On a two-server finite queuing system with ordered entry and deterministic arrivals

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Abstract. Consider a two-server, ordered entry, queuing system with heterogeneous servers and finite waiting rooms in front of the servers. Service times are negative exponentially distributed. The arrival process is deterministic. A matrix solution for the steady state probabilities of the number of customers in the system is derived. The overflow probability will be used to formulate the stability condition of a closed-loop conveyor system with two work stations.

Keywords: Queues

1. Introduction

Consider a two-server queuing system with heterogeneous exponential servers. Both servers have a finite waiting room capacity. The arrival process is deterministic. Every arriving customer tries to join the queue in front of server 1 first (i.e. 'ordered entry' selection); if the waiting room is full he tries to join the queue in front of server 2. If both waiting rooms are full the customer is lost. Having joined a queue a customer will eventually leave the system after being served. Overflow models of the above type, with two or more servers, have been considered by several authors. For Poisson arrivals we refer to Disney [3,4] and Gupta [5]. In Disney [4] a matrix-geometric solution is given for the joint distribution of the queue sizes, while Gupta gives the probability generating function. Several authors have dealt with the queue $GI/M/S/N$ (i.e. renewal input, exponential servers, and finite capacity). Since the overflow process from this queue is again a renewal process the results are relevant for the model considered here. These overflow models are of interest for e.g. communication networks, in which it is often assumed that the service times are exponential, whereas the input process of a particular queue result from superposition of several output or overflow processes from other queues and may be approximated by a renewal process. For some relevant contributions to the queue $GI/M/S/N$, see Çinlar and Disney [1], Rath and Sheng [11], De Smit [2] and McNickle [8]. The latter author considers a simple network model in which the overflows of a $GI/M/S/N$ queue form the input to a single exponential server queue with infinite waiting room.

For the particular model considered here we give a matrix solution for the joint distribution of the queue sizes under steady-state conditions, which is easily programmed. The result will be used to formulate the stability condition of a certain type of closed-loop conveyor.

2. The mathematical model

Let the subsequent arrivals occur at the epochs $t_n = nd$, $n = 0, 1\dots$ $(d > 0)$. We suppose that the service times at station $j(=1, 2)$ are independent negative exponentially distributed with parameter μ_j, both

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families of random variables being independent of each other and independent of the state of the system. The waiting rooms at the servers 1 and 2 have capacities $N-1$ and $M-1$ respectively.

Let $x_j(n)$ denote the number of customers in the queue plus in service at server j ($j=1,2$) at epoch $nd-0$, $n \geqslant 0$, i.e. just before the arrival of a customer. The process $\{(x_1(n), x_2(n)), n \geqslant 0\}$ is a vector Markov chain, with finite state space, which is irreducible and aperiodic. Consequently, the chain is ergodic and possesses a unique stationary probability distribution. Let this distribution be denoted by

$$\pi(i,j) = \lim_{n\to\infty} \Pr\{x_1(n)=i, x_2(n)=j | x_1(0)=i_0, x_2(0)=j_0\} \tag{1}$$

for $i=0,1,\ldots,N$ and $j=0,1,\ldots,M$.

Let the states (i,j) of the process $\{(x_1(n), x_2(n))\}$ be ordered lexicographically, i.e. in the order (0, 0), (0, 1),...,(0, M), (1, 0), (1, 1),...,(1, M),..., ..., (N, 0), (N, 1),...,(N, M). Then the one-step transition matrix Q of the process can be written in the following block-partitioned form:

$$Q = \begin{pmatrix} q_1A & p_0A & 0 & \ldots & 0 \\ q_2A & p_1A & p_0A & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ q_NA & p_{N-1}A & p_{N-2}A & \ldots & p_0A \\ q_NB & p_{N-1}B & p_{N-2}B & \ldots & p_0B \end{pmatrix} \tag{2}$$

where

$$p_j = \frac{(\mu_1 d)^j}{j!} e^{-\mu_1 d}, \qquad q_j = 1 - \sum_{k=0}^{j-1} p_k \qquad (j \geqslant 1), \tag{3}$$

and

$$A = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ \bar{q}_1 & \bar{p}_0 & 0 & \ldots & 0 \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \bar{q}_{M-1} & \bar{p}_{M-2} & \bar{p}_{M-3} & \cdots & 0 \\ \bar{q}_M & \bar{p}_{M-1} & \bar{p}_{M-2} & \cdots & \bar{p}_0 \end{pmatrix}, \qquad B = \begin{pmatrix} \bar{q}_1 & \bar{p}_0 & 0 & \ldots & 0 \\ \bar{q}_2 & \bar{p}_1 & \bar{p}_0 & \ldots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \bar{q}_M & \bar{p}_{M-1} & \bar{p}_{M-2} & \cdots & \bar{p}_0 \\ \bar{q}_M & \bar{p}_{M-1} & \bar{p}_{M-2} & \cdots & \bar{p}_0 \end{pmatrix} \tag{4}$$

where $\bar{p}_j$ and $\bar{q}_j$ are given by (3) with μ_1 replaced by μ_2.

