In this section we derive a system of equations for the functions $Z_j(r, \phi, \eta)$, $j = 1, 2, \dots, N$. This system turns out to be of Wiener–Hopf-type.

Let

$$A_{ij}(\phi) = E(\exp(-\phi A_n) 1(Y_n = j) | Y_{n-1} = i)$$

and $A(\phi)$ the $N \times N$ -matrix with elements $A_{ij}(\phi)$, $i = 1, 2, \dots, N$; $j = 1, 2, \dots, N$. We prove the following lemma.

LEMMA 2.1. For $\operatorname{Re} \phi > 0$ the functions A_{ij} satisfy the following system of equations

$$A_{ij}(\phi) = \frac{q_i + \lambda_i}{\phi + q_i + \lambda_i} \left[\frac{\lambda_i}{q_i + \lambda_i} \delta_{ij} + \sum_{k \neq i}^N \frac{Q_{ik}}{q_i + \lambda_i} A_{kj}(\phi) \right],$$

$$i = 1, 2, \dots, N; \quad j = 1, 2, \dots, N; \quad (2.1)$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$; or in matrix notation

$$M(\phi)A(\phi) = \Lambda \quad (2.2)$$

where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$ and $M(\phi) = \phi I + \Lambda - Q$.

PROOF. Let $T_{n-1} + J_n$ be the first epoch after T_{n-1} at which either a customer arrives or the Markov chain $(X_t, t \geq 0)$ jumps to a new state. Then, given that $Y_{n-1} = i$, J_n is exponentially distributed with parameter $q_i + \lambda_i$ and at $T_{n-1} + J_n$ with probability $\lambda_i/(q_i + \lambda_i)$ a customer arrives and with probability $Q_{ik}/(q_i + \lambda_i)$ the Markov chain $(X_t, t \geq 0)$ jumps to state k ($k \neq i$). By conditioning we thus obtain (2.1).

The process $((W_n, T_n, Y_n), n = 1, 2, \dots)$ is easily seen to be a Markov process, since $((Y_n, T_n), n = 1, 2, \dots)$ is a Markov renewal process. Denote $p_j = P(X_0 = j)$, $j = 1, 2, \dots, N$; and let p be the N -dimensional column vector with components p_j . Define for $j = 1, 2, \dots, N$, $\operatorname{Re} \phi > 0$, $\operatorname{Re} \eta > 0$, $|r| < 1$,

$$V_j(r, \phi, \eta) = \sum_{n=1}^{\infty} r^{n+1} E((1 - \exp(\phi[W_n + S_n - A_{n+1}]^-)) \exp(-\eta T_{n+1}) 1(Y_{n+1} = j)),$$

and let $V(r, \phi, \eta)$ be the N -dimensional column vector with components $V_j(r, \phi, \eta)$. The

$N \times N$ -matrix $H(r, \phi, \eta)$ is defined by

$$H(r, \phi, \eta) = I - rA^T(\eta - \phi)B(\phi)$$

where $B(\phi) = \text{diag}(B_1(\phi), B_2(\phi), \dots, B_N(\phi))$.

Then we find the following system of Wiener-Hopf-type equations.

THEOREM 2.1.

$$H(r, \phi, \eta)Z(r, \phi, \eta) = r\rho + V(r, -\phi, \eta) \quad (2.3)$$

for $\text{Re } \phi = 0$, $\text{Re } \eta > 0$ and $|r| < 1$.

PROOF. Using the identity

$$\exp(-\phi x^+) + \exp(-\phi x^-) = \exp(-\phi x) + 1,$$

we have for $\text{Re } \phi = 0$, $\text{Re } \eta > 0$, $|r| < 1$,

$$\begin{aligned} & E(\exp(-\phi W_{n+1} - \eta T_{n+1})1(Y_{n+1} = j)) \\ &= E(\exp(-\phi[W_n + S_n - A_{n+1}]^+ - \eta T_{n+1})1(Y_{n+1} = j)) \\ &= E(\exp(-\phi(W_n + S_n - A_{n+1}) - \eta T_{n+1})1(Y_{n+1} = j)) \\ &+ E((1 - \exp(-\phi[W_n + S_n - A_{n+1}]^-))\exp(-\eta T_{n+1})1(Y_{n+1} = j)) \quad \text{and} \\ & E(\exp(-\phi(W_n + S_n - A_{n+1}) - \eta T_{n+1})1(Y_{n+1} = j)) \\ &= \sum_{i=1}^N E(\exp(-\phi W_n - \eta T_n)1(Y_n = i)) \\ &\quad \cdot E(\exp(-\phi S_n - (\eta - \phi)A_{n+1})1(Y_{n+1} = j) | Y_n = i) \\ &= \sum_{i=1}^N E(\exp(-\phi W_n - \eta T_n)1(Y_n = i))B_i(\phi)A_{ij}(\eta - \phi), \end{aligned}$$

so for $\text{Re } \phi = 0$,

$$Z_j(r, \phi, \eta) = rP(X_0 = j) + r \sum_{i=1}^N Z_i(r, \phi, \eta)B_i(\phi)A_{ij}(\eta - \phi) + V_j(r, -\phi, \eta),$$

which proves the theorem.

3. Solution of the system of Wiener-Hopf-type equations. We solve the system (2.3) by a factorization method which is similar to the method applied in [3]. First we prove an auxiliary lemma. Let $\rho = \sum_{i=1}^N \pi_i \lambda_i \beta_i$ and $\lambda = \sum_{i=1}^N \pi_i \lambda_i$, and define

$$L(r, \phi, \eta) = (\eta - \phi)I + (I - rB(\phi))\Lambda - Q^T. \quad (3.1)$$

LEMMA 3.1. *The N eigenvalues of $\Lambda - Q^T$ all lie in the right half-plane $\text{Re } \phi > 0$; for $|r| < 1$, $\text{Re } \eta > 0$ or $|r| < 1$, $\text{Re } \eta > 0$, $\det L(r, \phi, \eta)$ has exactly N zeros in the right half-plane $\text{Re } \phi > 0$.*

If $\rho < 1$, $\det L(1, \phi, 0)$ has exactly $N - 1$ zeros in the right half-plane $\text{Re } \phi > 0$ and a simple zero at 0.

PROOF. For $d > 0$ let C_d be the closed contour consisting of the part of the imaginary axis running from $-id$ to id and the half-circle with centre at the origin and radius d running clockwise from id to $-id$.

Let $d_0 = |\eta| + 2\max_{1 \leq i \leq N}(\lambda_i + q_i)$. Then on C_d with $d > d_0$ we have for $|r| < 1$,

$\operatorname{Re} \eta > 0$, or $|r| < 1$, $\operatorname{Re} \eta > 0$,

$$|\eta - \phi + q_i + \lambda_i| > q_i + \lambda_i |r B_i(\phi)|, \quad i = 1, 2, \dots, N. \quad (3.2)$$

From Theorem 1 of Appendix 2 of [3] it follows that

$$\det[(\eta - \phi)I + (I - rB(\phi))\Lambda - Q^T]$$

has the same number of zeros (counted according to their order) inside C_d as

$$\det[(\eta - \phi)I + \operatorname{diag}(q_1 + \lambda_1, q_2 + \lambda_2, \dots, q_N + \lambda_N)] = \prod_{i=1}^N (\eta - \phi + q_i + \lambda_i).$$

This proves the first part of the lemma. Moreover we see that $\det M(0) \neq 0$ and $\det L(r, 0, \eta) \neq 0$ for $|r| < 1$, $\operatorname{Re} \eta > 0$, or $|r| < 1$, $\operatorname{Re} \eta > 0$. To prove the second part consider the matrix

$$L(r, \phi, 0) = -\phi I + (I - rB(\phi))\Lambda - Q^T.$$

The matrix $L^*(r, \phi)$ is obtained from $L(r, \phi, 0)$ by adding all rows to the first row so that

$$L_{ij}^*(r, \phi) = -\phi + (1 - rB_j(\phi))\lambda_j, \quad j = 1, 2, \dots, N.$$

