Stackelberg Strategies in Linear-Quadratic Stochastic Differential Games[1]

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Abstract. This paper obtains the Stackelberg solution to a class of two-player stochastic differential games described by linear state dynamics and quadratic objective functionals. The information structure of the problem is such that the players make independent noisy measurements of the initial state and are permitted to utilize only this information in constructing their controls. Furthermore, by the very nature of the Stackelberg solution concept, one of the players is assumed to know, in advance, the strategy of the other player (the leader). For this class of problems, we first establish existence and uniqueness of the Stackelberg solution and then relate the derivation of the leader's Stackelberg solution to the optimal solution of a nonstandard stochastic control problem. This stochastic control problem is solved in a more general context, and its solution is utilized in constructing the Stackelberg strategy of the leader. For the special case Gaussian statistics, it is shown that this optimal strategy is affine in observation of the leader. The paper also discusses numerical aspects of the Stackelberg solution under general statistics and develops algorithms which converge to the unique Stackelberg solution.

Key Words. Stochastic differential games, Stackelberg solution, linear-quadratic games, nonzero-sum games.

1. Introduction

The Stackelberg solution concept, originally introduced by H. Von Stackelberg in the context of static economic equilibrium (Ref. 1), has entered the control literature through the works of Chen, Cruz, and Simaan

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443

(Refs. 2–4), and has found applications in hierarchical control and differen-
tial game problems (Refs. 5–7). Our interest in this paper lies in solving a
class of stochastic nonzero-sum differential games with (nonnested) static
information under the Stackelberg solution concept.

The first result in the literature on a stochastic Stackelberg problem
can be found in Ref. 8, where the authors obtain the feedback Stackelberg
solution of a two-player linear-quadratic-Gaussian (LQG) nonzero-sum
discrete-time dynamic game problem under a nested information pattern.
With regard to nonnested information patterns, Refs. 5 and 9 solve a class
of static stochastic Stackelberg game problems when the underlying statis-
tics of the random variables are not necessarily Gaussian. Some of these
results are then utilized in Ref. 10 to obtain the feedback Stackelberg
solution of the LQG problem of Ref. 8 under a one-step-delay information
sharing pattern.

In this paper, we consider the two-player continuous-time LQ nonzero-
sum differential game under general statistics and under a (nonnested)
static information for both players. We establish existence and uniqueness
of the Stackelberg solution, and determine it as the solution of an integral
equation.

The special case of Gaussian statistics is also treated in the paper, and
it is shown that the Stackelberg strategies are then affine in the available
static information. Finally, the paper includes a discussion on the computa-
tional aspects of the solution.

In Section 2, we formulate our problem and prove existence and
uniqueness of Stackelberg strategies. Furthermore, we outline the underly-
ing idea of our method of solution. This leads to a nonstandard stochastic
control problem where the state equation contains an extra term involving
conditional expectation of the control term, and where some of the com-
ponents of the state vector have their initial conditions while others have
their terminal conditions specified. In Section 3, we solve a general stochas-
tic control problem of this type; and, in Section 4, we use these results to
obtain the Stackelberg strategies. The special case of Gaussian statistics is
treated in Section 5, while numerical aspects are discussed in Section 6.
The paper ends with a conclusions section.

2. Problem Formulation and Existence Consideration

Let $\{x_t, t \geq t_0\}$ denote an n-dimensional stochastic process satisfying
the Itô stochastic differential equation

$$dx_t = [A(t)x_t + B^1(t)u_t^1 + B^2(t)u_t^2] \, dt + F(t) \, dW_t,$$

$$t \geq t_0, \qquad x_{t_0} = x_0, \tag{1}$$

and whose sample paths are continuous. Here, x_0 is a random vector with known statistics, such that $E|x_0| < \infty$, and $\{W_t, t \geqslant t_0\}$ is an n-dimensional standard Brownian motion. $A(\cdot)$, $B^1(\cdot)$, $B^2(\cdot)$, $F(\cdot)$ are appropriate dimensional matrices with continuous entries on $[t_0, t_f]$. $\{u_t^1, t \geqslant t_0\}$ and $\{u_t^2, t \geqslant t_0\}$ are, respectively, r_1-dimensional and r_2-dimensional stochastic processes denoting the controls of decision makers DM1 and DM2, respectively.

The decision makers make independent noisy measurements of the initial state x_0, which we denote by y^i, $i = 1, 2$. We assume that the conditional joint distribution of (y^1, y^2), given x_0, is *a priori* known. The information available to each DM, therefore, is static in nature. Let H^i, $i = 1, 2$, denote the class of second-order stochastic processes defined on $[t_0, t_f]$ which are y^i-measurable.

The decision law (or strategy) γ^i of DMi is a real Borel-measurable mapping

$$[t_0, t_f] \times \mathbb{R}^{m_i} \to \mathbb{R}^{r_i},$$

where m_i denotes the dimension of y^i, $i = 1, 2$. Denoting the space of all Borel-measurable functions

$$f : [t_0, t_f] \times \mathbb{R}^{m_i} \to \mathbb{R}^{r_i},$$

with the further restriction that

$$E \int_{t_0}^{t_f} \|f(t, y^i)\|^2 \, dt < \infty,$$

by Γ^i, we require each permissible γ^i to be in Γ^i. Clearly, $\gamma^i(\cdot, y^i)$ is in H^i. It should be noted that, for each pair of elements in $H^1 \times H^2$, the stochastic differential equation (1) admits a unique solution whose sample paths are continuous (Ref. 11).

For each $\{\gamma^1 \in \Gamma^1, \gamma^2 \in \Gamma^2\}$, we now define the quadratic objective functionals of DM1 and DM2, respectively, as

$$J_1(\gamma^1, \gamma^2) = E\Big\{ x_{t_f}' Q_{1f} x_{t_f} + \int_{t_0}^{t_f} (x_t' Q_1(t) x_t + u_t^{1'} u_t^1$$
$$+ u_t^{2'} R(t) u_t^2) \, dt \,|\, u_t^i = \gamma^i(t, y^i), i = 1, 2 \Big\}, \qquad (2)$$

$$J_2(\gamma^1, \gamma^2) = E\Big\{ x_{t_f}' Q_{2f} x_{t_f} + \int_{t_0}^{t_f} (x_t' Q_2(t) x_t + u_t^{2'} u_t^2) \, dt \,|\,$$
$$u_t^i = \gamma^i(t, y^i), i = 1, 2 \Big\}, \qquad (3)$$

with prime denoting transpose of a matrix, where

$$Q_{1f} \geqslant 0, \qquad Q_{2f} \geqslant 0, \qquad Q_1(\cdot) \geqslant 0, \qquad Q_2(\cdot) \geqslant 0, \qquad R(\cdot) \geqslant 0,$$

where the last three matrix-valued functions have continuous entries on
$[t_0, t_f]$ and the expectation operation is taken over the underlying statistics.

We note at this point that, because of the hierarchical nature of the
solution concept to be introduced below, which allows DM2 to have access
to the strategy of DM1, Eqs. (2) and (3) define the most general quadratic
convex functionals for the decision makers when the cross terms involving
the two strategies do not explicitly appear.

