Lie Algebraic Approach to τ -Functions and its Equations

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Abstract. A general construction of equations satisfied by the components of τ -functions is given by considering the tensor products of modules. As an example, the homogeneous basic realization of $\hat{A}_1$ is given, leading to NLS equations.

0. Introduction

Some years ago, Date, Jimbo, Kashiwara, and Miwa [2, 6] showed that solutions of many soliton equations can be considered as group orbits of highest weight vectors V_Λ of $L(\Lambda)$.

Kac [7] gave a Lie algebraic interpretation for the case that the representation $L(\Lambda)$ remains irreducible when considered as an $\mathfrak{z}$ -module. Here $\mathfrak{z}$ is a Heisenberg subalgebra. One can extend this construction to the case where $L(\Lambda)$ does not remain irreducible as an $\mathfrak{z}$ -module.

Modified equations can be found by considering $L(\Lambda) \otimes L(\tilde{\Lambda})$ ($\Lambda \neq \tilde{\Lambda}$) instead of $L(\Lambda) \otimes L(\Lambda)$.

1. Irreducible Highest Weight Modules

We start with a given affine Kac–Moody algebra $\mathcal{G}(A)$ with a $(n+1) \times (n+1)$ matrix A , where $\mathcal{G}(A)$ has the generators $e_0, \dots, e_n$ and $f_0, \dots, f_n$ and a center c . The principal grading δ is defined by

$$\delta(e_i) = -\delta(f_j) = 1 \quad (i, j = 0 \dots n).$$

We suppose that we have a Heisenberg subalgebra $\mathfrak{z}$ in $\mathcal{G}(A)$ with a basis $\{p_i, q_j, c\}$ ($i, j > 0$) such that

$$[p_i, q_j] = \delta_{ij} c \quad (i, j \in \mathbb{N}, i, j > 0).$$

We also suppose that $\delta(p_i) = -\delta(q_i) > 0$.

Take an integrable irreducible highest weight module $L(\Lambda)$ (over $\mathcal{G}(A)$) with highest weight vector V_Λ , and denote by Π the representation of $\mathcal{G}(A)$ it affords. One has $\Pi(c) = m \cdot \text{Id}$ (cf. Lemma 9.3 of [7]).

We define the grading δ_L on $L(\Lambda)$ by

$$\delta_L(\Pi(g)V_\Lambda) = -\delta(g) \quad (g \in \mathcal{U}(\mathcal{G}(A)) \cdot).$$

Because $\delta(p_i) > 0$, $\Pi(p_i)$ is locally nilpotent on $L(\Lambda)$. The highest weight module $L(\Lambda)$ splits up as a direct sum of the irreducible $\mathfrak{g}$ -modules, all isomorphic to each other (see Lemma 14.4 of [7]):

$$L(\Lambda) \simeq \bigoplus_{i \in I} R \otimes_{\mathbb{C}} u_i, \quad R = \mathbb{C}[x_1, x_2, \dots]$$

(1.1)

and

$$\Pi(p_i) = m \frac{\partial}{\partial x_i} \otimes 1; \quad \Pi(q_i) = x_i \otimes 1; \quad \Pi(c) = m \cdot \text{Id}.$$

$1 \otimes u_i$ are the vacuum vectors of the irreducible components with eigenvalue m .

The $\mathcal{G}(A)$ -module $L(\Lambda)$ carries a unique nondegenerate contravariant Hermitian form H_1 such that $H_1(V_\Lambda, V_\Lambda) = 1$ (see Lemma 11.5 of [7]).

2. The τ -Functions and Equations

To make τ -functions, one has to identify a group G associated to $\mathcal{G}(A)$. For example, one could take the group generated by $\exp(te_i)$ and $\exp(tf_i)$ ($i = 0 \dots n, t \in \mathbb{C}$). The τ -function $\tau(g)$ is defined by

$$\tau(g) = \Pi(g)V_\Lambda \quad (g \in G). \quad (2.1)$$

To find equations satisfied by $\tau(g)$, one considers $L(\Lambda) \otimes L(\Lambda)$. This module is completely reducible (§10.7 of [7]). The module generated by $V_\Lambda \otimes V_\Lambda$ is denoted by L_{high} and the submodule $L_{\text{high}}^\perp$ by L_{low} . In general, L_{low} is not irreducible.

Orthogonality is taken with respect to H defined by

$$H(V_1 \otimes W_1, V_2 \otimes W_2) = H_1(V_1, V_2) \cdot H_1(W_1, W_2).$$

It is now essential that $\tau(g) \otimes \tau(g) \in L_{\text{high}}$, from which, by definition, follows

$$H(V \otimes W, \tau(g) \otimes \tau(g)) = 0 \quad (V \otimes W \in L_{\text{low}}). \quad (2.2)$$

This equation will lead to bilinear equations under one more assumption. We assume that

$$-\omega_0(q_i) = a_i p_i \quad (a_i \in \mathbb{C}, \omega_0 \text{ anti-linear Cartan involution}). \quad (2.3)$$

According to (1.1), we can write $v = \sum P_i(x)u_i$ ($v \in L(\Lambda)$, we omit $\otimes_{\mathbb{C}}$) and also $\tau(g) = \sum \tau_k(x)u_k$. For $w \in L(\Lambda) \otimes L(\Lambda)$ we write

$$w = \sum_{i,j} P_i(x^{(0)})u_i \otimes Q_j(x^{(1)})u_j = \sum_{i,j} P_i(x^{(0)})Q_j(x^{(1)})u_i \otimes u_j.$$

Suppose

$$H_1(u_i, u_j) = \varepsilon_{ij} \quad \text{and} \quad \tilde{D}_x = \left(a_1 \frac{\partial}{\partial x_1}, a_2 \frac{\partial}{\partial x_2}, \dots \right).$$