Let $\mu(r)$ be the function defined by

$$\det L(r, \mu(r), 0) = \det L^*(r, \mu(r)) = 0 \quad (3.3)$$

and $\mu(1) = 0$. Using the implicit function theorem we shall prove that there is a neighbourhood of 1 in which the function $\mu(r)$ is uniquely defined and continuously differentiable. Since we have assumed that there exists a $\delta > 0$ such that the $B_i(\phi)$ can be continued analytically to the region $\operatorname{Re} \phi > -\delta$, there exists in $\mathbb{R} \times \mathbb{C}$ a neighbourhood of $(1, 0)$ in which $\det L^*(r, \phi)$ is continuously differentiable with respect to both r and ϕ . Moreover $\det L^*(1, 0) = 0$. For small ϕ we have

$$L_{ij}^*(1, \phi) = -\phi(1 - \beta_j \lambda_j) + o(\phi), \quad \phi \rightarrow 0$$

so that

$$\det L^*(1, \phi) = -\phi \det L^* + o(\phi), \quad \phi \rightarrow 0, \quad \text{where}$$

$$L_{ij}^* = 1 - \beta_j \lambda_j, \quad j = 1, 2, \dots, N;$$

$$L_{ij}^* = -Q_{ji}, \quad i = 2, 3, \dots, N; \quad j = 1, 2, \dots, N.$$

Consequently

$$\frac{\partial}{\partial \phi} \det L^*(1, \phi) \Big|_{\phi=0} = -\det L^*. \quad (3.4)$$

We immediately see that

$$L^* \pi = (1 - \rho) e_1 \quad (3.5)$$

where $e_1 = (1, 0, \dots, 0)^T$. Since $Q^T \pi = 0$ and Q^T has rank $N - 1$, π is the only solution of (3.5) and hence $\det L^* \neq 0$. We see that all conditions of the implicit function theorem are satisfied so that in a neighbourhood of 1 the function $\mu(r)$ is uniquely defined and continuously differentiable. We consider this function for r close to 1, so

$$\mu(r) = -(1 - r)\mu'(1) + o(1 - r), \quad r \rightarrow 1, \quad \text{and}$$

$$L_{ij}^*(r, \mu(r)) = (1 - r)[\mu'(1) + (1 - \beta_j \mu'(1))\lambda_j] + o(1 - r), \quad r \rightarrow 1.$$

Since $\det L^*(r, \mu(r)) = 0$,

$$0 = \frac{d}{dr} \det L^*(r, \mu(r)) \Big|_{r=1} = -\det L^\circ, \quad \text{where}$$

$$L^\circ_{ij} = \mu'(1) + (1 - \beta_j \mu'(1)) \lambda_j, \quad j = 1, 2, \dots, N; \quad \text{and} \quad (3.6)$$

$$L^\circ_{ij} = -Q_{ji}, \quad i = 2, 3, \dots, N; \quad j = 1, 2, \dots, N.$$

Now we must have $L^\circ \pi = 0$ (otherwise $\det L^\circ$ would not equal 0) and in particular $\sum_{j=1}^N L^\circ_{ij} \pi_j = 0$, or with (3.6) $\mu'(1) = -\lambda/(1 - \rho)$, so that

$$\mu(r) = (1 - r)\lambda/(1 - \rho) + o(1 - r), \quad r \rightarrow 1. \quad (3.7)$$

The inequalities $|\phi + \lambda_i + q_i| > q_i + \lambda_i |r B_i(\phi)|$, $i = 1, 2, \dots, N$, hold for $r < 1$, $\text{Re } \phi = 0$, and for $r = 1$, $\text{Re } \phi = 0$, $\phi \neq 0$. This implies (see Lemma 1 of Appendix 2 of [3]) that, for $r < 1$, $\det H(r, \phi, 0)$ has no zeros on the imaginary axis $\text{Re } \phi = 0$, while for $r = 1$ it can have no zeros on the imaginary axis except at 0. According to the first part of the present theorem there are for $r < 1$ exactly N zeros in the right half-plane $\text{Re } \phi > 0$. These zeros are continuous functions of r . From (3.7) we see that for $\rho < 1$ and $r < 1$ and sufficiently close to 1 the function $\mu(r)$ is one of these zeros. Since $\det H(1, \phi, 0)$ has a simple zero at 0 and no zeros elsewhere on the imaginary axis we see that for $r \uparrow 1$ the zero $\mu(r)$ is the only zero which tends to the imaginary axis while the other zeros remain in the right half-plane $\text{Re } \phi > 0$. This proves the second part of the theorem.

From (2.2) we have $A(\phi) = M^{-1}(\phi)\Lambda$ so that

$$H(r, \phi, \eta) = I - r\Lambda(M^T(\eta - \phi))^{-1}B(\phi). \quad (3.8)$$

Consequently

$$\begin{aligned} L(r, \phi, \eta) &= M^T(\eta - \phi)\Lambda^{-1}H(r, \phi, \eta)\Lambda \quad \text{and} \\ \det L(r, \phi, \eta) &= \det M(\eta - \phi)\det H(r, \phi, \eta). \end{aligned} \quad (3.9)$$

Let $\mu_1(r, \eta), \dots, \mu_N(r, \eta)$ be the N zeros in the right half-plane of $\det L(r, \phi, \eta)$, with $\mu_1(1, 0) = 0$, and let $\nu_1, \dots, \nu_N$ be the N eigenvalues of $\Lambda - Q^T$. Note that for $\text{Re } \eta > 0$, $\mu_i(0, \eta) = \nu_i + \eta$, $i = 1, 2, \dots, N$. We next impose the following condition.

Condition 3.1. $\mu_1(r, \eta), \dots, \mu_N(r, \eta)$ and $\nu_1 + \eta, \dots, \nu_N + \eta$ are all distinct.

As in [3] we see that this condition will always be satisfied if $\nu_1, \nu_2, \dots, \nu_N$ are distinct, except possibly for countably many values of η if r is fixed or countably many values of r if η is fixed. Hence from (3.9) it follows that $\mu_1(r, \eta), \dots, \mu_N(r, \eta)$ are the zeros in the right half-plane $\text{Re } \phi > 0$ of $\det H(r, \phi, \eta)$.

For $i = 1, 2, \dots, N$, let D_i be a (nonunique) nonzero row vector satisfying

$$D_i H(r, \mu_i(r, \eta), \eta) = 0, \quad (3.10)$$

and let D be the $N \times N$ -matrix the i th row of which is D_i . For $i = 1, 2, \dots, N$, let R^i be a right eigenvector of $\Lambda - Q^T$ corresponding to the eigenvalue ν_i , while R is the matrix the i th column of which is R^i . Then

$$\Lambda - Q^T = R \text{diag}(\nu_1, \nu_2, \dots, \nu_N) R^{-1}. \quad (3.11)$$

Define the $N \times N$ -matrix S by

$$S_{ij} = \frac{1}{\nu_j + \eta - \mu_i(r, \eta)} D_i \Lambda R^j. \quad (3.12)$$

Note that D , R and S depend on r and η . We assume that the following condition holds.

Condition 3.2. $\det S \neq 0$.

Again we note that this condition will always be satisfied except possibly for countably many values of η if r is fixed or countably many values of r if η is fixed. Define

$$K(r, \phi, \eta) = I - CJ(r, \phi, \eta)D, \quad \text{where} \quad (3.13)$$

$$C = \Lambda RS^{-1} \quad \text{and} \quad J(r, \phi, \eta) = \text{diag} \left(\frac{1}{\phi - \mu_1(r, \eta)}, \dots, \frac{1}{\phi - \mu_N(r, \eta)} \right).$$

We can now state and prove the factorization theorem.

THEOREM 3.1. *If Conditions 3.1 and 3.2 hold then*

$$(i) \quad \det K(r, \phi, \eta) = \prod_{i=1}^N \left[\frac{\phi - \nu_i - \eta}{\phi - \mu_i(r, \eta)} \right],$$

(ii) for $\text{Re } \phi = 0$,

$$H(r, \phi, \eta) = H^-(r, \phi, \eta)H^+(r, \phi, \eta), \quad \text{where}$$

(a) $H^-(r, \phi, \eta) = K^{-1}(r, \phi, \eta)$ is analytic for $\text{Re } \phi < 0$ and bounded and continuous for $\text{Re } \phi \leq 0$;

(b) $H^+(r, \phi, \eta) = K(r, \phi, \eta)H(r, \phi, \eta)$ is analytic for $\text{Re } \phi > 0$ and bounded and continuous for $\text{Re } \phi \geq 0$.