To introduce the hierarchical (Stackelberg) solution concept, let us
first note that, for each $\bar{\gamma}^1 \in \Gamma^1$, there exists a unique $\bar{\gamma}^2 \in \Gamma^2$ that minimizes
$J_2(\bar{\gamma}^1, \gamma^2)$ over Γ^2. This, then, implies the existence of a unique map
$T: \Gamma^1 \to \Gamma^2$ such that

$$J_2(\gamma^1, T\gamma^1) \leq J_2(\gamma^1, \gamma^2),$$

for all $\gamma^2 \in \Gamma^2$ and for every $\gamma^1 \in \Gamma^1$. Now, consider the minimization of the
functional $J_1(\gamma^1, T\gamma^1)$ over Γ^1, and let $\gamma^{1^0} \in \Gamma^1$ denote one minimizing
solution; that is,

$$J_1(\gamma^{1^0}, T\gamma^{1^0}) \leq J_1(\gamma^1, T\gamma^1), \qquad \forall \gamma^1 \in \Gamma^1. \tag{4}$$

Then, we say that the pair $(\gamma^{1^0}, \gamma^{2^0} = T\gamma^{1^0})$ provides a *Stackelberg solution*
for the game problem under consideration, with DM1 as the leader. Our
objective in this paper is to study the existence, uniqueness, and (most
importantly) characterization of the Stackelberg solution of the nonzero-
sum linear-quadratic stochastic differential game formulated above.

The existence and uniqueness questions may be answered most readily
by converting the problem into an equivalent hierarchical static game
problem in Hilbert space. Barring the stochastic aspect, our approach is
identical to that in Ref. 3. We introduce an inner product $\langle \cdot, \cdot \rangle_i$ on H^i
through the relation

$$\langle u, v \rangle_i = E\left\{ \int_{t_0}^{t_f} u_t(\omega)' v_t(\omega) \, dt \right\} \tag{5a}$$

for each pair $\{u \in H^i, v \in H^i\}$, $\omega \in \Omega$, with $(\Omega, \mathcal{B}, \mathcal{P})$ denoting the underlying
probability space. With these inner products, H^i, $i = 1, 2$, become Hilbert
spaces. Let $L_{2f}([t_0, t_f] \times \Omega)$ denote the completion of the space of continuous
functions from $[t_0, t_f] \times \Omega$ into $\mathbb{R}^n$, under the inner product

$$\langle x, z \rangle_f = E\left\{ x_{t_f}(\omega)' z_{t_f}(\omega) + \int_{t_0}^{t_f} x_t(\omega)' z_t(\omega) \, dt \right\}. \tag{5b}$$

Further, let $\mathcal{D}_i$, $i = 1, 2$, be the operators mapping $L_{2f}([t_0, t_f] \times \Omega)$ into itself,

defined by

$$(\mathcal{Q}_i x)_t(\omega) = \begin{cases} Q_i(t)x_t(\omega), & t_0 \leqslant t < t_f, \\ Q_{if}x_{t_f}(\omega), & t = t_f. \end{cases} \qquad (6)$$

Define Volterra operators

$$\mathcal{L}_i: H^i \to L_{2f}([t_0, t_f] \times \Omega), \qquad i = 1, 2,$$

by

$$(\mathcal{L}_i u)_t(\omega) = \int_{t_0}^{t} \Phi(t, s)B^i(s)u_s(\omega) \, ds, \qquad (7a)$$

where $\Phi(t, s)$ is the state transition matrix function satisfying

$$d\Phi(t, s)/dt = A(t)\Phi(t, s), \qquad \Phi(s, s) = I. \qquad (7b)$$

Further, let

$$r \in L_{2f}([t_0, t_f] \times \Omega)$$

be defined by

$$r_t(\omega) = \Phi(t, t_0)x_0(\omega) + \int_{t_0}^{t} \Phi(t, s)F(s) \, dW_s(\omega). \qquad (7c)$$

The solution of (1) can then be expressed as

$$x = r + \mathcal{L}_1 \gamma^1(\cdot, y^1) + \mathcal{L}_2 \gamma^2(\cdot, y^2). \qquad (8)$$

Then, the quadratic objective functionals of DM1 and DM2 respectively become

$$J_1(\gamma^1, \gamma^2) = \langle \mathcal{Q}_1(\mathcal{L}_1 \gamma^1(\cdot, y^1) + \mathcal{L}_2 \gamma^2(\cdot, y^2) + r), \mathcal{L}_1 \gamma^1(\cdot, y^1)$$
$$+ \mathcal{L}_2 \gamma^2(\cdot, y^2) + r\rangle_f + \langle \gamma^1(\cdot, y^1), \gamma^1(\cdot, y^1)\rangle_1$$
$$+ \langle \gamma^2(\cdot, y^2), R\gamma^2(\cdot, y^2)\rangle_2, \qquad (9)$$

$$J_2(\gamma^1, \gamma^2) = \langle \mathcal{Q}_2(\mathcal{L}_1 \gamma^1(\cdot, y^1) + \mathcal{L}_2 \gamma^2(\cdot, y^2) + r), \mathcal{L}_1 \gamma^1(\cdot, y^1)$$
$$+ \mathcal{L}_2 \gamma^2(\cdot, y^2) + r\rangle_f + \langle \gamma^2(\cdot, y^2), \gamma^2(\cdot, y^2)\rangle_2, \qquad (10)$$

where

$$R: H^2 \to H^2$$

is defined by

$$(Ru)_t(\omega) = R(t)u_t(\omega), \qquad t_0 \leqslant t \leqslant t_f. \qquad (11)$$

For fixed $\gamma^1 \in \Gamma^1$, minimization of $J_2(\gamma^1, \gamma^2)$ is equivalent to the control

problem of minimizing

$$J_2(u^2) = \langle \mathscr{Q}_2(\mathscr{L}_1\gamma^1(\,\cdot\,,\,y^1) + \mathscr{L}_2 u^2 + r),\; \mathscr{L}_1\gamma^1(\,\cdot\,,\,y^1) + \mathscr{L}_2 u^2 + r \rangle_f + \langle u^2,\, u^2 \rangle_2,$$

(12)

for $u^2 \in H^2$. Now, $I + \mathscr{L}_2^* \mathscr{Q}_2 \mathscr{L}_2$ being strongly positive and having a bounded inverse ($\mathscr{L}_2^*$ denotes the adjoint of $\mathscr{L}_2$ in H^2), $J_2(u^2)$ has the unique minimum

$$u^2 = -(I + \mathscr{L}_2^* \mathscr{Q}_2 \mathscr{L}_2)^{-1} \mathscr{L}_2^* \mathscr{Q}_2 (\mathscr{L}_1\gamma^1(\,\cdot\,,\,y^1) + r).$$

(13)

Substituting this expression for $\gamma^2(\,\cdot\,,\,y^2)$ in (9) and rearranging terms, we get, in terms of the notation discussed above,

$$J_1(\gamma^1,\, T\gamma^1) = \langle \gamma^1(\,\cdot\,,\,y^1),\; (I + \bar{\mathscr{R}})\gamma^1(\,\cdot\,,\,y^1) \rangle_1 + 2\langle \gamma^1(\,\cdot\,,\,y^1),\; \tilde{\mathscr{L}}_1^* \mathscr{Q}_1 \tilde{r}$$
$$+ \mathscr{L}_1^* \mathscr{Q}_2 \mathscr{L}_2 (I + \mathscr{L}_2^* \mathscr{Q}_2 \mathscr{L}_2)^{-1} R (I + \mathscr{L}_2^* \mathscr{Q}_2 \mathscr{L}_2)^{-1} \mathscr{L}_2^* \mathscr{Q}_2 r \rangle_1 + J_{10},$$

(14)

where J_{10} is independent of γ^1,

$$\bar{\mathscr{R}} = \tilde{\mathscr{L}}_1^* \mathscr{Q}_1 \tilde{\mathscr{L}}_1 + \mathscr{L}_1^* \mathscr{Q}_2 \mathscr{L}_2 (I + \mathscr{L}_2^* \mathscr{Q}_2 \mathscr{L}_2)^{-1} R (I + \mathscr{L}_2^* \mathscr{Q}_2 \mathscr{L}_2)^{-1} \mathscr{L}_2^* \mathscr{Q}_2 \mathscr{L}_1,$$

$$\tilde{\mathscr{L}}_1 = \mathscr{L}_1 - \mathscr{L}_2 (I + \mathscr{L}_2^* \mathscr{Q}_2 \mathscr{L}_2)^{-1} \mathscr{L}_2^* \mathscr{Q}_2 \mathscr{L}_1,$$

$$\tilde{r} = (I - \mathscr{L}_2 (I + \mathscr{L}_2^* \mathscr{Q}_2 \mathscr{L}_2)^{-1} \mathscr{L}_2^* \mathscr{Q}_2) r,$$

and $\tilde{\mathscr{L}}_1^*$ is the adjoint of $\tilde{\mathscr{L}}_1$ under the inner product (5b). Again, $I + \bar{\mathscr{R}}$ being strongly positive and having a bounded inverse, taking variations in Γ^1, we can readily show that there exists a unique γ^{1^o} for which $J_1(\gamma^1,\, T\gamma^1)$ has a minimum. Then, this guarantees existence of a unique Stackelberg solution for the nonzero-sum stochastic differential game above.