Observe that A is a nonsingular lower-triangular $(M+1)\times(M+1)$ matrix. The matrix B is an $(M+1)\times(M+1)$ matrix with zeros on all but the first superdiagonals. The latter type of matrices are called 'almost left triangular'.

Observe also that A and B are stochastic matrices; the elements of BA^{-1} on its first superdiagonal and the last element of its last row all equal one, all other elements being zero.

Introducing the row vectors

$$\pi_i = (\pi(i,0), \pi(1,1), \ldots, \pi(i,M)), \quad i=0,1,\ldots,N, \tag{5}$$

the steady-state equations of the process can now be written as

$$\pi_0 = \sum_{j=0}^{N-1} q_{j+1}\pi_j A + q_N \pi_N B, \tag{6a}$$

$$\pi_i = \sum_{j=i-1}^{N-1} p_{j-i+1}\pi_j A + p_{N-i}\pi_N B, \quad i=1,2,\ldots,N. \tag{6b}$$

together with the normalizing condition

$$\sum_{i=0}^{N} \sum_{j=0}^{M} \pi(i,j) = 1. \tag{7}$$

Our problem is now to determine the vectors π_j, $i = 0, 1, \ldots, N$ satisfying (5), (6) and (7).

3. A solution for the steady-state probabilities

In view of the almost left triangular block structure of Q we introduce the $(M+1) \times (M+1)$ matrices C_j, $j = 0, 1, \ldots, N$, which relate the vector π_i to the vector π_N according to

$$\pi_i = \pi_N C_{N-i}, \quad i = 0, 1, \ldots, N, \tag{8}$$

where $C_0 = I$ is the identity matrix of rank $M+1$.

Upon substituting (8) into (6b) we obtain

$$C_j = \sum_{k=1}^{j+1} p_{j-k+1} C_k A + p_j B, \quad j = 0, 1, \ldots, N-1. \tag{9}$$

Rewriting this as

$$C_{j+1} = \frac{1}{p_0} \left\{ C_j A^{-1} - \sum_{k=1}^{j} p_{j+1-k} C_k - p_j B A^{-1} \right\} \tag{10}$$

and noting that the inverse A^{-1} exists, we obtain a recursive scheme from which the matrices C_j can be determined since $C_0 = I$.

By adding all the equations in (6) we obtain

$$\sum_{i=0}^{N} \pi_i = \sum_{i=0}^{N} \pi_i A + \pi_N (B - A). \tag{11}$$

Substitution of (8) then yields the following relation for the row vector π_N:

$$\pi_N \{ C(I - A) - B + A \} = 0, \tag{12}$$

where

$$C = \sum_{j=0}^{N} C_j. \tag{13}$$

Lemma 1. *Let* $D = C(I - A) - B + A = (d_{i,j})$, $i = 0, 1, \ldots, M$; $j = 0, 1, \ldots, M$.

(a) *D is almost left triangular, i.e.* $d_{i,j} = 0$ *for* $i + 2 \leqslant j \leqslant M$, $i = 0, 1, \ldots, M-2$;

(b) *The matrix $\bar{D}$ obtained from D by deleting its first column and its last row is a lower triangular Toeplitz matrix, i.e. its entries satisfy* $\bar{d}_{i,j} = d_{i,j} = d_{i-j+1}$, $i = 0, 1, \ldots, M-1$; $j = 1, 2, \ldots, M$;

(c) *The elements* $d_{i,i+1} = d_0$, $i = 0, 1, \ldots, M-1$ *on the first superdiagonal of D (c.q. the main diagonal of $\bar{D}$) are negative, that is* $d_0 < 0$;

(d) *The rank of D equals M.*

Proof. The assertions (a) and (b) follows from (10) and the particular form of A and B. The main observations are that:

(1) any matrix satisfying (a) and (b) when right multiplied by a matrix having the same form as A yields a matrix which also satisfies (a) and (b) and

(2) A^{-1} is of the same form as A, i.e. it is lower triangular and moreover the matrix obtained from A^{-1} by deleting its first column and last row is a Toeplitz matrix.