PROOF. From (3.13) we see that $\mu_i(r, \eta)$ is either a simple pole of $\det K(r, \phi, \eta)$ or $\det K(r, \phi, \eta)$ is analytic at $\mu_i(r, \eta)$. Since $\lim_{|\phi| \rightarrow \infty} K(r, \phi, \eta) = I$ we have

$$\det K(r, \phi, \eta) = g(r, \phi, \eta) / \prod_{i=1}^N (\phi - \mu_i(r, \eta)), \quad (3.14)$$

where $g(r, \phi, \eta)$ is a polynomial of ϕ of degree N . From (3.9) and Condition 3.1 it follows that $\det H(r, \phi, \eta)$ is of the form

$$\det H(r, \phi, \eta) = f(r, \phi, \eta) \prod_{i=1}^N \left[\frac{\phi - \mu_i(r, \eta)}{\phi - \eta - \nu_i} \right], \quad (3.15)$$

where $f(r, \phi, \eta)$ is bounded away from 0 for $\text{Re } \phi > 0$.

Expression (3.11) implies

$$[M^T(\eta - \phi)]^{-1} = R \text{diag} \left(\frac{1}{\nu_1 + \eta - \phi}, \dots, \frac{1}{\nu_N + \eta - \phi} \right) R^{-1}, \quad (3.16)$$

so that from (3.8)

$$\begin{aligned} H^+(r, \phi, \eta) &= K(r, \phi, \eta)H(r, \phi, \eta) \\ &= K(r, \phi, \eta) + r \sum_{i=1}^N \frac{1}{\phi - \eta - \nu_i} K(r, \phi, \eta) \Lambda R^i R_i^{-1} B(\phi) \end{aligned} \quad (3.17)$$

where R_i^{-1} is the i th row of R^{-1} . Note that R_i^{-1} is a left eigenvector of $\Lambda - Q^T$ corresponding to the eigenvalue ν_i . Moreover from (3.13) we have

$$\begin{aligned} K(r, \nu_i + \eta, \eta) \Lambda R^i &= \Lambda R^i - \sum_{j=1}^N C^j \frac{1}{\nu_i + \eta - \mu_j(r, \eta)} D_j \Lambda R^i \\ &= \Lambda R^i - \sum_{j=1}^N C^j S_{ji} = 0, \end{aligned}$$

where C^j is the j th column of C . Now (3.17) yields

$$\lim_{\phi \rightarrow \nu_i + \eta} (\phi - \nu_i - \eta) H^+(r, \phi, \eta) = rK(r, \nu_i + \eta, \eta) \Lambda R^i R_i^{-1} B(\nu_i + \eta) = 0.$$

We see that $H^+(r, \phi, \eta)$ has no poles at $\nu_i + \eta$, $i = 1, 2, \dots, N$; but from (3.14) and (3.15)

$$\det H^+(r, \phi, \eta) = f(r, \phi, \eta) g(r, \phi, \eta) / \prod_{i=1}^N (\phi - \nu_i - \eta),$$

and hence

$$g(r, \phi, \eta) = \prod_{i=1}^N (\phi - \nu_i - \eta),$$

which proves part (i) of the theorem.

With (3.13) we have

$$H^+(r, \phi, \eta) = H(r, \phi, \eta) - C \operatorname{diag} \left(\frac{1}{\phi - \mu_1(r, \eta)}, \dots, \frac{1}{\phi - \mu_N(r, \eta)} \right) D H(r, \phi, \eta)$$

so that

$$\lim_{\phi \rightarrow \mu_i(r, \eta)} (\phi - \mu_i(r, \eta)) H^+(r, \phi, \eta) = -C^i D_i H(r, \mu_i(r, \eta), \eta) = 0,$$

and the proof of part (ii) is complete.

REMARK. The Wiener-Hopf factors H^- and H^+ satisfying

$$H(r, \phi, \eta) = H^-(r, \phi, \eta) H^+(r, \phi, \eta)$$

are unique up to multiplication by a constant matrix. This means that our factorization is equivalent to the one given by Arjas [1, Lemma 2.2]. In fact the factorizations are identical since in both cases $\lim_{\phi \rightarrow -\infty} H^-(r, \phi, \eta) = I$. Arjas gives a probabilistic interpretation to the factors but his method does not lead to explicit computationally tractable results. Denote $U_n = S_1 + \dots + S_n$, $n = 1, 2, \dots$; and $M = \inf\{n > 0 \mid T_{n+1} > U_n\}$, then M is the number of customers during the first busy period, U_M the length of the first busy period and T_{M+1} the length of the first busy cycle. From Arjas [1] we now have the following probabilistic interpretation of our Wiener-Hopf factors

$$H_{ji}^-(r, \phi, \eta) = \delta_{ij} - E(r^M \exp(-\phi(U_M - T_{M+1}) - \eta T_{M+1}) 1(Y_{M+1} = j) \mid Y_1 = i) \quad \text{and}$$

$$\begin{aligned} & [H^+(r, \phi, \eta)]_{ji}^{-1} \\ &= \delta_{ij} + \sum_{n=1}^{\infty} r^n E(\exp(-\phi(U_n - T_{n+1}) - \eta T_{n+1}) 1(M > n, Y_{n+1} = j) \mid Y_1 = i). \end{aligned}$$

We can now give the solution of the equations (2.3).

THEOREM 3.2. *If Conditions 3.1 and 3.2 hold then for $\operatorname{Re} \phi > 0$,*

$$K(r, \phi, \eta) H(r, \phi, \eta) Z(r, \phi, \eta) = rK(r, 0, \eta) p$$

if $|r| < 1$, $\operatorname{Re} \eta > 0$, or $|r| < 1$, $\operatorname{Re} \eta > 0$.

PROOF. With a standard Wiener-Hopf argument (see [3]) we have for $\operatorname{Re} \phi > 0$,

$$H^+(r, \phi, \eta) Z(r, \phi, \eta) = H^+(r, 0, \eta) Z(r, 0, \eta).$$

Moreover from (2.3)

$$H(r, 0, \eta) Z(r, 0, \eta) = r p \tag{3.18}$$

so that the theorem is proved.

4. Waiting time and number of customers in the system.

4.1. *Actual waiting time.* The process $((W_n, Y_n), n = 1, 2, \dots)$ is regenerative where for any $i = 1, 2, \dots, N$; the state $(0, i)$ can be chosen to be the regenerative state. We easily see that the return times are aperiodic, so that

$$\lim_{n \rightarrow \infty} P(W_n \leq x, Y_n = i)$$

for $x > 0$ exists. If this limit is 0 no limiting distribution exists, if it is positive (W_n, Y_n) converges weakly to a stationary random vector (W, Y) . From (3.13) we see that

$$\lim_{r \uparrow 1} (1 - r)K(r, 0, 0)p = C^1 \lim_{r \uparrow 1} \frac{1 - r}{\mu_1(r, 0)} D_1 p, \quad (4.1)$$

where C^1 is the first column of C . Since $H(1, 0, 0) = I - A^T(0)$, where $A(0)$ is the transition matrix of the Markov chain $(Y_n, n = 1, 2, \dots)$, we see from (3.10) that $D_1 = D_{11}(1, 1, \dots, 1)$, so that $D_1 p = D_{11}$. From the proof of Lemma 3.1 we have

$$\begin{aligned} \lim_{r \uparrow 1} \frac{1 - r}{\mu_1(r, 0)} &= (1 - \rho)/\lambda, & \text{for } \rho < 1, \\ &= 0, & \text{for } \rho > 1. \end{aligned} \quad (4.2)$$

Part (i) of Theorem 3.1 implies that $C_{i1} \neq 0$ for at least one i , if Conditions 3.1 and 3.2 hold for $r = 1$ and $\eta = 0$.