Using (2.3), one finds

$$H_1 \left(\sum_i P_i(x) u_i, \sum_j Q_j(x) u_j \right) = \sum_{i,j} \varepsilon_{ij} \bar{P}_i(m\tilde{D}_x) Q_j(x) |_{x=0}. \quad (2.4)$$

Changing variables

$$2x_j = x_j^{(0)} + x_j^{(1)} \quad \text{and} \quad 2y_j = x_j^{(0)} - x_j^{(1)}$$

leads to

$$\Pi \otimes \Pi(p_i) = m \frac{\partial}{\partial x_i}; \quad \Pi \otimes \Pi(q_j) = 2x_j; \quad \Pi \otimes \Pi(c) = 2m \cdot \text{Id}.$$

Using these variables, L_{low} can be written as $L_{\text{low}} = \mathbb{C}[x] \otimes \text{Hir}$, where the elements of Hir are of the form

$$\sum_{i,j} P_{ij}(y) u_i \otimes u_j \in \text{Hir} \Leftrightarrow Q(x) \sum_{i,j} P_{ij}(y) u_i \otimes u_j \in L_{\text{low}}.$$

Substituting this in (2.2), and using (2.4) and the arbitrariness of Q , yields

$$\sum_{i,j,k,l} \bar{P}_{ij}(m\tilde{D}_y) \tau_k(x+y) \tau_l(x-y) \varepsilon_{ik} \varepsilon_{jl} |_{y=0} = 0.$$

These are equations in Hirota's bilinear form. The sums on i and j are finite and, therefore, so are the sums on k and l , since $\varepsilon_{ik} \neq 0$ for only finitely many k (i fixed). We have assumed the analyticity of τ .

3. Example

As an example, we give the homogeneous basic realization of $\hat{A}_1$ (see [5]). So we take $\mathcal{P}(A) = \hat{A}_1$ and $p_i = \frac{1}{2}h \otimes t^i$, $q_j = (1/j)h \otimes t^{-j}$, and $\Lambda = \Lambda_0$. Then $a_j = 2/j$ (see (2.3)) and $m = 1$ (see (1.1)).

One can find u_i so that

$$L(\Lambda_0) \simeq \sum_{i \in \mathbb{Z}} R \otimes_{\mathbb{C}} u_i; \quad \delta_L(u_i) = 2i^2 - i;$$

$$1 \otimes u_0 = \mathbf{V}_{\Lambda_0}; \quad H_1(u_i, u_j) = \delta_{ij}.$$

Moreover,

$$\Pi(e \otimes t') Q(x) u_k = (-1)^{k+1} \sum_m p_{m-i-2k-1}(x) p_m(-\tilde{D}) Q(x) u_{k+1},$$

$$\Pi(f \otimes t') Q(x) u_k = (-1)^k \sum_m p_{m-i+2k-1}(-x) p_m(\tilde{D}) Q(x) u_{k-1},$$

where $p_i(x)$ is defined by

$$\sum p_i(x) k^i = \exp \left[\sum_{i \geq 1} x_i k^i \right].$$

For τ_0 , τ_1 , and τ_{-1} , one finds (among others) the following three equations

$$\begin{aligned} D_1^2(\tau_0 \cdot \tau_0) + 2\tau_1 \cdot \tau_{-1} &= 0; & (D_1^2 + D_2)(\tau_0 \cdot \tau_1) &= 0; \\ (D_1^2 + D_2)(\tau_{-1} \cdot \tau_0) &= 0. \end{aligned}$$

Introducing $\tau_1/\tau_0 = q$ and $\tau_{-1}/\tau_0 = q^*$ yields

$$q_t = q_{xx} - 2q^2 q^*; \quad -q_t^* = q_{xx}^* - 2q^{*2} q. \quad (3.1)$$

4. Remarks

(1) By considering two different modules $L(\Lambda)$ and $L(\tilde{\Lambda})$, both realized in the ‘same’ way, one can derive modified equations. For example, the principal realizations of $L(\Lambda_0)$ and $L(\Lambda_1)$ lead to the MKdV (after considering $L(\Lambda_0) \otimes L(\Lambda_1)$) (cf. [9]). In the same way, the homogeneous realizations of $L(\Lambda_0)$ and $L(\Lambda_1)$ lead to the NLS with a derivative coupling (cf. [10]), that is, Equation (3.1) with $qq_x(q^*q_x^*)$ instead of $q^2(q^{*2})$.

(2) It is already known that some connections exist between the NLS equation and the Lie algebra $\hat{A}_1$ (cf. [1–4, 6]).

The approach followed here tries to explain, using only Lie algebras, how the equations for the τ -functions arise. This derivation is quite different from [2, 6]. In those papers, the equations are consequences of a bilinear identity for the wavefunctions w and w^* , while in this Letter the equations reflect the orthogonality relations for τ -functions.

This also opens possibilities for attacking other algebras (for example $\hat{E}_8$). Our approach is an extension of Kac [7, Ch. 14] where irreducible $\mathfrak{g}$ -modules are treated.

Wakimoto and Yamada [11] suggest yet another interesting possibility by extending the algebra with a Virasoro algebra.

Drinfeld and Sokolov [3] associate a hierarchy of KdV-like equations to every vertex of a Dynkin diagram. This Dynkin diagram belongs to a Kac–Moody Lie algebra. An interesting problem is to investigate the relation between their approach (leading to Hamiltonian forms for the equations) and the τ -function approach [8].

Finally, the Estabrook–Wahlquist prolongation method applied to the NLS equation leads to a prolongation algebra closely connected with $\hat{A}_1$ [4].

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