To prove (c) let c_j, $j = 1, 2, \ldots, N$ be an element on the first (nonzero) superdiagonal of C_j. Taking $j = 0$ in (10) and noting that $C_0 = I$ it follows that $c_1 = -1$. In the same way we obtain for $j = 1$ that $c_2 = -1/p_0^2$, so that $c_2 < c_1 = -1$. Observing that the elements on the first superdiagonal of BA^{-1} are equal to 1 it follows from (10) that

$$c_{j+1} = p_0^{-1}\left\{ c_j/p_0 - \sum_{k=2}^{j} p_{j+1-k}c_k \right\}, \quad j = 2, 3, \ldots, N-1, \tag{14}$$

noting that the matrices C_j, $j = 1, 2, \ldots, N$ satisfy the properties (a) and (b) of the lemma.

Now we proceed by induction in order to prove that $c_N < c_{N-1} < \ldots < c_2 < c_1 = -1$. Suppose that $c_j < c_{j-1} < \ldots < c_1 = -1$, which is true for $j = 2$. By (14) we then have

$$c_{j+1} < p_0^{-1}\left\{ p_0^{-1} - \sum_{k=1}^{j-1} p_k \right\} c_j. \tag{15}$$

Observing that $p_0^{-1}\{ p_0^{-1} - \Sigma_{k=1}^{j-1} p_k \} > 1$ and noting that $c_j < 0$, it follows that $c_{j+1} < c_j < \ldots < c_1 = -1$. Consequently $c_N < c_{N-1} < \ldots < c_1 = -1$ by induction on j. Hence $c = \Sigma_{j=1}^{N} c_j < 0$ and since $d_0 = c(1 - p_0) - p_0$ we also have $d_0 < 0$.

Finally to prove part (d) first note that $De = 0$ where e is a column vector all whose $M + 1$ elements equal 1. Hence the rank of D is at most M. However, by deleting its first column and last row we obtain the matrix $\overline{D}$ which in view of parts (b) and (c) is nonsingular, since $\det(\overline{D}) = d_0^M \neq 0$. Consequently the rank of D equals M.

Since D is of rank M it is possible to determine the elements $\pi(N, j)$, $j = 0, 1, \ldots, M-1$ of the vector π_N in terms of $\pi(N, M)$ using the equation $\pi_N D = 0$, cf. (12).

Therefore, introducing the row vector g, let

$$\pi_N = g\pi(N, M) = (g_M, g_{M-1}, \ldots, g_1, 1)\pi(N, M). \tag{16}$$

It follows from the preceding lemma that the vector $(g_M, g_{M-1}, \ldots g_1)$ is the unique solution of the equation

$$(g_M, g_{M-1}, \ldots, g_1)\overline{D} = -(d_{M,1}, d_{M,2}, \ldots, d_{M,M}) \tag{17}$$

from which it is readily verified that the numbers g_j are determined recursively by

$$g_j = \left\{ d_{M,M-j+1} + \sum_{k=1}^{j-1} g_k d_{M-k,M-j+1} \right\} / |d_0|, \quad j = 1, 2, \ldots, M, \tag{18}$$

where $d_0 = d_{i,i+1}$.

By definition we have, cf. (8) and (16),

$$\pi_i = \pi(N, M) g C_{n-i}, \quad i = 0, 1, \ldots, N, \tag{19}$$

so that all probabilities can be expressed in terms of $\pi(N, M)$. The latter probability obviously follows from the normalizing condition $\Sigma\Sigma\pi(i, j) = 1$ which can be written as $\pi(N, M)gCe = 1$ so that

$$\pi(N, M) = \{ gCe \}^{-1}. \tag{20}$$

We summarize our result in the following theorem.

Theorem 1. *The unique stationary distribution* $\{ \pi_i, i = 0, 1, \ldots, N \} = \{ \pi(i, j), i = 0, 1, \ldots, N; j = 0, 1, \ldots, M \}$ *satisfying* (6) *and* (7) *is given by*

$$\pi_i = gC_{N-i}/(gCe)$$

where the vector $g=(g_M, g_{M-1},\dots,g_1, 1)$ *is uniquely determined by* (18) *and the* $(M+1)\times(M+1)$ *matrices* C_j, $j=0, 1,\dots,N$ *are uniquely determined by* (10) *with* $C_0=I$.