Let us now outline our method of solution. For each $\gamma^1 \in \Gamma_1$, the control $u^2 \in H^2$ that minimizes $J_2(u^2)$ is given by (see Ref. 12 for details):

$$u_t^2 = -B^2(t)' \{ S^2(t) E[x_t | y^2] + k_t^2 \},$$

where

$$\dot{S}^2(t) = -A(t)' S^2(t) - S^2(t) A(t) - Q_2(t) + S^2(t) B^2(t) B^2(t)' S^2(t),$$
$$S^2(t_f) = Q_{2f}, \quad (15)$$

$$\dot{k}_t^2 = -(A(t)' - S^2(t) B^2(t) B^2(t)') k_t^2 - S^2(t) B^1(t) E[\gamma^1(t,\, y^1) | y^2],$$
$$k_{t_f}^2 = 0; \quad (16a)$$

and let us denote $E[x_t | y^2]$ by $\hat{x}_t^2$. Then,

$$d\hat{x}_t^2 = (A(t) - B^2(t) B^2(t)' S^2(t)) \hat{x}_t^2 \, dt - B^2(t) B^2(t)' k_t^2 \, dt$$
$$+ B^1(t) E[\gamma^1(t,\, y^1) | y^2] \, dt,$$

(16b)

$$\hat{x}_{t_0}^2 = E[x_{t_0}|y^2] = E[x_0|y^2]. \tag{16c}$$

Furthermore, if

$$e_t = x_t - \hat{x}_t^2,$$

then

$$de_t = A(t)e_t\,dt + B^1(t)\gamma^1(t, y^1)\,dt$$

$$- B^1(t)E[\gamma^1(t, y^1)|y^2]\,dt + F(t)\,dW_t, \tag{16d}$$

$$e_{t_0} = x_0 - E[x_0|y^2]. \tag{16e}$$

In terms of these newly introduced variables $\hat{x}_t^2$, e_t, k_t^2, we can express J_1 as follows:

$$J_1(\gamma^1, T\gamma^1) = \hat{x}_{t_f}^{2'}Q_{1f}\hat{x}_{t_f}^2 + e_{t_f}'Q_{1f}e_{t_f} + \int_{t_0}^{t_f} (\hat{x}_t^{2'}Q_1(t)\hat{x}_t^2 + e_t'Q_1(t)e_t$$

$$+ \{\hat{x}_t^{2'}S^2(t) + k_t^{2'}\}B^2(t)R(t)B^2(t)'\{S^2(t)\hat{x}_t^2 + k_t^2\}$$

$$+ \gamma^1(t, y^1)'\gamma^1(t, y^1))\,dt. \tag{17}$$

Therefore, the leader is faced with a nonstandard stochastic optimization problem, the general formulation and solution of which is discussed in the next section.

3. Digression: Nonstandard Stochastic Control Problem

Let us consider a stochastic system whose state evolves according to the following stochastic differential equation:

$$dx_t = \{Ax_t + B\gamma(t, y) + CE[\gamma(t, y)|z]\}\,dt + dw_t, \tag{18}$$

where x_t is an n-dimensional stochastic process; y and z are p-dimensional and q-dimensional random vectors, respectively; $\gamma: [t_0, t_f] \times \mathbb{R}^p \to \mathbb{R}^m$, and all admissible γ belong to Γ, with Γ being a linear vector space; w_t is n-dimensional Brownian motion, and A, B, C are appropriate dimensional matrices. We are interested in the situation where some components of x_t are specified at the initial time t_0 and the rest are specified at the terminal time t_f. Let us instead consider a slightly more general set up. Thus, there exist matrices T_0 and T_f such that

$$T_0 x_{t_0} = \pi, \qquad T_f x_{t_f} = \chi \tag{19a}$$

are given. To ensure that all the components of x_t are specified at one of the endpoints, we impose the restriction that

$$T_0'T_0 + T_f'T_f = I. \tag{19b}$$

Furthermore, in order to avoid some cumbersome expressions in the analysis to follow, we assume that the system matrix is structured so that

$$T_f A T_0' = 0, \tag{19c}$$

which is a condition that is fulfilled for the Stackelberg problem under consideration (see Section 4). Finally, we assume that Γ is chosen such that, for each $\gamma \in \Gamma$, there exists a unique solution of (18) satisfying the boundary conditions (19a).

Our problem then is to determine $\gamma \in \Gamma$ for which

$$J(\gamma) = E\left\{ x_{t_f}' Q_f x_{t_f} + \int_{t_0}^{t_f} [x_t' Q(t) x_t + \gamma(t, y)' \gamma(t, y)] \, dt \right\} \tag{20}$$

attains a minimum, where

$$Q_f \geq 0, \qquad Q(t) \geq 0, \qquad \text{for a.e. } t \in [t_0, t_f].$$

We use standard variational argument to solve this problem. If we perturb γ to $\gamma + \epsilon\beta$, suppose that the state x_t is perturbed to x_t^ϵ. Then,

$$dx_t^\epsilon = \{ A x_t^\epsilon + B\gamma(t, y) + \epsilon B\beta(t, y) + CE[\gamma(t, y) | z]$$

$$+ \epsilon CE[\beta(t, y) | z] \} \, dt + dw_t,$$

$$T_0 x_{t_0}^\epsilon = \pi, \qquad T_f x_{t_f}^\epsilon = \chi.$$

We can clearly decompose x_t^ϵ as $x_t + \epsilon\eta_t$, where

$$\dot{\eta}_t = A\eta_t + B\beta(t, y) + CE[\beta(t, y) | z],$$

$$T_0 \eta_{t_0} = 0, \qquad T_f \eta_{t_f} = 0.$$

Taking first variation in J, a necessary condition for minimum is that

$$\delta_1 J(\gamma; \beta) = E\left\{ \eta_{t_f}' Q_f x_{t_f} + \int_{t_0}^{t_f} [\eta_t' Q(t) x_t + \beta(t, y)' \gamma(t, y)] \, dt \right\} = 0, \qquad \forall \beta \in \Gamma.$$

Since the second variation is positive, the above condition is also sufficient.