Denote

$$\begin{aligned} \mu_i &= \mu_i(1, 0), \quad i = 1, 2, \dots, N; \quad \text{where } \mu_1 = 0; \\ H(\phi) &= H(1, \phi, 0); \\ \tilde{K}(\phi) &= I - C \operatorname{diag}\left(0, \frac{1}{\phi - \mu_2}, \frac{1}{\phi - \mu_3}, \dots, \frac{1}{\phi - \mu_N}\right) D; \\ H^+(\phi) &= \tilde{K}(\phi)H(\phi) - \frac{1}{\phi} C^1 D_1 H(\phi), \quad \phi \neq 0, \\ &= \tilde{K}(0)H(0) - C^1 D_1 H'(0), \quad \phi = 0; \end{aligned}$$

and let $Z(\phi)$ be the N -dimensional column vector with components

$$Z_j(\phi) = E(\exp(-\phi W)1(Y = j)), \quad j = 1, 2, \dots, N;$$

then from Theorem 3.2 we have for $\operatorname{Re} \phi > 0$,

$$\begin{aligned} H^+(\phi)Z(\phi) &= C^1 D_{11}(1 - \rho)/\lambda & \text{for } \rho < 1, \\ &= 0 & \text{for } \rho > 1. \end{aligned} \quad (4.3)$$

Let Z_0 be the column vector with components $Z_j = P(W = 0, Y = j) = \lim_{\phi \rightarrow \infty} Z_j(\phi)$. It is easily seen that if $Z_j > 0$ for some j , then $Z_j > 0$ for all $j = 1, 2, \dots, N$; and consequently

$$\begin{aligned} Z_0 &= C^1 D_{11}(1 - \rho)/\lambda & \text{for } \rho < 1, \\ &= 0 & \text{for } \rho > 1; \end{aligned} \quad (4.4)$$

where all components of $C^1 D_{11}$ are strictly positive. The moments $E(W^k 1(Y = j))$, $k = 1, 2, \dots$, are obtained by differentiating (4.3) setting $\phi = 0$ and solving the resulting linear equations. In order to obtain the first two moments we solve $Z(0)$,

$Z'(0)$ and $Z''(0)$ from the equations

$$\begin{aligned} H^+(0)Z(0) &= Z_0, \quad (H^+(0))'Z(0) + H^+(0)Z'(0) = 0, \quad \text{and} \quad (4.5) \\ (H^+(0))''Z(0) + 2(H^+(0))'Z'(0) + H^+(0)Z''(0) &= 0, \quad \text{where} \\ H^+(0) &= \tilde{K}(0)H(0) - C^1D_1H'(0), \\ (H^+(0))' &= \tilde{K}'(0)H(0) + \tilde{K}(0)H'(0) - \frac{1}{2}C^1D_1H''(0) \quad \text{and} \\ (H^+(0))'' &= \tilde{K}''(0)H(0) + 2\tilde{K}'(0)H'(0) + \tilde{K}(0)H''(0) - \frac{1}{3}C^1D_1H'''(0). \end{aligned}$$

Then

$$\begin{aligned} E(W1(Y=j)) &= -Z'_j(0), \quad j = 1, 2, \dots, N; \\ E(W^21(Y=j)) &= Z''_j(0), \quad j = 1, 2, \dots, N. \end{aligned} \quad (4.6)$$

4.2. *Virtual waiting time.* Let W_t^* be the virtual waiting time at time t , then

$$W_t^* = [W_{N_t} + S_{N_t} + T_{N_t} - t]^+, \quad (4.7)$$

where N_t is the number of customers arriving during $[0, t]$. Consequently for $\text{Re } \phi > 0$,

$$\begin{aligned} E(\exp(-\phi W_t^*)1(X_t = k)) \\ &= \sum_{n=1}^{\infty} E(\exp(-\phi[W_n + S_n + T_n - t]^+)1(T_n < t < T_{n+1}, X_t = k)) \\ &= \sum_{n=1}^{\infty} \sum_{l=1}^N \int_0^t P(A_2 > t-u, X_{t-u} = k | X_0 = l) \\ &\quad \cdot d_u E(\exp(-\phi[W_n + S_n + T_n - t]^+)1(Y_n = l, T_n \leq u)). \end{aligned} \quad (4.8)$$

Let $P(t)$ be the transition matrix of the Markov chain $(X_t, t > 0)$ and

$$\tilde{P}(\phi) = \int_0^{\infty} \exp(-\phi t)P(t)dt, \quad \text{Re } \phi > 0,$$

then

$$\tilde{P}(\phi) = (\phi I - Q)^{-1}. \quad (4.9)$$

Since

$$P(A_2 > t, X_t = k | X_0 = l) = P_{lk}(t) - \sum_{m=1}^N \int_0^t P_{mk}(t-u)dP(A_2 \leq u, Y_1 = m | Y_0 = l)$$

we have from (2.2) and (4.9)

$$\begin{aligned} \int_0^{\infty} \exp(-\eta t)P(A_2 > t, X_t = k | X_0 = l)dt \\ &= \tilde{P}_{lk}(\eta) - \sum_{m=1}^N \tilde{P}_{mk}(\eta)A_{lm}(\eta) = [M^{-1}(\eta)]_{lk}. \end{aligned} \quad (4.10)$$

From the identity

$$\exp(-\phi x^+) = \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \frac{\phi}{\phi - \xi} \exp(-\xi x), \quad \text{Re } \phi > \text{Re } \xi > 0, \quad (4.11)$$

where the path of integration is a line parallel to the imaginary axis, we have

$$\begin{aligned} & E(\exp(-\phi[W_n + S_n + T_n - t]^+)1(Y_n = l, T_n < u)) \\ &= \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \frac{\phi}{\phi - \xi} \exp(\xi t) B_l(\xi) E(\exp(-\xi(W_n + T_n))1(Y_n = l, T_n < u)). \end{aligned} \quad (4.12)$$

Combining (4.8), (4.10) and (4.12) yields

$$\begin{aligned} & \int_0^\infty \exp(-\eta t) E(\exp(-\phi W_t^*)1(X_t = k)) dt \\ &= \sum_{n=1}^\infty \sum_{l=1}^N \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \frac{\phi}{\phi - \xi} M^{-1}(\eta - \xi)_{lk} B_l(\xi) \\ & \quad \cdot E(\exp(-\xi W_n - \eta T_n)1(Y_n = l)). \end{aligned} \quad (4.13)$$

If $Z^*(\eta, \phi)$ is the column vector with components

$$Z_k^*(\eta, \phi) = \int_0^\infty \exp(-\eta t) E(\exp(-\phi W_t^*)1(X_t = k)) dt,$$

then (4.13) becomes

$$Z^*(\eta, \phi) = \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \frac{\phi}{\phi - \xi} [M^T(\eta - \xi)]^{-1} B(\xi) Z(1, \xi, \eta), \quad (4.14)$$

$\operatorname{Re} \phi > \operatorname{Re} \xi > 0, \operatorname{Re} \eta > 0$.

The interchanges of the order of integration and summation which were made in the above derivation are justified by absolute convergence and Fubini's theorem. From Theorem 3.2 we have

$$\Lambda[M^T(\eta - \phi)]^{-1} B(\phi) Z(1, \phi, \eta) = Z(1, \phi, \eta) - K^{-1}(1, \phi, \eta) K(1, 0, \eta) p.$$

Inserting this into (4.14) and applying contour integration gives for $\operatorname{Re} \eta > 0, \operatorname{Re} \phi > 0$,

$$Z^*(\eta, \phi) = \Lambda^{-1} Z(1, \phi, \eta) - \Lambda^{-1} p. \quad (4.15)$$

The process $((W_t^*, X_t), t \geq 0)$ is regenerative, where the regeneration points are the epochs at which the process enters a state $(0, i)$ for some $i, 1 \leq i \leq N$. Since the times between regeneration points are nonarithmetic $\lim_{t \rightarrow \infty} E(\exp(-\phi W_t^*)1(X_t = i))$ exists. Denote this limit by $Z_i^*(\phi)$ and let $Z^*(\phi)$ be the N -dimensional column vector with components $Z_i^*(\phi)$. Similar to (4.2) we have

$$\begin{aligned} \lim_{\eta \downarrow 0} \frac{\eta}{\mu_1(1, \eta)} &= 1 - \rho & \text{for } \rho < 1, \\ &= 0 & \text{for } \rho > 1, \end{aligned}$$

so that

$$\begin{aligned} \lim_{\eta \downarrow 0} \eta K(1, 0, \eta) p &= C^1 D_{11}(1 - \rho) & \text{for } \rho < 1, \\ &= 0 & \text{for } \rho > 1. \end{aligned}$$