Remark 1. Consider the case $N=M=1$. From (10) we obtain $C_1=p_0^{-1}A^{-1}-BA^{-1}$, so that $C=I+p_0^{-1}A^{-1}-BA^{-1}$. Hence $D=C(I-A)+A-B=I-p_0^{-1}I+p_0^{-1}A^{-1}-BA^{-1}$ and $Ce=p_0^{-1}e$. It is readily verified that $d_{00}=1$, $d_{01}=-1$, $d_{10}=-\bar{q}_1/(p_0\bar{p}_0)$ and $d_{11}=\bar{q}_1/(p_0\bar{p}_0)$.

Therefore from (18) we obtain $g_1=\bar{q}_1/(p_0\bar{p}_0)$ *and consequently* $(g_1, 1)Ce=(1-q_1\bar{p}_0)/\bar{p}_0\bar{p}_0^2$, so that the overflow probability $\pi(1, 1)$ is given by

$$\pi(1, 1)=\frac{\bar{p}_0 p_0^2}{1-q_1\bar{p}_0},\quad N=M=1. \tag{21}$$

Taking $\mu_1=\mu_2$ in (21) one obtains the loss probability for the D/M/2/2 loss system, see Takács [12].

Remark 2. It can be verified from (10) that all the elements of the vector Ce are identical. In implementing the solution for computer computation the memory requirement and the number of elementary computations can be reduced taking into account that the matrices C_j, $j=1, 2,\dots,N$ and D satisfy part (a) and (b) of Lemma 1.

Remark 3. It is interesting to notice that the transition matrices $Q^{(s)}$ and $Q^{(s-1)}$ for the models with s and $s-1$ servers, respectively, are connected in the following way when the states are ordered lexicographically (assuming equal waiting room capacities and $\mu_1=\mu_2=\cdots=\mu_s$).

$$Q^{(s)}=\begin{pmatrix} q_1A^{(s-1)\times} & p_0A^{(s-1)\times} & 0 & \dots & 0 \\ q_2A^{(s-1)\times} & p_1A^{(s-1)\times} & p_0A^{(s-1)\times} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ q_NA^{(s-1)\times} & p_{N-1}A^{(s-1)\times} & p_{N-2}A^{(s-1)\times} & & p_0A^{(s-1)\times} \\ q_NQ^{(s-1)\times} & p_{N-1}Q^{(s-1)\times} & p_{N-2}Q^{(s-1)\times} & \dots & p_0Q^{(s-1)\times} \end{pmatrix} \tag{22}$$

where $A^{(s-1)\times}$ is the $(s-1)$-fold Kronecker product of A with itself. For the definition of the Kronecker product see e.g. Marcus and Minc [7, p. 9]. Taking $s=2$ in (22) we obtain (2) once we note that $Q^{(1)}=B$ is indeed the transition matrix of the single server case. From (22) we see that $Q^{(s)}$ has the same block structure as Q so that the preceding method can be generalized for $s>2$, at least in principal.

Remark 4. Finally it should be mentioned that the single server case with transition matrix B can be handled by the method of Raju and Bhat, in particular see Theorem 2.2.1 in [9, p. 252]). We also note the similarity between their results and our results, see Raju and Bhat [10, section 3].

We end this section with some numerical results and a worked example. In Fig. 1 the overflow probability $\pi(N, M)$ is shown as a function of M for some values of N and three combinations of $\mu_1 d$ and $\mu_2 d$. The values of $\mu_1 d$ and $\mu_2 d$ for these combinations are chosen so that $\mu_2>\mu_1$ $(\mu_1+\mu_2>d^{-1})$, $\mu_1=\mu_2$ $(\mu_1+\mu_2=d^{-1})$ and $\mu_2<\mu_1$ $(\mu_1+\mu_2<d^{-1})$.

Note the asymptotic behaviour for $N\to\infty$(with fixed M) and $M\to\infty$ (with fixed N). Observe that the dependence on M for given N decreases when $\mu_1+\mu_2\downarrow 0$, while for large values of $\mu_1+\mu_2$ the dependence on N, for fixed M, decreases.