Define η_t^1 and η_t^2 by

$$\dot{\eta}_t^1 = A\eta_t^1 + B\beta(t, y), \qquad T_0 \eta_{t_0}^1 = 0, \qquad T_f \eta_{t_f}^1 = 0, \tag{21a}$$

$$\dot{\eta}_t^2 = A\eta_t^2 + C\beta(t, y), \qquad T_0 \eta_{t_0}^2 = 0, \qquad T_f \eta_{t_f}^2 = 0. \tag{21b}$$

Then,

$$\eta_t = \eta_t^1 + E[\eta_t^2 | z]$$

and the necessary and sufficient condition for minimum may be reexpressed as

$$
\begin{aligned}
\delta_1 J(\gamma;\beta) = E\Big\{ &\eta_{t_f}^{1'} Q_f E[x_{t_f}|y] + E[\eta_{t_f}^{2'}|z] Q_f E[x_{t_f}|z] \\
&+ \int_{t_0}^{t_f} [\eta_t^{1'} Q(t) E[x_t|y] + E[\eta_t^{2'}|z] Q(t) E[x_t|z] \\
&+ \beta(t,y)'\gamma(t,y)]\, dt \Big\} = E\Big\{ \eta_{t_f}^{1'} Q_f E[x_{t_f}|y] + \eta_{t_f}^{2'} Q_f E[E[x_{t_f}|z]|y] \\
&+ \int_{t_0}^{t_f} [\eta_t^{1'} Q(t) E[x_t|y] + \eta_t^{2'} Q(t) E[E[x_t|z]|y] \\
&+ \beta(t,y)'\gamma(t,y)]\, dt \Big\} = 0, \qquad \forall \beta \in \Gamma.
\end{aligned}
\tag{22}
$$

Let

$$
\hat{x}_t^y = E[x_t|y] \quad \text{and} \quad \hat{\hat{x}}_t^y = E[E[x_t|z]|y].
$$

Then,

$$
\dot{\hat{x}}_t^y = A\hat{x}_t^y + B\gamma(t,y) + CE[E[\gamma(t,y)|z]|y],
\tag{23a}
$$

$$
T_0 \hat{x}_{t_0}^y = E[\pi|y], \qquad T_f \hat{x}_{t_f}^y = E[\chi|y],
\tag{23b}
$$

$$
\dot{\hat{\hat{x}}}_t^y = A\hat{\hat{x}}_t^y + (B+C)E[E[\gamma(t,y)|z]|y],
\tag{23c}
$$

$$
T_0 \hat{\hat{x}}_{t_0}^y = E[E[\pi|z]|y], \qquad T_f \hat{\hat{x}}_{t_f}^y = E[E[\chi|z]|y],
\tag{23d}
$$

and

$$
\begin{aligned}
\delta_1 J(\gamma;\beta) = E\Big\{ &\eta_{t_f}^{1'} Q_f \hat{x}_{t_f}^y + \eta_{t_f}^{2'} Q_f \hat{\hat{x}}_{t_f}^y + \int_{t_0}^{t_f} [\eta_t^{1'} Q(t)\hat{x}_t^y \\
&+ \eta_t^{2'} Q(t)\hat{\hat{x}}_t^y + \beta(t,y)'\gamma(t,y)]\, dt \Big\} = 0, \qquad \forall \beta \in \Gamma.
\end{aligned}
\tag{24}
$$

Note now that, under (19c), $T_f\eta_t^1$ and $T_f\eta_t^2$ satisfy, respectively,

$$
T_f\dot{\eta}_t^1 = T_f A T_f' T_f \eta_t^1 + T_f B\beta(t,y), \qquad T_f\eta_{t_f}^1 = 0, \cdot
$$

$$
T_f\dot{\eta}_t^2 = T_f A T_f' T_f \eta_t^2 + T_f C\beta(t,y), \qquad T_f\eta_{t_f}^2 = 0,
$$

so that

$$
T_f\eta_t^1 = -\int_t^{t_f} \Phi_f(t,\sigma) T_f B(\sigma)\beta(\sigma,y)\, d\sigma,
\tag{25a}
$$

$$
T_f\eta_t^2 = -\int_t^{t_f} \Phi_f(t,\sigma) T_f C(\sigma)\beta(\sigma,y)\, d\sigma,
\tag{25b}
$$

where

$$
(d/dt)\Phi_f(t,\sigma) = T_f A T_f' \Phi_f(t,\sigma), \qquad \Phi_f(\sigma,\sigma) = I.
\tag{25c}
$$

Since

$$T_0'T_0 + T_f'T_f = I,$$

(21a) and (21b) may be written as

$$\dot{\eta}_t^1 = A(T_0'T_0 + T_f'T_f)\eta_t^1 + B\beta(t, y),$$
$$\dot{\eta}_t^2 = A(T_0'T_0 + T_f'T_f)\eta_t^2 + C\beta(t, y);$$

and, multiplying further by T_0, we have

$$T_0\dot{\eta}_t^1 = T_0AT_0'T_0\eta_t^1 + T_0AT_f'T_f\eta_t^1 + T_0B\beta(t, y), \qquad T_0\eta_{t_0}^1 = 0,$$
$$T_0\dot{\eta}_t^2 = T_0AT_0'T_0\eta_t^2 + T_0AT_f'T_f\eta_t^2 + T_0C\beta(t, y), \qquad T_0\eta_{t_0}^2 = 0,$$

which imply that

$$T_0\eta_t^1 = \int_{t_0}^t \Phi_0(t, \sigma)[T_0B\beta(\sigma, y) + T_0A(\sigma)T_f'T_f\eta_\sigma^1]\, d\sigma, \tag{26a}$$

$$T_0\eta_t^2 = \int_{t_0}^t \Phi_0(t, \sigma)[T_0C\beta(\sigma, y) + T_0A(\sigma)T_f'T_f\eta_\sigma^2]\, d\sigma, \tag{26b}$$

where

$$(d/dt)\Phi_0(t, \sigma) = T_0AT_0'\,\Phi_0(t, \sigma), \qquad \Phi_0(\sigma, \sigma) = I. \tag{26c}$$

Therefore,

$$\eta_t^1 = T_0'T_0\eta_t^1 + T_f'T_f\eta_t^1$$

$$= T_0' \int_{t_0}^t \Phi_0(t, \sigma)T_0B(\sigma)\beta(\sigma, y)\, d\sigma - T_f' \int_t^{t_f} \Phi_f(t, \sigma)T_fB(\sigma)\beta(\sigma, y)\, d\sigma$$

$$- T_0' \int_{t_0}^t \Phi_0(t, \tau)T_0A(\tau)T_f' \int_\tau^{t_f} \Phi_f(\tau, \sigma)T_fB(\sigma)\beta(\sigma, y)\, d\sigma\, d\tau.$$

Now,

$$\int_t^{t_f} \Phi_f(t, \sigma)T_fB(\sigma)\beta(\sigma, y)\, d\sigma = \Phi_f(t, t_f)\left[\int_{t_0}^{t_f} \Phi_f(t_f, \sigma)T_fB(\sigma)\beta(\sigma, y)\, d\sigma\right.$$

$$\left. - \int_{t_0}^t \Phi_f(t_f, \sigma)T_fB(\sigma)\beta(\sigma, y)\, d\sigma\right] = \int_{t_0}^{t_f} \Phi_f(t, \sigma)T_fB(\sigma)\beta(\sigma, y)\, d\sigma$$

$$- \int_{t_0}^t \Phi_f(t, \sigma)T_fB(\sigma)\beta(\sigma, y)\, d\sigma.$$

Furthermore,

$$-T_0' \int_{t_0}^{t} \Phi_0(t, \tau) T_0 A(\tau) T_f' \int_{\sigma}^{t_f} \Phi_f(\tau, \sigma) T_f B(\sigma) \beta(\sigma, y) \, d\sigma \, d\tau$$

$$= -T_0' \int_{t_0}^{t_f} d\sigma \int_{t_0}^{t} d\tau \, \Phi_0(t, \tau) T_0 A(\tau) T_f' \Phi_f(\tau, \sigma) T_f B(\sigma) \beta(\sigma, y) \, d\sigma$$

$$+ T_0' \int_{t_0}^{t} d\sigma \int_{\sigma}^{t} d\tau \, \Phi_0(t, \tau) T_0 A(\tau) T_f' \Phi_f(\tau, \sigma) T_f B(\sigma) \beta(\sigma, y) \, d\sigma$$

$$= -\int_{t_0}^{t_f} d\sigma \int_{t_0}^{t} d\tau \, T_0' \Phi_0(t, \tau) T_0 A(\tau) T_f' \Phi_f(\tau, \sigma) T_f B(\sigma) \beta(\sigma, y) \, d\sigma$$

$$+ \int_{t_0}^{t} d\sigma \int_{\sigma}^{t} d\tau \, T_0' \Phi_0(t, \tau) T_0 A(\tau) T_f' \Phi_f(\tau, \sigma) T_f B(\sigma) \beta(\sigma, \cdot y) \, d\sigma.$$