From Theorem 3.2 and (4.3) it therefore follows that

$$\lim_{\eta \downarrow 0} \eta Z(1, \phi, \eta) = \lambda Z(\phi). \quad (4.16)$$

We see that (W_t^*, X_t) converges weakly to a random vector (W^*, X) iff $\rho < 1$, and

from (4.15) we have for $\text{Re } \phi > 0$,

$$Z^*(\phi) = \lambda \Lambda^{-1} Z(\phi). \quad (4.17)$$

Let Z_0^* be the column vector with components $Z_j^* = P(W^* = 0, X = j)$, then

$$Z_0^* = \lambda \Lambda^{-1} Z_0 = \Lambda^{-1} C^{-1} D_{11} (1 - \rho), \quad (4.18)$$

while with (4.17) the moments of W^* are easily expressed in those of W .

4.3. *Number of customers in the system at arrival epochs.* Let C_n be the number of customers in the system at $T_n -$, i.e. just before the arrival of the n th customer. Since

$$\{C_n < j\} = \Omega, \quad n = 1, 2, \dots, j+1; \quad (4.19)$$

$$\{C_{n+j+1} < j\} = \{T_n + W_n + S_n < T_{n+j+1}\}, \quad n = 1, 2, \dots;$$

we find by straightforward calculations

$$\begin{aligned} \sum_{n=1}^{\infty} r^n E(s^{C_n} 1(Y_n = k)) &= r \sum_{l=1}^N p_l [I - rsA(0)]_{lk}^{-1} + r(1-s) \sum_{l=1}^N \sum_{j=0}^{\infty} (rs)^j \\ &\times \sum_{n=1}^{\infty} r^n P(T_n + W_n + S_n < T_{n+j+1}, Y_n = i, Y_{n+j+1} = k). \end{aligned} \quad (4.20)$$

We use the identity

$$1(x < 0) + \frac{1}{2} 1(x = 0) = \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \exp(-\xi x) \quad (4.21)$$

where $\int_{-i\infty+0}^{i\infty+0}$ stands for $\lim_{R \rightarrow \infty} \int_{-iR+\delta}^{iR+\delta}$ with $\delta > 0$ and sufficiently small. Then we have

$$\begin{aligned} &P(T_n + W_n + S_n < T_{n+j+1}, Y_n = i, Y_{n+j+1} = k) \\ &= \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} E(\exp(-\xi(W_n + S_n + T_n - T_{n+j+1}))) 1(Y_n = i, Y_{n+j+1} = k) \\ &= \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} E(\exp(-\xi W_n) 1(Y_n = i)) B_i(\xi) [A^{j+1}(-\xi)]_{ik}. \end{aligned}$$

Inserting this into (4.20) yields

$$\begin{aligned} \sum_{n=1}^{\infty} r^n E(s^{C_n} 1(Y_n = k)) &= r \sum_{l=1}^N p_l [I - rsA(0)]_{lk}^{-1} \\ &+ r(1-s) \sum_{i=1}^N \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} Z_i(r, \xi, 0) B_i(\xi) \\ &\times [(I - rsA(-\xi))^{-1} A(-\xi)]_{ik}. \end{aligned}$$

Let $U_k(s, n) = E(s^{C_n} 1(Y_n = k))$ and $U(s, n)$ the N -dimensional column vector with components $U_k(s, n)$, then

$$\begin{aligned} \sum_{n=1}^{\infty} r^n U(s, n) &= r(\Lambda - Q^T)((1 - rs)\Lambda - Q^T)^{-1} p \\ &+ r(1-s) \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} \Lambda(-\xi I + (1 - rs)\Lambda - Q^T)^{-1} \\ &\times B(\xi) Z(r, \xi, 0). \end{aligned} \quad (4.23)$$

The process $((C_n, Y_n), n = 1, 2, \dots)$ is regenerative with the same regeneration points as the process $((W_n, Y_n), n = 1, 2, \dots)$, because the events $\{C_n = 0\}$ and $\{W_n = 0\}$ are identical. Therefore (C_n, Y_n) converges weakly to a stationary vector (C, Y) iff $\rho < 1$. For $\rho < 1$ let $U_k(s) = E(s^C 1(Y = k))$ and $U(s)$ the column vector with components $U_k(s)$. Then

$$U(s) = (1-s) \frac{1}{2\pi i} \int_{-i\infty-0}^{i\infty+0} \frac{d\xi}{\xi} \Lambda(-\xi I + (1-s)\Lambda - Q^T)^{-1} B(\xi) Z(\xi). \quad (4.24)$$

As $\{C_n = 0\}$ and $\{W_n = 0\}$ are identical we have

$$U(0) = Z_0, \quad \text{where} \quad U_k(0) = P(C = 0, Y = k).$$

This also follows from (4.24) by using (4.21).

Since we have assumed that there exists a $\delta > 0$ such that $B(\phi)$ can be continued analytically to the region $\text{Re } \phi > -\delta$, we see from (4.3) that the same applies to $Z(\phi)$. Expression (4.17) implies

$$Z(0) = \lambda^{-1} \Lambda Z^*(0) = \lambda^{-1} \Lambda \pi, \quad (4.25)$$

so that

$$\begin{aligned} & (1-s)\Lambda((1-s)\Lambda - Q^T)^{-1} Z(0) \\ &= \lambda^{-1} \Lambda((1-s)\Lambda - Q^T)^{-1} (1-s)\Lambda \pi \\ &= \lambda^{-1} \Lambda \pi + \lambda^{-1} \Lambda((1-s)\Lambda - Q^T)^{-1} Q^T \pi = \lambda^{-1} \Lambda \pi. \end{aligned}$$

Consequently from (4.24) we have for $|s| < 1$,

$$U(s) = \lambda^{-1} \Lambda \pi + \frac{1}{2\pi i} \int_{-i\infty-0}^{i\infty-0} \frac{d\xi}{\xi} (1-s)\Lambda(-\xi I + (1-s)\Lambda - Q^T)^{-1} B(\xi) Z(\xi). \quad (4.26)$$

We easily see that

$$\begin{aligned} & \frac{\partial^n}{\partial s^n} (1-s)\Lambda(-\xi I + (1-s)\Lambda - Q^T)^{-1} \\ &= n! \left[\Lambda(-\xi I + (1-s)\Lambda - Q^T)^{-1} \right]^{n+1} (\xi I + Q^T) \Lambda^{-1}. \end{aligned} \quad (4.27)$$

The integral in (4.26) converges uniformly in s for $0 < s < 1$, so that the first derivative of $U(s)$ is found by differentiating the integrand with respect to s . Similarly we find the higher derivatives so that for the n th derivative of $U(s)$ we obtain from (4.26) and (4.27),

$$\begin{aligned} U^{(n)}(s) &= \frac{n!}{2\pi i} \int_{-i\infty-0}^{i\infty-0} \frac{d\xi}{\xi} \left[\Lambda(-\xi I + (1-s)\Lambda - Q^T)^{-1} \right]^{n+1} \\ &\quad \cdot (\xi I + Q^T) \Lambda^{-1} B(\xi) Z(\xi), \quad \text{and} \end{aligned} \quad (4.28)$$

$$U^{(n)}(1) = \frac{(-1)^{n+1} n!}{2\pi i} \int_{-i\infty-0}^{i\infty-0} \frac{d\xi}{\xi} \left[\Lambda(\xi I + Q^T)^{-1} \right]^n B(\xi) Z(\xi). \quad (4.29)$$