Example. Consider the case $\mu_1=\mu_2=0.5$ with $N=1$ and $M=3$. From (3) and (4) we obtain

$$A=\begin{pmatrix} 1.000 & 0 & 0 & 0 \\ 0.394 & 0.607 & 0 & 0 \\ 0.090 & 0.303 & 0.607 & 0 \\ 0.014 & 0.076 & 0.303 & 0.607 \end{pmatrix},\qquad B=\begin{pmatrix} 0.394 & 0.607 & 0 & 0 \\ 0.090 & 0.303 & 0.607 & 0 \\ 0.014 & 0.076 & 0.303 & 0.607 \\ 0.014 & 0.076 & 0.303 & 0.607 \end{pmatrix}.$$

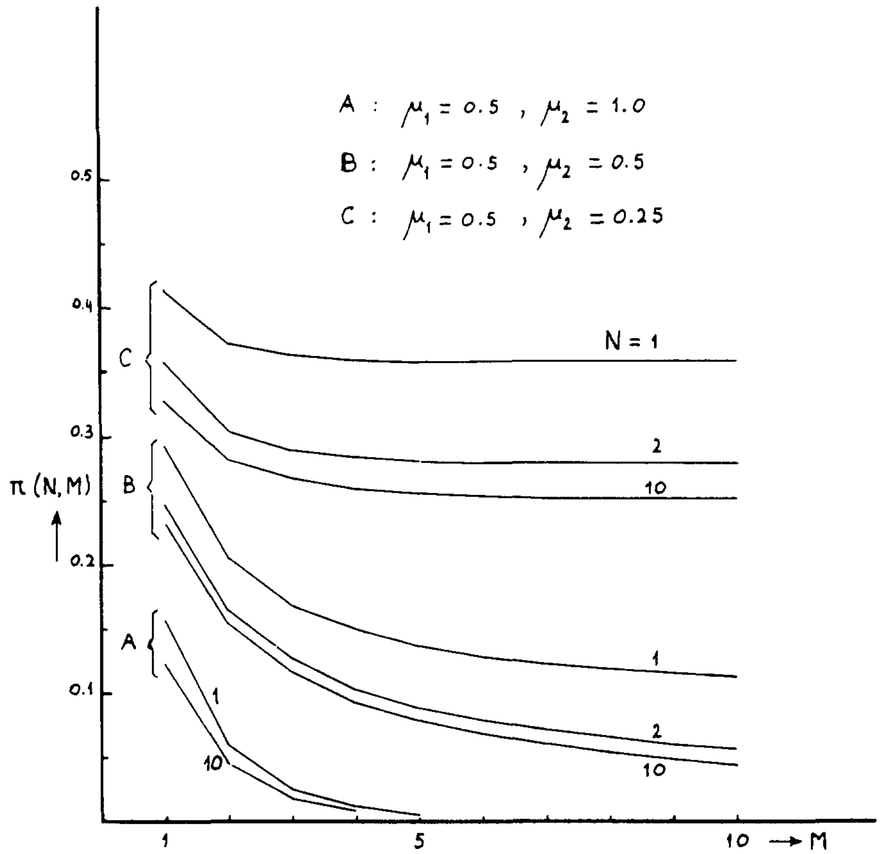

Fig. 1.

From (10) we have $C_1 = \{A^{-1} - p_0 BA^{-1}\}/p_0$, giving

$$C_1 = \begin{pmatrix} 1.649 & -1 & 0 & 0 \\ -1.070 & 2.718 & -1 & 0 \\ 0.290 & -1.359 & 2.718 & -1 \\ -0.050 & 0.340 & -1.359 & 1.718 \end{pmatrix}.$$

Since $C_0 = I$ (the identity matrix) and $e^{\mathrm{T}} = (1, 1, 1, 1)$ it follows from (13) that $(Ce)^{\mathrm{T}} = (C_0 e + C_1 e)^{\mathrm{T}} = 1.649$ e^{T}. Moreover, from $D = C(I - A) + A - B$ we obtain

$$D = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1.070 & 2.070 & -1 & 0 \\ 0.290 & -1.360 & 2.070 & -1 \\ -0.050 & 0.340 & -1.360 & 1.070 \end{pmatrix}$$

The Toeplitz matrix $\overline{D}$ is obtained from D by deleting its first column and its last row. The vector $g = (g_3, g_2, g_1, 1)$ follows from (18) yielding $g = (0.654, 0.854, 1.070, 1)$. Next $\pi(1, 3)$ is calculated using relation (20) giving $\pi(1, 3) = \{1.649 \times (0.654 + 0.854 + 1.070 + 1)\}^{-1} = 0.169$.

Now from Theorem 1 we finally have, cf. (5), (8), (16),

$$\pi_1 = \pi(1, 3) g = (0.111, 0.145, 0.181, 0.169)$$

and

$$\pi_0 = \pi_1 C_1 = (0.072, 0.094, 0.118, 0.110).$$

4. An application to the stability condition of a conveyor system

The reason for analyzing the present model lies in a study of stability conditions for a certain class of closed-loop conveyors.