Then,

$$\eta_t^1 = \int_{t_0}^{t} P^1(t, \sigma) B(\sigma) \beta(\sigma, y) \, d\sigma - \int_{t_0}^{t_f} P^2(t, \sigma) B(\sigma) \beta(\sigma, y) \, d\sigma,$$

where

$$P^1(t, \sigma) = T_0' \Phi_0(t, \sigma) T_0 + T_f' \Phi_f(t, \sigma) T_f$$

$$+ \int_{\sigma}^{t} [T_0' \Phi_0(t, \tau) T_0 A(\tau) T_f' \Phi_f(\tau, \sigma) T_f] \, d\tau, \qquad (27a)$$

$$P^2(t, \sigma) = T_f' \Phi_f(t, \sigma) T_f + \int_{t_0}^{t} [T_0' \Phi_0(t, \tau) T_0 A(\tau) T_f' \Phi_f(\tau, \sigma) T_f] \, d\tau. \qquad (27b)$$

Analogously,

$$\eta_t^2 = \int_{t_0}^{t} P^1(t, \sigma) C(\sigma) \beta(\sigma, y) \, d\sigma - \int_{t_0}^{t_f} P^2(t, \sigma) C(\sigma) \beta(\sigma, y) \, d\sigma.$$

Let us now define

$$P(t, \sigma) = P^1(t, \sigma) - P^2(t, \sigma) = T_0' \Phi_0(t, \sigma) T_0$$

$$- \int_{t_0}^{\sigma} d\tau \, T_0' \Phi_0(t, \tau) T_0 A(\tau) T_f' \Phi_f(\tau, \sigma) T_f, \qquad (27c)$$

and note that

$$\int_{t_0}^{t_f} \eta_t^{1'} Q(t) \hat{x}_t^y \, dt = \int_{t_0}^{t_f} d\sigma \beta'(\sigma, y) B'(\sigma) \int_{\sigma}^{t_f} P^1(t, \sigma)' Q(t) \hat{x}_t^y \, dt$$

$$- \int_{t_0}^{t_f} d\sigma \, \beta'(\sigma, y) B'(\sigma) \int_{t_0}^{t_f} P^2(t, \sigma)' Q(t) \hat{x}_t^y \, dt$$

$$= \int_{t_0}^{t_f} d\sigma \, \beta'(\sigma, y) B'(\sigma) \left[\int_{\sigma}^{t_f} P(t, \sigma)' Q(t) \hat{x}_t^y \, dt - \int_{t_0}^{\sigma} P^2(t, \sigma)' Q(t) \hat{x}_t^y \, dt \right].$$

$$(28a)$$

Analogously,

$$\int_{t_0}^{t_f} \eta_t^{2'} Q(t) \hat{x}_t^y \, dt = \int_{t_0}^{t_f} d\sigma \, \beta'(\sigma, y) C'(\sigma) \left[\int_{\sigma}^{t_f} P(t, \sigma)' Q(t) \hat{x}_t^y \, dt \right.$$

$$\left. - \int_{t_0}^{\sigma} P^2(t, \sigma)' Q(t) \hat{x}_t^y \, dt \right].$$

$$(28b)$$

Utilizing (28a)–(28b) in (24), together with the expressions for $\eta_{t_f}^1$ and $\eta_{t_f}^2$, we finally obtain

$$\delta_1 J(\gamma; \beta) = E \left\{ \int_{t_0}^{t_f} \beta(t, y)' l(t, y) \, dt \right\} = 0, \qquad \forall \beta \in \Gamma, \qquad (29a)$$

where

$$l(t, y) = \gamma(t, y) + B(t)' P(t_f, t)' Q_f \hat{x}_{t_f}^y + C(t)' P(t_f, t)' Q_f \hat{\hat{x}}_{t_f}^y$$

$$+ B(t)' \int_t^{t_f} P(\sigma, t)' Q(\sigma) \hat{x}_\sigma^y \, d\sigma + C(t)' \int_t^{t_f} P(\sigma, t)' Q(\sigma) \hat{\hat{x}}_\sigma^y \, d\sigma$$

$$- B(t)' \int_{t_0}^t P^2(\sigma, t)' Q(t) \hat{x}_\sigma^y \, d\sigma - C(t)' \int_{t_0}^t P^2(\sigma, t)' Q(\sigma) \hat{\hat{x}}_\sigma^y \, d\sigma.$$

$$(29b)$$

It now readily follows from (29a) that a necessary and sufficient condition for $\gamma \in \Gamma$ to provide a minimizing solution is that

$$l(t, y) = 0, \qquad \forall t \in [t_0, t_f], \qquad \text{a.e. } \mathscr{P}_y.$$

The following lemma summarizes the result.

Lemma 3.1. A control strategy $\gamma \in \Gamma$ provides a minimizing solution to the stochastic optimum control problem formulated in this section iff it

satisfies the integral equation

$$\gamma(t, y) = -B(t)'P(t_f, t)'Q_f \hat{x}_{t_f}^y - C(t)'P(t_f, t)'Q_f \hat{x}_{t_f}^y$$

$$-B(t)' \int_t^{t_f} P(\sigma, t)'Q(\sigma)\hat{x}_\sigma^y \, d\sigma - C(t)' \int_t^{t_f} P(\sigma, t)'Q(\sigma)\hat{x}_\sigma^y \, d\sigma$$

$$+B(t)' \int_{t_0}^t P^2(\sigma, t)Q(\sigma)\hat{x}_\sigma^y \, d\sigma + C(t)' \int_{t_0}^t P^2(\sigma, t)'Q(\sigma)\hat{x}_\sigma^y \, d\sigma,$$

$$(30)$$

where $\hat{x}_t^y$ and $\hat{x}_t^y$ satisfy (23a)–(23b), $P(t, \sigma)$ and $P^2(t, \sigma)$ are given by (27a)–(27c), and $\Phi_0(t, \sigma)$ and $\Phi_f(t, \sigma)$ satisfy (26c) and (25c), respectively.

4. Determination of Stackelberg Strategies

We now go back to our original problem as discussed in Section 2. We have to minimize the cost functional given by Eq. (17), subject to the constraints (16a) through (16c). To this end, we first define the new state vector

$$\underline{x}_t = \begin{bmatrix} \hat{x}_t^2 \\ e_t \\ k_t^2 \end{bmatrix}$$

and the matrices

$$\underline{A}(t) = \begin{bmatrix} A(t) - B^2(t)B^2(t)'S^2(t) & 0 & -B^2(t)B^2(t)' \\ 0 & A(t) & 0 \\ 0 & 0 & -(A(t)' - S^2(t)B^2(t)B^2(t)') \end{bmatrix},$$

$$\underline{B}(t) = \begin{bmatrix} 0 \\ B^1(t) \\ 0 \end{bmatrix}, \qquad \underline{C}(t) = \begin{bmatrix} B^1(t) \\ -B^1(t) \\ -S^2(t)B^1(t) \end{bmatrix}, \qquad \underline{F}(t) = \begin{bmatrix} 0 \\ F(t) \\ 0 \end{bmatrix},$$

$$\underline{Q}_{1f} = \text{diag}(Q_{1f}, Q_{1f}, 0),$$

$$\underline{Q}_1(t) = \begin{bmatrix} Q_1(t) + S^2(t)B^2(t)R(t)B^2(t)'S^2(t) & 0 & S^2(t)B^2(t)R(t)B^2(t)' \\ 0 & Q_1(t) & 0 \\ B^2(t)R(t)B^2(t)'S^2(t) & 0 & B^2(t)R(t)B^2(t)' \end{bmatrix}.$$

Then, we have the following minimization problem for player 1: Minimize

$$J_1(\gamma^1, T\gamma^1) = \underline{x}_{t_f}'\underline{Q}_{1f}\underline{x}_{t_f} + \int_{t_0}^{t_f} [\underline{x}_t'\underline{Q}_1(t)\underline{x}_t + \gamma^1(t, y^1)'\gamma^1(t, y^1)] \, dt,$$

subject to

$$dx_t = \{A(t)x_t + B(t)\gamma^1(t, y^1) + C(t)E[\gamma^1(t, y^1) | y^2]\} \, dt + F(t) \, dW_t,$$

where $\hat{x}_{t_0}^2$, e_{t_0}, $k_{t_f}^2$ are given.