Let $\tau_1, \tau_2, \dots, \tau_N$ be the eigenvalues of $-Q^T$ with $\tau_1 = 0$ and $\text{Re } \tau_i > 0, i = 2, \dots, N$. Assume that they are distinct. The right eigenvector of $-Q^T$ corresponding with τ_i is

denoted by V^j and V is the $N \times N$ -matrix the j th column of which is V^j . Let V_j^{-1} be the j th row of V^{-1} . Note that we can take $V^1 = \pi$ and $V_1^{-1} = \mathbf{1}$, where $\mathbf{1}$ is the N -dimensional row vector $(1, 1, \dots, 1)$. With these notations we may write

$$(\xi I + Q^T)^{-1} = V \operatorname{diag} \left(\frac{1}{\xi}, \frac{1}{\xi - \tau_2}, \dots, \frac{1}{\xi - \tau_N} \right) V^{-1}. \quad (4.30)$$

Inserting (4.30) into (4.29) for $n = 1$ and applying contour integration yields

$$\begin{aligned} U'(1) &= \frac{1}{2\pi i} \int_{-\infty-0}^{\infty-0} \frac{d\xi}{\xi^2} \Lambda \pi \mathbf{1} B(\xi) Z(\xi) \\ &\quad + \sum_{j=2}^N \frac{1}{2\pi i} \int_{-\infty-0}^{\infty-0} \frac{d\xi}{\xi} \Lambda V^j V_j^{-1} \frac{1}{\xi - \tau_j} B(\xi) Z(\xi) \\ &= -\Lambda \pi \mathbf{1} [B'(0)Z(0) + B(0)Z'(0)] \\ &\quad + \sum_{j=2}^N \Lambda V^j V_j^{-1} \tau_j^{-1} [B(0)Z(0) - B(\tau_j)Z(\tau_j)], \quad \text{or} \end{aligned} \quad (4.31)$$

$$U'(1) = \Lambda \pi [\rho/\lambda + EW] + \sum_{j=2}^N \bar{V}_j \tau_j^{-1} [\lambda^{-1} \Lambda \pi - B(\tau_j)Z(\tau_j)], \quad \text{where} \quad (4.32)$$

$$\bar{V}_j = \Lambda V^j V_j^{-1}.$$

In the same way we may find explicit expressions for $U^{(n)}(1)$, $n = 2, 3, \dots$; from (4.29) and (4.30).

For $U''(1)$ we find

$$\begin{aligned} U''(1) &= \Lambda \pi \left[\sum_{i=1}^N \pi_i \lambda_i \beta_i^{(2)} + 2\lambda \sum_{i=1}^N \beta_i E(W1(Y=i)) + \lambda EW^2 \right] \\ &\quad + 2V_1^* [Z'(0) - \lambda^{-1} \beta \Lambda \pi] \\ &\quad + 2 \left[\left(\sum_{j=2}^N \bar{V}_j \tau_j^{-1} \right)^2 - \sum_{j=2}^N (\bar{V}_1 \bar{V}_j + \bar{V}_j \bar{V}_1) \tau_j^{-2} \right] \lambda^{-1} \Lambda \pi \\ &\quad + 2 \sum_{j=2}^N \bar{V}_j^2 \tau_j^{-1} [B'(\tau_j)Z(\tau_j) + B(\tau_j)Z'(\tau_j)] \\ &\quad + 2 \sum_{j=2}^N [V_j^* \tau_j^{-1} - \bar{V}_j^2 \tau_j^{-2}] B(\tau_j)Z(\tau_j), \end{aligned}$$

where $\beta_i^{(2)} = B_i''(0)$ and

$$V_j^* = \sum_{\substack{k=1 \\ k \neq j}}^N (\bar{V}_j \bar{V}_k + \bar{V}_k \bar{V}_j) \frac{1}{\tau_j - \tau_k}, \quad j = 1, 2, \dots, N.$$

The vectors $Z(\tau_j)$ and $Z'(\tau_j)$ can be solved from (4.3). For the first two moments of C we now have

$$E(C1(Y=k)) = U'_k(1) \quad E(C^2 1(Y=k)) = U''_k(1) + U'_k(1). \quad (4.34)$$

4.4. Number of customers in the system in continuous time. The number of custom-

ers in the system at time t is denoted by C_t^* . From the relation

$$\begin{aligned}\{C_t^* < j\} &= \{N_t < j\} \cup \bigcup_{n=1}^{\infty} \{T_n + W_n + S_n < t, N_t = n + j\} \\ &= \{t < T_{j+1}\} \cup \bigcup_{n=1}^{\infty} \{T_n + W_n + S_n < t, T_{n+j} < t < T_{n+j+1}\}\end{aligned}$$

we find analogous to the derivation of (4.23), for $\operatorname{Re} \eta > \operatorname{Re} \xi$, $|s| < 1$,

$$\begin{aligned}\int_0^{\infty} \exp(-\eta t) U^*(s, t) dt &= s(\eta I + (1-s)\Lambda - Q^T)^{-1} p \\ &\quad + (1-s) \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} ((\eta - \xi)I + (1-s)\Lambda - Q^T)^{-1} \\ &\quad \times B(\xi) Z(1, \xi, \eta),\end{aligned}\quad (4.35)$$

where $U^*(s, t)$ is the N -dimensional column vector with components

$$U_k^*(s, t) = E(s^{C_t^*} 1(X_t = k)).$$

The process $((C_t^*, X_t), t \geq 0)$ is regenerative with the same regeneration epochs as $((W_t^*, X_t), t \geq 0)$. Consequently (C_t^*, X_t) converges for $t \rightarrow \infty$ to a stationary vector (C^*, X) iff $\rho < 1$. For $\rho < 1$ let $U_k^*(s) = E(s^{C^*} 1(X = k))$ and $U^*(s)$ the column vector with components $U_k^*(s)$. With (4.16) we find from (4.35) for $|s| < 1$,

$$U^*(s) = (1-s) \frac{1}{2\pi i} \int_{-i\infty+0}^{i\infty+0} \frac{d\xi}{\xi} (-\xi I + (1-s)\Lambda - Q^T)^{-1} B(\xi) \lambda Z(\xi). \quad (4.36)$$

By comparing (4.36) with (4.24) we see that

$$U^*(s) = \lambda \Lambda^{-1} U(s). \quad (4.37)$$

The probabilities $P(C^* = 0, X = j)$ and the moments $E(C^{*n} 1(X = j))$ are now easily expressed in the corresponding quantities for (C, Y) .

5. Numerical algorithm. The analysis in the previous sections implies a numerical algorithm for the computation of several quantities of interest. If $\rho < 1$ this algorithm yields exact results for the steady state distributions of:

- the actual waiting time W ;
- the virtual waiting time W^* ;
- the total number of customers in the system at arrival epochs C ;
- the total number of customers in the system in continuous time C^* .

In particular we can calculate the probabilities that these random variables are equal to zero as well as all their moments. We shall restrict ourselves to the first two moments.

In this section we give a summary of some important definitions and results, a description of the numerical algorithm and some numerical examples.

For a complete description of the model the following quantities have to be specified:

- N : the number of states of the underlying Markov chain $(X_t, t \geq 0)$;
- Q : the generator of this Markov chain;
- $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$, where λ_i is the arrival rate while the underlying Markov chain is in state i ;
- $B(\phi) = \operatorname{diag}(B_1(\phi), B_2(\phi), \dots, B_N(\phi))$, where $B_i(\phi)$ is the Laplace-Stieltjes transform

of the service time of a customer arriving while the underlying Markov chain is in state i .

Moreover we denote

$$\beta = \text{diag}(\beta_1, \beta_2, \dots, \beta_N),$$

$$\beta^{(k)} = \text{diag}(\beta_1^{(k)}, \beta_2^{(k)}, \dots, \beta_N^{(k)}), \quad k = 2, 3, \dots;$$

where β_i is the first moment and $\beta_i^{(k)}$ the k th moment corresponding to $B_i(\phi)$, i.e. $\beta_i = -B_i'(0)$ and $\beta_i^{(k)} = (-1)^k B_i^{(k)}(0)$.