Consider a closed chain, moving with constant speed, equipped with r equidistantly spaced hooks. The time to make one revolution equals T. Somewhere along the chain units arrive according to a renewal process with arrival rate λ. The units are connected to a hook, with one unit per hook, for further transport to a work station. There are two work stations along the conveyor, which are characterized by our previously discussed two-server system, i.e. the first station has storage capacity $N-1$ and the second $M-1$ and the processing times at the stations are exponentially distributed with parameters μ_1 and μ_2. Loaded hooks are automatically unloaded when passing a station, provided that the storage is not full.

Now imagine that the system is highly congested so that several loaded hooks cannot be unloaded at either one of the stations. In this case units recirculate one or more times before unloading into a storage takes place. Consequently, at the loading point a queue may built up.

The question arises under which condition the system is stable, i.e. that the queue at the loading point reaches a statistical equilibrium in the long-run. To answer this question let λ^* be the maximal attainable departure rate from the system. It will be intuitively clear that the system is stable if $\lambda < \lambda^*$, a condition that can be proved along the lines of Lavenberg [6].

To obtain λ^* imagine a saturated system in which there is an unbounded queue in front of the loading point. Since every hook arriving at the first station is loaded, the arrival process at this station is deterministic with interarrival time $\tau = T/r$. The arrival process at the second station is the overflow process at the first station, translated in time over the travel time between the two stations. Hence, we may well asume that this travel time is zero. Then it is readily seen from its definition, that the process $\{(x_1(n), x_2(n)), n \geqslant 1\}$ describes the states of the stations just before the arrival of a hook at the epochs $n\tau - 0$, $n = 1, 2, \ldots$.

Now the number of units unloaded from the hook arriving at epoch $n\tau - 0$ equals $\min(1, N+M-x_1(n) - x_2(n))$. Therefore, the expected number of unloaded units per hook equals

$$E\{\min(1, N+M-x_1(n)-x_2(n))\} = \Pr\{x_1(n)+x_2(n) \leqslant N+M-1\}$$
$$= 1 - \Pr\{x_1(n)+x_2(n) = N+M\}.$$

Consequently, under steady-state conditions, the expected number of unloaded units per unit of time in the saturated system equals $\{1-\pi(N, M)\}/\tau$, from which we conclude that $\lambda^* = \{1-\pi(N, M)\}/\tau$.

Hence the stability condition we are looking for becomes

$$\lambda\tau < 1 - \pi(N, M). \tag{23}$$

Table 1
Overflow probabilities $\pi(N, N)$ ($\mu = 1$)

τ	N					
	1	2	3	5	8	10
0.05	0.903	0.900	0.900	0.900	0.900	0.900
0.1	0.811	0.801	0.800	0.800	0.800	0.800
0.2	0.644	0.606	0.601	0.600	0.600	0.600
0.3	0.503	0.426	0.408	0.401	0.400	0.400
0.4	0.387	0.276	0.240	0.214	0.204	0.202
0.5	0.293	0.165	0.119	0.079	0.054	0.044
0.6	0.220	0.092	0.051	0.020	0.006	0.003
0.7	0.163	0.050	0.020	0.004	0.001	0.000
0.8	0.121	0.026	0.007	0.001	0.000	0.000
0.9	0.089	0.013	0.003	0.000	0.000	0.000
1.0	0.065	0.007	0.001	0.000	0.000	0.000

Table 1 gives some values of the overflow probability $\pi(N, M)$ for the case $N = M$ as a function of N and τ with $\mu_1 = \mu_2 = 1$.

Observe that $\pi(N, N) \simeq 1 - 2\tau$ for small τ. The table suggests, moreover, that for large but finite N $\pi(N, N) \to 1 - 2\tau$ if $\tau < 0.5$ and $\pi(N, N) \to 0$ if $\tau > 0.5$. The restriction on N to remain finite stems from the fact that our two-server model degenerates into the one server model D/M/1 when N is unbounded.

In view of this observation, noting that $\mu = 1$, we see that for small values of τ the stability condition is $\lambda < 2$, whereas for large values of τ we find $\lambda\tau < 1$. The former condition, which can be written as $\lambda/\mu = \rho < 2$, is the stability condition for the classical GI/G/2 queuing system and the latter stems from the GI/D/1 queue at the loading point. In the case of small τ only the servers determine the stability, while for large τ only the conveyor is the bottleneck, as could be expected.

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