This stochastic optimum control problem can readily be solved by applying the result of Lemma 3.1. To this end, x_t, A, B, C, F of Section 3 must be identified with $\underline{x}_t, \underline{A}, \underline{B}, \underline{C}, \underline{F}$. Furthermore,

$$T_0 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \qquad T_f = [0, 0, I],$$

where I stands for suitable-order identity matrix and

$$\pi = \begin{bmatrix} x_{t_0}^2 \\ e_{t_0} \end{bmatrix}, \qquad \chi = k_{t_f}^2 = 0.$$

Note that, for these specific structural forms,

$$T_0'T_0 + T_f'T_f = I \quad \text{and} \quad T_f A T_0' = 0;$$

therefore, conditions (19b) and (19c) are satisfied. Then,

$$T_0 \underline{A} T_0' = \begin{bmatrix} A - B^2 B^{2'} S^2 & 0 \\ 0 & A \end{bmatrix},$$

and $\Phi_0(t, \sigma)$ can be split into four submatrices which satisfy

$$(d/dt)\begin{bmatrix} \Phi_0^{11}(t, \sigma) & \Phi_0^{12}(t, \sigma) \\ \Phi_0^{21}(t, \sigma) & \Phi_0^{22}(t, \sigma) \end{bmatrix} = \begin{bmatrix} A - B^2 B^{2'} S^2 & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \Phi_0^{11}(t, \sigma) & \Phi_0^{12}(t, \sigma) \\ \Phi_0^{21}(t, \sigma) & \Phi_0^{22}(t, \sigma) \end{bmatrix},$$

$$\begin{bmatrix} \Phi_0^{11}(\sigma, \sigma) & \Phi_0^{12}(\sigma, \sigma) \\ \Phi_0^{21}(\sigma, \sigma) & \Phi_0^{22}(\sigma, \sigma) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Therefore,

$$\Phi^{12}(t, \sigma) = \Phi^{21}(t, \sigma) = 0, \tag{31a}$$

$$(d/dt)\Phi_0^{11}(t, \sigma) = (A - B^2 B^{2'} S^2)\Phi_0^{11}(t, \sigma), \qquad \Phi_0^{11}(\sigma, \sigma) = I, \tag{31b}$$

$$(d/dt)\Phi_0^{22}(t, \sigma) = A\Phi_0^{22}(t, \sigma), \qquad \Phi_0^{22}(\sigma, \sigma) = I. \tag{31c}$$

On the other hand,

$$T_f \underline{A} T_f' = -(A' - S^2 B^2 B^{2'}),$$

$$(d/dt)\Phi_f(t, \sigma) = -(A' - S^2 B^2 B^{2'})\Phi_f(t, \sigma), \qquad \Phi_f(\sigma, \sigma) = I.$$

Thus,

$$\Phi_f(t, \sigma) = \Phi_0^{11}(\sigma, t)'.$$

Let us denote $\Phi_0^{11}(t, \sigma)$ by $\psi(t, \sigma)$ and $\Phi_0^{22}(t, \sigma)$ by $\Phi(t, \sigma)$. Now,

$$T_0' \Phi_0 T_0 = \begin{bmatrix} \psi & 0 & 0 \\ 0 & \Phi & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad T_f' \Phi_f T_f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Phi_f \end{bmatrix},$$

$$T_0' \Phi_0(t, \tau) T_0 A(\tau) T_f' \Phi_f(\tau, \sigma) T_f = \begin{bmatrix} 0 & 0 & -\Psi(t, \tau)B^2(\tau)B^2(\tau)'\Psi(\sigma, \tau)' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and define

$$F(t, \sigma) = \int_{t_0}^{\sigma} d\tau \, \Psi(t, \tau)B^2(\tau)B^2(\tau)'\Psi(\sigma, \tau)'. \tag{32}$$

Then,

$$P(t, \sigma) = \begin{bmatrix} \Psi(t, \sigma) & 0 & F(t, \sigma) \\ 0 & \Phi(t, \sigma) & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$P^2(t, \sigma) = \begin{bmatrix} 0 & 0 & -F(\sigma, t)' \\ 0 & 0 & 0 \\ 0 & 0 & \Psi(\sigma, t)' \end{bmatrix}.$$

Let

$$\hat{k}_t^{21} = E[k_t^2 | y^1], \qquad \hat{x}_t^{21} = E[\hat{x}_t^2 | y^1], \qquad \hat{e}_t^1 = E[e_t | y^1].$$

Then, the following result follows directly from Lemma 3.1 by identifying y with y^1, z with y^2, and γ with γ^1.

Theorem 4.1. The two-person nonzero-sum stochastic differential game with static information, as formulated in Section 2, admits a unique Stackelberg solution with DM1 as the leader. The leader's corresponding Stackelberg strategy $\gamma^1 \in \Gamma^1$ is given by

$$\gamma^{10}(t, y^1) = -B^1(t)'[(\Psi(t_f, t)' - S^2(t)F(t_f, t)')Q_{1f}\hat{x}_{t_f}^{21} + \Phi(t_f, t)'Q_{1f}\hat{e}_{t_f}^1]$$

$$- B^1(t)' \int_t^{t_f} [(\Psi(\sigma, t)' - S^2(t)F(\sigma, t)')(Q_1(\sigma)\hat{x}_\sigma^{21} + S^2(\sigma)\hat{z}_\sigma)$$

$$+ \Phi(\sigma, t)'Q_1(\sigma)\hat{e}_\sigma^1] \, d\sigma$$

$$+ B^1(t)'S^2(t) \int_{t_0}^t [F(t, \sigma)(Q_1(\sigma)\hat{x}_\sigma^{21} + S^2(\sigma)\hat{z}_\sigma) - \Psi(t, \sigma)\hat{z}_\sigma] \, d\sigma, \tag{33}$$

where

$$\hat{z}_t = B^2(t)R(t)B^2(t)'[S^2(t)\hat{x}_t^{21} + \hat{k}_t^{21}], \tag{34a}$$

and $\hat{x}_t^{21}, \hat{e}_t^1, \hat{k}_t^{21}$ satisfy the differential equations

$$\dot{\hat{x}}_t^{21} = (A - B^2 B^{2'} S^2)\hat{x}_t^{21} + B^1 E[E[\gamma^{10}(t, y^1)|y^2|y^1] - B^2 B^{2'}\hat{k}_t^{21}, \tag{34b}$$

$$\dot{\hat{e}}_t^1 = A\hat{e}_t^1 + B^1(t)\gamma^{10}(t, y^1) - B^1(t)E[E[\gamma^{10}(t, y^1)|y^2]|y^1], \tag{34c}$$

$$\dot{\hat{k}}_t^{21} = -(A' - S^2 B^2 B^{2'})\hat{k}_t^{21} - S^2 B^1 E[E[\gamma^{10}(t, y^1)|y^2]|y^1], \tag{34d}$$

$$\hat{x}_{t_0}^{21} = E[E[x_0|y^2]|y^1], \tag{34e}$$

$$\hat{e}_{t_0}^1 = E[x_0|y^1] - E[E[x_0|y^2]|y^1], \tag{34f}$$

$$\hat{k}_{t_f}^{21} = 0. \tag{34g}$$

Furthermore, the corresponding unique optimal response strategy of DM2, the follower, is given by

$$\gamma^{20}(t, y^2) = -B^2(t)'\{S^2(t)\hat{x}_t^2 + k_t^2\}, \tag{35}$$

where the relevant terms are defined by (15) and (16), with γ^1 replaced by γ^{10}.