We have defined for $i = 1, 2, \dots, N$;

$$Z_i(\phi) = E(\exp(-\phi W)1(Y = i)), \quad Z_i^*(\phi) = E(\exp(-\phi W^*)1(X = i)),$$

$$U_i(s) = E(s^C 1(Y = i)), \quad U_i^*(s) = E(s^{C^*} 1(X = i)),$$

and $Z(\phi)$, $Z^*(\phi)$, $U(s)$ and $U^*(s)$ as the column vectors the i th components of which are $Z_i(\phi)$, $Z_i^*(\phi)$, $U_i(s)$ and $U_i^*(s)$ respectively. Furthermore we denote the stationary distribution of the Markov chain $(X, t > 0)$ by the N -dimensional column vector π , the mean arrival rate by $\lambda = \sum_{i=1}^N \pi_i \lambda_i$ and the traffic intensity by $\rho = \sum_{i=1}^N \pi_i \lambda_i \beta_i$.

From (4.17) and (4.37) we have for $i = 1, 2, \dots, N$:

$$Z_i^*(\phi) = (\lambda/\lambda_i)Z_i(\phi), \quad U_i^*(\phi) = (\lambda/\lambda_i)U_i(\phi). \quad (5.1)$$

This implies the intuitive fact that

$$E(\exp(-\phi W^*) | X = i) = E(\exp(-\phi W) | Y = i),$$

$$E(s^{C^*} | X = i) = E(s^C | Y = i). \quad (5.2)$$

For a convenient presentation of the numerical results we introduce the following additional notations:

the stationary distribution of the Markov chain $(Y_n, n = 1, 2, \dots)$ is denoted by the N -dimensional column vector γ ; from (5.1) we have $\gamma_i = \pi_i \lambda_i / \lambda$, $i = 1, 2, \dots, N$;

the mean expected service time $\beta = \sum_{i=1}^N \gamma_i \beta_i$;

the coefficient of variation $c_i^2 = \beta_i^{(2)} / \beta_i^2 - 1$;

$P0_i = P(W = 0 | Y = i) = P(C = 0 | Y = i) = P(W^* = 0 | X = i) = P(C^* = 0 | X = i)$;

$EW_i = E(W | Y = i) = E(W^* | X = i)$;

$EW_i^2 = E(W^2 | Y = i) = E(W^{*2} | X = i)$;

$\text{var } W_i = EW_i^2 - (EW_i)^2$;

$EC_i = E(C | Y = i) = E(C^* | X = i)$;

$EC_i^2 = E(C^2 | Y = i) = E(C^{*2} | X = i)$;

$\text{var } C_i = EC_i^2 - (EC_i)^2$.

Next we give a stepwise description of the numerical algorithm:

1. Compute π from the system $Q^T \pi = 0$ and the normalizing condition $1\pi = 1$; compute $\lambda = 1\Lambda\pi$, $\beta = 1\beta\gamma$ and $\rho = \lambda\beta$.

2. Determine the zeros $\mu_1 = 0, \mu_2, \dots, \mu_N$ in the right half-plane of

$$\det L(\phi) = \det(-\phi I + (I - B(\phi))\Lambda - Q^T),$$

and the corresponding nonzero row vectors $F_1, F_2, \dots, F_N$ satisfying $F_i L(\mu_i) = 0$, $i = 1, 2, \dots, N$. Compute the row vectors

$$D_i = F_i(-\mu_i I + \Lambda - Q^T), \quad i = 1, 2, \dots, N;$$

and let D be the $N \times N$ -matrix the i th row of which is D_i .

3. Determine the eigenvalues $\nu_1, \nu_2, \dots, \nu_N$ (in the right half-plane) of $\Lambda - Q^T$ and the corresponding right eigenvectors $R^1, R^2, \dots, R^N$; and let R be the $N \times N$ -matrix the i th column of which is R^i .

4. Compute the matrix S with $S_{ij} = D_i \Lambda R^j / (\nu_j - \mu_i)$, $i = 1, 2, \dots, N$; $j = 1, 2, \dots, N$; and the matrix $C = \Lambda R S^{-1}$.

Then for $i = 1, 2, \dots, N$;

$$P0_i = C_{i1} D_{11} (1 - \rho) / (\pi_i \lambda_i).$$

5. Solve $Z'(0)$ and $Z''(0)$ from (4.5) then for $i = 1, \dots, N$,

$$EW_i = -Z'_i(0)/\gamma_i; \quad EW_i^2 = Z''_i(0)/\gamma_i.$$

6. Determine the eigenvalues $\tau_1 = 0, \tau_2, \dots, \tau_N$ (in the right half-plane) of $-Q^T$ and the corresponding right eigenvectors $V^1 = \pi, V^2, \dots, V^N$; and let V be the $N \times N$ -matrix the i th column of which is V^i .

7. For $j = 1, 2, \dots, N$, solve $Z(\tau_j)$ and $Z'(\tau_j)$ from (4.3). Compute $U'(1)$ and $U''(1)$ using (4.32) and (4.33) then we have for $i = 1, 2, \dots, N$,

$$EC_i = U'_i(1)/\gamma_i; \quad EC_i^2 = [U''_i(1) + U'_i(1)]/\gamma_i.$$

The algorithm can be used whenever the following conditions are satisfied:

— $\lambda_i > 0$, $i = 1, 2, \dots, N$;

— $\tau_1, \tau_2, \dots, \tau_N$ are distinct;

— $\mu_1, \mu_2, \dots, \mu_N, \nu_1, \nu_2, \dots, \nu_N$ are all distinct;

— $\det S \neq 0$.

In the examples that we have considered we have chosen Λ and Q such that the first two conditions were satisfied. It turned out that for these choices the last two conditions were also satisfied.

It is seen that any model for which the conditions are not all satisfied can be approximated arbitrarily closely by a model for which they are. We have considered examples for which $\tau_1, \tau_2, \dots, \tau_N$ are all real and examples for which some of them are not. In the first case $\mu_1, \mu_2, \dots, \mu_N$ and $\nu_1, \nu_2, \dots, \nu_N$ turned out to be also real. Our numerical experience has led us to the following conjectures.

Conjecture 5.1. If $\lambda_i > 0$, $i = 1, 2, \dots, N$, and $\tau_1, \tau_2, \dots, \tau_N$ are distinct then $\mu_1, \mu_2, \dots, \mu_N, \nu_1, \nu_2, \dots, \nu_N$ are all distinct and $\det S \neq 0$.

Conjecture 5.2. If $\tau_1, \tau_2, \dots, \tau_N$ are all real then so are $\mu_1, \mu_2, \dots, \mu_N, \nu_1, \nu_2, \dots, \nu_N$.

We have not been able to prove these conjectures. If $\tau_1, \tau_2, \dots, \tau_N$ are all real the $\mu_1, \mu_2, \dots, \mu_N$ are easily calculated using an improved regula falsi. In the case that some of the τ_i are not real we have restricted ourselves to rational $B(\phi)$ so that $\mu_1, \mu_2, \dots, \mu_N$ are zeros of a polynomial and can be calculated using Bairstow's method.

EXAMPLE 5.1 (Burman and Smith [2], Model 2 with exponential holding times). $N = 6$, $\rho = 0.8$, $\lambda = 0.8$, $\beta = 1$.

					Q_{ij}						
i	λ_i	G_i	β_i	c_i^2	j	1	2	3	4	5	6
1	0.00	M	1.00	1.00		-0.5	0.5	0.0	0.0	0.0	0.0
2	0.32	M	1.00	1.00		0.1	-0.5	0.4	0.0	0.0	0.0
3	0.64	M	1.00	1.00		0.0	0.2	-0.5	0.3	0.0	0.0
4	0.96	M	1.00	1.00		0.0	0.0	0.3	-0.5	0.2	0.0
5	1.28	M	1.00	1.00		0.0	0.0	0.0	0.4	-0.5	0.1
6	1.60	M	1.00	1.00		0.0	0.0	0.0	0.0	0.5	-0.5

i	π_i	γ_i	$P0_i$	EW_i	$\text{var } W_i$	EC_i	$\text{var } C_i$
1	0.03125	0.00000	0.52063	3.6569	47.632	3.6569	43.975
2	0.15625	0.06250	0.34840	4.6157	54.328	4.6157	49.712
3	0.31250	0.25000	0.22507	5.6844	61.071	5.6844	55.386
4	0.31250	0.37500	0.14082	6.8466	67.716	6.8466	60.869
5	0.15625	0.25000	0.08558	8.0859	74.163	8.0859	66.077
6	0.03125	0.06250	0.05065	9.3872	80.355	9.3872	70.967

$P(W=0)=P(C=0)=0.15541$	$P(W^*=0)=P(C^*=0)=0.2000$
$EW=6.8852$	$EW^*=6.3082$
$\text{var } W=69.054$	$\text{var } W^*=65.992$
$EC=6.8852$	$EC^*=6.3082$
$\text{var } C=62.169$	$\text{var } C^*=59.683$

EXAMPLE 5.2 (Burman and Smith [2], Model 2 with constant holding times). $N=6$, $\rho=0.8$, $\lambda=0.8$, $\beta=1$.