5. Special Case of Gaussian Statistics

When the statistics of the random variables involved are jointly Gaussian, some of the expressions that determine the unique Stackelberg strategy of the leader can further be simplified; in particular, it can be shown that the Stackelberg strategy is affine in the available static observation. Toward this end, let us first assume that x_0, y^1, y^2 are jointly Gaussian with mean values $\bar{x}_0, \bar{y}^1, \bar{y}^2$, respectively.

Furthermore, let us introduce the appropriate dimensional matrices $\Lambda_{01}, \Lambda_{02}, \Lambda_{12}, \Lambda_{21}$ through the relations

$$E[x_0|y^i] = \bar{x}^0 + \Lambda_{0i}(y^i - \bar{y}^i), \qquad i = 1, 2, \tag{36a}$$

$$E[y^i|y^j] = \bar{y}^i + \Lambda_{ij}(y^j - \bar{y}^j), \qquad i, j = 1, 2, \quad i \neq j. \tag{36b}$$

Now, starting with expressions (14), we note that γ^{10} [which satisfies (33)] is also given by

$$\gamma^{10}(\cdot, y^1) = -(I + \bar{\mathcal{R}})^{-1}\{\mathcal{L}_1^* \mathcal{Q}_1 \tilde{r} + \mathcal{L}_1^* \mathcal{Q}_2 \mathcal{L}_2(I + \mathcal{L}_2^* \mathcal{Q}_2 \mathcal{L}_2)^{-1}$$

$$\times R(I + \mathcal{L}_2^* \mathcal{Q}_2 \mathcal{L}_2)^{-1}\mathcal{L}_2^* \mathcal{Q}_2 r\}, \tag{37}$$

where the relevant terms are defined following (14). It has already been shown in Ref. 12, Appendix 1, in a different context, that $\mathcal{L}_i^*$ is a linear operator which, when applied on an element of $L_{2f}([t_0, t_f] \times \Omega)$ under the Gaussianness assumption, yields an element of H^i which is affine in y^i. A similar analysis readily proves that $\tilde{\mathcal{L}}_1^*$ also possesses the same property. Then, $r_t(\omega)$ being a Gaussian vector for each $t \in [t_0, t_f]$, we arrive at the conclusion that, under the Gaussianness assumption, $\gamma^{10}(\cdot, y^1)$, as given by (37), is affine in y^1. In particular, also in view of (33), it can be written as

$$\gamma^{10}(t, y^1) = -B^1(t)'[K(t)(y^1 - \bar{y}^1) + l(t)], \tag{38}$$

where $K(\cdot)$ is an $(n \times m_1)$-dimensional matrix-valued function and $l(\cdot)$ is an n-dimensional vector-valued function, both of them being independent of $y^1 - \bar{y}^1$.

Now, to determine the expressions yielding $K(\cdot)$ and $l(\cdot)$, we substitute the functional form (38) into (33) and also into (34a) through (34d). Firstly, we obtain from (34d)

$$\hat{k}_t^{21} = K^k(t)\Lambda_{12}\Lambda_{21}(y^1 - \bar{y}^1) + l^k(t), \tag{39a}$$

where $K^k(\cdot)$ and $l^k(\cdot)$ uniquely satisfy

$$\dot{K}^k = -(A - B^2 B^{2'} S^2)' K^k + S^2 B^1 B^{1'} K(t), \qquad K^k(t_f) = 0, \tag{39b}$$

$$\dot{l}^k = -(A - B^2 B^{2'} S^2)' l^k + S^2 B^1 B^{1'} l(t), \qquad l^k(t_f) = 0. \tag{39c}$$

Using this solution, together with (38), in (34b), we further obtain

$$\hat{x}_t^{21} = K^x(t)(y^1 - \bar{y}^1) + l^x(t), \tag{40a}$$

where $K^x(\cdot)$ and $l^x(\cdot)$ uniquely satisfy

$$\dot{K}^x = (A - B^2 B^{2'} S^2) K^x - B^1 B^{1'} K(t)\Lambda_{12}\Lambda_{21} - B^2 B^{2'} K^k(t)\Lambda_{12}\Lambda_{21},$$

$$K^x(t_0) = \Lambda_{02}\Lambda_{21}, \tag{40b}$$

$$\dot{l}^x = (A - B^2 B^{2'} S^2) l^x - B^1 B^{1'} l(t) - B^2 B^{2'} l^k(t), \qquad l^x(t_0) = \bar{x}_0. \tag{40c}$$

Thirdly, substitution of the structural form (38) into (34c) yields

$$\hat{e}_t^1 = K^e(t)(y^1 - \bar{y}^1), \tag{41a}$$

where $K^e(\cdot)$ uniquely satisfies the differential equation

$$\dot{K}^e = A K^e - B^1 B^{1'} K(t) + B^1 B^{1'} K(t)\Lambda_{12}\Lambda_{21}, \qquad K^e(t_0) = \Lambda_{01} - \Lambda_{02}\Lambda_{21}. \tag{41b}$$

If we use all these in (33), we finally obtain the following two integral

equations whose unique solutions yield $K(\cdot)$ and $l(\cdot)$, respectively:

$$K(t) = \Psi(t_f, t)' - S^2(t)F(t_f, t)'Q_{1f}K^x(t_f) + \Phi(t_f, t)'Q_{1f}K^e(t_f)$$

$$+ \int_t^{t_f} \{[\Psi(\sigma, t)' - S^2(t)F(\sigma, t)']$$

$$\times [Q_1(\sigma)K^x(\sigma) - S^2(\sigma)B^2RB^{2'}(S^2(\sigma)K^x(\sigma)$$

$$+ K^k(\sigma)\Lambda_{12}\Lambda_{21})] + \Phi(\sigma, t)'Q_1(\sigma)K^e(\sigma)\} \, d\sigma - S^2(t) \int_{t_0}^t \{F(t, \sigma)$$

$$\times [Q_1(\sigma)K^x(\sigma) + S^2(\sigma)B^2RB^{2'}(S^2(\sigma)K^x(\sigma) + K^k(\sigma)\Lambda_{12}\Lambda_{21})]$$

$$- \Psi(t, \sigma)B^2RB^{2'}[S^2(\sigma)K^x(\sigma) + K^k(\sigma)\Lambda_{12}\Lambda_{21}]\} \, d\sigma, \qquad (42)$$

$$l(t) = [\Psi(t_f, t)' - S^2(t)F(t_f, t)']Q_{1f}l^x(t)$$

$$+ \int_{t_g}^{t_f} \{[\Psi(\sigma, t)' - S^2(t)F(\sigma, t)']$$

$$\times [Q_1(\sigma)l^x(\sigma) + S^2(\sigma)B^2RB^{2'}(S^2(\sigma)l^x(\sigma) + l^k(\sigma))]\} d\sigma$$

$$- S^2(t) \int_{t_0}^t \{F(t, \sigma)[Q_1(\sigma)l^x(\sigma) + S^2(\sigma)B^2RB^{2'}(S^2(\sigma)l^x(\sigma)$$

$$+ l^k(\sigma))] - \Psi(t, \sigma)B^2RB^{2'}[S^2(\sigma)l^x(\sigma) + l^k(\sigma)]\} \, d\sigma. \qquad (43)$$

The following theorem now summarizes the main result of this section.