					Q_y						
i	λ_i	G_i	β_i	c_i^2	j	1	2	3	4	5	6
1	0.00	D	1.00	0.00		-0.5	0.5	0.0	0.0	0.0	0.0
2	0.32	D	1.00	0.00		0.1	-0.5	0.4	0.0	0.0	0.0
3	0.64	D	1.00	0.00		0.0	0.2	-0.5	0.3	0.0	0.0
4	0.96	D	1.00	0.00		0.0	0.0	0.3	-0.5	0.2	0.0
5	1.28	D	1.00	0.00		0.0	0.0	0.0	0.4	-0.5	0.1
6	1.60	D	1.00	0.00		0.0	0.0	0.0	0.0	0.5	-0.5

i	π_i	γ_i	$P0_i$	EW_i	$\text{var } W_i$	EC_i	$\text{var } C_i$
1	0.03125	0.00000	0.62504	1.8316	17.335	2.0289	18.539
2	0.15625	0.06250	0.38863	2.5815	21.352	2.8907	22.428
3	0.31250	0.25000	0.22526	3.4974	25.593	3.8846	26.466
4	0.31250	0.37500	0.12291	4.5572	29.857	4.9941	30.511
5	0.15625	0.25000	0.06370	5.7322	33.988	6.1987	34.452
6	0.03125	0.06250	0.03161	6.9954	37.900	7.4783	38.219

$P(W=0)=P(C=0)=0.14460$	$P(W^*=0)=P(C^*=0)=0.20000$
$EW=4.6149$	$EW^*=4.0919$
$\text{var } W=31.033$	$\text{var } W^*=29.079$
$EC=5.0417$	$EC^*=4.4919$
$\text{var } C=31.792$	$\text{var } C^*=29.985$

EXAMPLE 5.3 (Van Hoorn and Seelen [5, Table 1] with $\rho=0.6$, $\theta=0.2$). $N=2$, $\rho=0.6$, $\lambda=0.6$, $\beta=1$.

					Q_{ij}		
i	λ_i	G_i	β_i	c_i^2	j	1	2
1	0.92399	H_2	1	4		-0.0446	0.0446
2	0.17263	H_2	1	4		0.0588	-0.0588

i	π_i	γ_i	$P0_i$	EW_i	$\text{var } W_i$	EC_i	$\text{var } C_i$
1	0.56880	0.87594	0.23754	6.9817	101.443	5.8880	61.741
2	0.43120	0.12406	0.61430	3.3583	59.084	2.7138	36.651

$P(W = 0) = P(C = 0) = 0.28428$ $EW = 6.5321$ $\text{var } W = 97.614$ $EC = 5.4942$ $\text{var } C = 59.723$	$P(W^* = 0) = P(C^* = 0) = 0.40000$ $EW^* = 5.4193$ $\text{var } W^* = 86.398$ $EC^* = 4.5193$ $\text{var } C^* = 53.394$

EXAMPLE 5.4. A case in which the eigenvalues of $-Q^T$ are real. $N = 8$, $\rho = 0.9$, $\lambda = 0.9$, $\beta = 1$.

					Q_{ij}								
i	λ_i	G_i	β_i	c_i^2	j	1	2	3	4	5	6	7	8
1	0.41976	M	0.41253	1.0000		-12	6	2	1	1	0	1	1
2	0.83952	D	1.03132	0.0000		3	-8	2	2	1	0	0	0
3	1.34323	E_2	1.23758	0.5000		1	4	-9	3	0	1	0	0
4	1.13335	H_2	1.15250	1.3533		0	0	3	-5	2	0	0	0
5	0.62964	D	0.59301	0.0000		0	1	0	2	-7	3	1	0
6	0.16790	E_2	0.92819	0.5000		0	0	0	1	3	-6	1	1
7	1.00742	M	1.46963	1.0000		1	1	1	2	1	2	-10	2
8	1.67904	E_3	0.54144	0.3333		0	0	0	0	1	1	2	-4

i	π_i	γ_i	$P0_i$	EW_i	$\text{var } W_i$	EC_i	$\text{var } C_i$
1	0.04734	0.02208	0.09988	9.3642	110.64	9.3399	100.46
2	0.12903	0.12036	0.09670	9.4119	110.71	9.3693	100.49
3	0.13060	0.19492	0.09225	9.5092	111.07	9.4282	100.58
4	0.25741	0.32415	0.09653	9.4452	111.04	9.3896	100.56
5	0.17540	0.12271	0.10533	9.2581	110.41	9.2895	100.44
6	0.13820	0.02578	0.11379	9.1813	110.22	9.2109	100.36
7	0.05041	0.05643	0.10213	9.3044	110.56	9.3325	100.53
8	0.07160	0.13357	0.09146	9.2550	110.16	9.4645	100.74

$P(W = 0) = P(C = 0) = 0.09695$ $EW = 9.3888$ $\text{var } W = 110.762$ $EC = 9.3835$ $\text{var } C = 100.558$	$P(W^* = 0) = P(C^* = 0) = 0.10000$ $EW^* = 9.3554$ $\text{var } W^* = 110.683$ $EC^* = 9.3499$ $\text{var } C^* = 100.517$

EXAMPLE 5.5. A case in which some eigenvalues of $-Q^T$ are nonreal. $N = 5$, $\rho = 0.95$, $\lambda = 0.95$, $\beta = 1$.

i	λ_i	G_i	β_i	c_i^2	Q_{ij}					
					j	1	2	3	4	5
1	0.65242	M	1.27265	1.0000		-4	1	1	1	1
2	0.97863	E_2	1.07380	0.5000		1	-5	2	0	2
3	0.32621	H_2	0.17101	1.0757		3	3	-9	1	2
4	1.30484	E_3	2.14760	0.3333		1	2	1	-8	4
5	1.63105	M	0.63633	1.0000		3	2	1	0	-6

i	π_i	γ_i	$P0_i$	EW_i	$\text{var } W_i$	EC_i	$\text{var } C_i$
1	0.33906	0.23285	0.05202	19.7130	432.73	19.6788	412.15
2	0.25521	0.26290	0.04906	19.7556	432.70	19.7315	412.18
3	0.12552	0.04310	0.05319	19.6556	432.54	19.6619	412.14
4	0.05807	0.07976	0.04935	19.9290	433.49	19.7195	412.15
5	0.22214	0.38138	0.04635	19.7702	432.74	19.8150	412.33

$P(W = 0) = P(C = 0) = 0.04892$	$P(W^* = 0) = P(C^* = 0) = 0.05000$
$EW = 19.7608$	$EW^* = 19.7419$
$\text{var } W = 432.78$	$\text{var } W^* = 432.75$
$EC = 19.7471$	$EC^* = 19.7227$
$\text{var } C = 412.23$	$\text{var } C^* = 412.20$

The algorithm requires that all λ_i be strictly positive. Therefore in Example 5.1 we have chosen $\lambda_1 = 10^{-12}$. In Examples 5.4 and 5.5 the q_i are relatively large compared to the λ_i , i.e. the Markov chain is "fast" compared to the arrival process. This explains why $P0_i$, EW_i , $\text{var } W_i$, EC_i and $\text{var } C_i$ do not vary much for different values of i . In Example 5.3 the Markov chain is relatively slow whereas in Examples 5.1 and 5.2 the q_i and the λ_i are of the same order of magnitude. The computer time needed for the calculations is very modest. For each of the Examples 5.4 and 5.5 the required CPU-time on a DEC 2060 was about 10 seconds.

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