Theorem 5.1. When all the random vectors are jointly Gaussian distributed, the two-person nonzero-sum stochastic differential game of Section 2 admits a unique Stackelberg strategy for the leader, which is given by (38), with $K(\cdot)$ and $l(\cdot)$ solved from (42) and (43), respectively, also in view of the relations (39)–(41). The corresponding unique optimal response strategy of the follower is again given by (35).

Proof. As discussed prior to the statement of the theorem, this result follows directly from Theorem 4.1 after verification of the optimal affine structure (38) for γ^{1^0} under the Gaussianness assumption. Substitution of (38) into (33) and an appropriate decomposition readily yields Eqs. (42) and (43).   □

Remark 5.1. It should be noted that, under the Gaussian statistics, the unique Stackelberg strategy of the leader, as presented in Theorem 5.1, is composed of two parts: (i) a linear transformation on the zero-mean part of the leader's observation, which depends on the statistics of the

Gaussian random variables involved; and (ii) a linear transformation on the mean value ($\bar{x}_0$) of the initial state, which is independent of the statistics of the random variables. Hence, we can say that the Stackelberg solution also features a separation property. In fact, the latter part of the Stackelberg strategy of the leader, which involves a linear transformation on $\bar{x}_0$, also constitutes a Stackelberg solution to the deterministic version of the non-zero-sum differential game of Section 2, in which the leader has open-loop information and $\bar{x}_0$ stands for the initial state vector. In that respect, the solution presented here, i.e.,

$$\gamma^{1^0}(t, \bar{x}_0) = -B^1(t)'l(t),$$

is less restrictive than the one given in Ref. 3, since it does not require the linear transformation to be differentiable. Whenever $l(\cdot)$ is differentiable, then the two solutions coincide.

6. Computational Aspects of the Stackelberg Solution

In this section, we develop an algorithm to solve numerically for the leader's Stackelberg strategy from Eq. (33) and under general statistics. To this end, let us first observe that the solutions of the differential equations (34b) through (34d) can be obtained through affine transformations on γ^{1^0}; and, when these are substituted into (33), together with the linear relation (34a), the right-hand side of (33) becomes an affine function of γ^{1^0}. Hence, there exists a linear operator $\mathscr{B}: \Gamma^1 \to \Gamma^1$ and $b \in \Gamma^1$, such that (33) can be written equivalently as

$$\gamma^{1^0}(t, y^1) = (\mathscr{B}\gamma^{1^0})(t, y^1) + b(t, y^1). \tag{44}$$

Explicit expressions for the operator $\mathscr{B}$ and the function b can readily be obtained in terms of the parameters of the problem.

Now, by an abuse of notation, we can also think of $\mathscr{B}$ as a mapping from H^1 into H^1 and of b as an element of H^1. Then, Eq. (44) can be written equivalently as

$$u^1 = \mathscr{B}u^1 + b, \tag{45}$$

which is known to admit the unique solution

$$u^{1^0} = \gamma^{1^0}(t, y^1).$$

Then, a natural algorithm to obtain u^{1^0} is the following.

Algorithm

Step 1. Start with any $u^1 \in H^1$; substitute this into (34a) through (34d) for γ^{1o}, and solve for the corresponding unique $\hat{z}_t$, $\hat{x}_t^{21}$, $\hat{e}_t^1$, $\hat{k}_t^{21}$.

Step 2. Substitute these into the right-hand side of (33) to generate a new $u^1 \in H^1$.

Step 3. Use the new u^1 obtained at Step 2 to replace the starting choice at Step 1, and iterate.

Proposition 6.1. Let there exist a $0 < \rho < 1$ such that

$$\langle u, \mathcal{B}^* \mathcal{B} u \rangle_1 \leq \rho \langle u, u \rangle_1, \qquad \forall u \in H^1. \tag{46}$$

Then, the preceding algorithm converges to the unique Stackelberg control u^{1o} of the leader, regardless of the initial choice.

Proof. The result readily follows, since H^1 is a Hilbert space and $\mathcal{B}$ is a linear contraction operator under (46). □

If the condition of Proposition 6.1 is not satisfied, then the proposed algorithm in general will not converge. However, in such a situation, it is still possible to develop a convergent algorithm by starting, this time, with Eq. (37), which we rewrite as

$$u^1 = -\bar{\mathcal{R}} u^1 + \tilde{\tilde{r}}, \tag{47a}$$

where

$$\tilde{\tilde{r}} \triangleq \mathcal{\hat{L}}_1^* \mathcal{Q}_1 \tilde{r} + \mathcal{L}_1^* \mathcal{Q}_2 \mathcal{L}_2 (I + \mathcal{L}_2^* \mathcal{Q}_2 \mathcal{L}_2)^{-1} R (I + \mathcal{L}_2^* \mathcal{Q}_2 \mathcal{L}_2)^{-1} \mathcal{L}_2^* \mathcal{Q}_2 r. \tag{47b}$$

Now, if the operator norm of $\bar{R}$ is less than unity, i.e.,

$$\|\bar{\mathcal{R}}\|_0 < 1,$$

we have the algorithm

$$u_{j+1}^1 = -\bar{\mathcal{R}} u_j^1 + \tilde{\tilde{r}}, \tag{48}$$

which is directly obtainable from (47a) and is convergent, since $\bar{\mathcal{R}}$ is a linear contraction operator and H^1 is a Hilbert space. If the condition

$$\|\bar{\mathcal{R}}\|_0 < 1$$

is not satisfied, then we add $\theta I u^1$ to both sides of (47a) to obtain

$$(1 + \theta) I u^1 = (\theta I - \bar{\mathcal{R}}) u^1 + \tilde{\tilde{r}}, \tag{49}$$

where $I: H^1 \to H^1$ is the identity operator and θ is a scalar satisfying

$$2\theta \geq \|\bar{\mathcal{R}}\|_0.$$

Then, the algorithm

$$u^1_{j+1} = [1/(1+\theta)](\theta I - \bar{\mathscr{R}})u^1_j + [1/(1+\theta)]\tilde{\bar{r}} \qquad (50)$$

converges to the unique solution $u^{1^o} \in H^1$, a result which follows from a convergence proof given in Ref. 13 for a similar algorithm, since $\bar{\mathscr{R}}$ is a nonnegative self-adjoint operator. It should be noted, however, that this algorithm is computationally more difficult than the first one proposed in this section, since it involves the inversion of an integral operator in determining $\tilde{\bar{r}}$ (though, this inversion has to be done only once, and not at every iteration point).

7. Conclusions

In this paper we have obtained the Stackelberg solution of a class of stochastic linear-quadratic differential game problems in which both players have access to static information concerning the initial state of the game. Existence and uniqueness of the solution is established, and the leader's optimal strategy is shown to satisfy a particular integral equation. For the special case when all the random variables have jointly Gaussian statistics, this integral equation admits a solution which is affine in the observation of the leader. We have also discussed in the paper numerical aspects of the Stackelberg solution under general statistics, and have developed algorithms which converge to the unique Stackelberg control of the leader. As a by-product of our analysis, we have also obtained the solution of a nonstandard stochastic control problem, which is formulated and solved in Section 3 in more general terms than required in the solution of the Stackelberg problem.

One natural extension of the results of this paper would be to obtain the feedback Stackelberg solution of linear-quadratic-Gaussian continuous-time differential games in which the players make sampled observations of the state and share this information with a one-step delay (as formulated in Ref. 12 within a team framework). In other words, a counterpart of the results of Ref. 10 in continuous-time can be obtained by utilizing the theory developed in this paper under static information. A repeated application of Theorem 5.1 would lead to the conclusion that the feedback Stackelberg strategies in such a differential game will again be affine in the information available to the players; derivation of these strategies, however, requires a rather involved analysis, details of which will be presented in a future paper.

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