COMMENT

On Bäcklund transformations and identities in bilinear form

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Abstract. For bilinear equations of the form $P(D)f \cdot f = 0$ we find all possibilities for rewriting $g^2 P(D)f \cdot f - f^2 P(D)g \cdot g = 0$ in the form $Q(D)f \cdot g = 0$. This is the first step in finding a Bäcklund transformation.

1. Introduction

In this comment we study Bäcklund transformations in bilinear form. This technique was introduced by Hirota [1, 2]. Let us start with a brief sketch of the method, and formulate the questions that we want to answer. We suppose that we are given an equation in bilinear form

$$P(D)f \cdot f = 0. \quad (1)$$

Here P is a polynomial in, say, n variables, i.e.

$$P(D_1, \dots, D_n) = \sum c_\alpha D^\alpha$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $D^\alpha = D_n^{\alpha_n} \dots D_1^{\alpha_1}$, where

$$D_n^{\alpha_n} \dots D_1^{\alpha_1} f \cdot g = \frac{\partial^{\alpha_n}}{\partial y_n^{\alpha_n}} \dots \frac{\partial^{\alpha_1}}{\partial y_1^{\alpha_1}} f(x_1 + y_1, \dots, x_n + y_n) \\ \times g(x_1 - y_1, \dots, x_n - y_n) \Big|_{y_1 = \dots = y_n = 0}.$$

To find a Bäcklund transformation, we apply the following trick: consider the equation

$$g^2 P(D)f \cdot f - f^2 P(D)g \cdot g = 0. \quad (2)$$

Then we note that for a solution f, g to (2), the following holds:

$$f \text{ is a solution of (1)} \Leftrightarrow g \text{ is a solution of (1)}.$$

Suppose that we could rewrite (2) in the following form:

$$Q(D)[Q_1(D)(f \cdot g) \cdot Q_2(D)(f \cdot g)] = 0 \quad (3)$$

Then by suitably splitting Q we derive two equations:

$$\begin{aligned} P_1(D)f \cdot g &= 0 \\ P_2(D)f \cdot g &= 0. \end{aligned} \tag{4}$$

The system (4) can be called a Bäcklund transformation for $P(D)f \cdot f = 0$. Namely, suppose g is given; then a solution f to equation (4) will also satisfy $P(D)f \cdot f = 0$.

The process of sensibly splitting Q is strongly equation-dependent; it seems unclear in general how to perform it. Two other general questions remain:

1. Is there for any P a solution Q ?
2. Is this solution unique? If not, can one find all possibilities?

In this comment we answer these questions. The answer to the first is yes; the proof is already essentially in Hirota [1, 2]. This solution is not at all unique. We give all possibilities in terms of a generating identity. The proof that these are all the possibilities is the most difficult part, and not fully included here. For complete proofs, see [3].

2. Algebraic background; partial solution

Let J denote the space of multi-indices $(i_1, \dots, i_n)$. In practice we will encounter $f^{(i)}$ and $g^{(j)}$ ($i, j \in J$), which will substitute for the partial derivatives. We will deal with the polynomial algebra $\mathcal{A} = \mathbb{R}[f^{(i)}, g^{(j)}]$, ($i, j \in J$). In this algebra, we have derivations ∂_r ($r = 1 \dots n$) which act in the obvious way. In particular

$$\partial_r(f^{(i)}) = f^{(i+1_r)} \text{ and } \partial_r(g^{(j)}) = g^{(j+1_r)}$$

where $1_r = (0, \dots, 0, 1, 0, \dots, 0)$, the r th basis vector.

Corresponding to ∂_r , we introduce the linear map $\mathcal{D}_r$ (the Hirota derivative) as a map $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$, by

$$\mathcal{D}_r(a \otimes b) = \partial_r(a) \otimes b - a \otimes \partial_r(b) \quad (r = 1, \dots, n)$$

The derivative ∂_r is also extended to $\mathcal{A} \otimes \mathcal{A}$ by

$$\partial_r(a \otimes b) = \partial_r(a) \otimes b + a \otimes \partial_r(b).$$

Obviously, we can define a Hirota derivative corresponding to any derivation of $\mathcal{A}$.

Lemma 1. Let $\mathcal{D} = \mathcal{D}_r$ and $\partial = \partial_r$ and $\mathcal{A}$ be as above. Then

$$\exp(\epsilon \mathcal{D})(a \otimes b) = \exp(\epsilon \partial)a \otimes \exp(-\epsilon \partial)b.$$

The equality is meant as formal power series, and follows by computing homogeneous terms with reference to ϵ^i , $i = 0, 1, 2, \dots$.

Note that our definition of $\mathcal{D}_r$ differs from the usual one in the following sense: the image is again in $\mathcal{A} \otimes \mathcal{A}$ and not in $\mathcal{A}$. This is a major difference, as we will see shortly. To get the usual Hirota derivatives D_r we have to project the image of $\mathcal{D}_r$ to $\mathcal{A}$: let $\pi : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$, $\pi(\sum a_i \otimes b_i) = \sum a_i b_i$, denote this projection. Then

$$D_r(a \otimes b) := \pi \mathcal{D}_r(a \otimes b)$$

and more generally

$$D_r^k(a \otimes b) := \pi \mathcal{D}_r^k(a \otimes b), \quad k = 1, 2, \dots$$

The exchange formula is central in this comment.

Theorem 1 (cf Hirota). Let $\partial_\alpha = \sum \alpha_i \partial_i$, $\partial_\beta = \sum \beta_i \partial_i$ and $\partial_\gamma = \sum \gamma_i \partial_i$ and let D_α etc. be the corresponding (projected) Hirota derivatives. Then we have for all $a, b, c, d \in \mathcal{A}$

$$\begin{aligned} \exp(D_\alpha)(\exp(D_\beta)(a \otimes b) \otimes \exp(D_\gamma)(c \otimes d)) \\ = \exp(\tfrac{1}{2}(D_\beta - D_\gamma))(\exp(\tfrac{1}{2}(D_\beta + D_\gamma) + D_\alpha)(a \otimes d) \\ \otimes \exp(\tfrac{1}{2}(D_\beta + D_\gamma) - D_\alpha)(c \otimes b)) \end{aligned} \quad (5)$$

Proof. The proof is based on lemma 1: expressing all the $\mathcal{D}$ in their corresponding ∂ and projecting on $\mathcal{A}$ gives the result immediately.

Note that this identity solves our first question by taking $a = b = f$ and $c = d = g$. Using the fact that $\exp(\sum \epsilon_i \mathcal{D}_i) = \prod \exp(\epsilon_i \mathcal{D}_i)$, we see that for three multi-indices $k, l, m \in J$, the coefficient of $\alpha^k \beta^l \gamma^m$ in (5) expresses

$$D^k(D^l(f \cdot f) \cdot D^m(g \cdot g))$$

in terms of

$$D^{\cdot\cdot}(D^{\cdot\cdot}(f \cdot g) \cdot D^{\cdot\cdot}(g \cdot f)).$$

Moreover we note that such an expression is not unique. If we take $a = c = f$ and $b = d = g$, then again comparing the coefficients of $\alpha^k \beta^l \gamma^m$ in (5), we see that $D^k(D^l(f \cdot g) \cdot D^m(f \cdot g))$ can be re-expressed. These observations solve a part of our problem; however, they do not solve the most difficult part. It is important to find *all* possibilities, hence all identities of the form

$$D^k(D^l(f \cdot g) \cdot D^m(f \cdot g)) = \sum D^{\cdot\cdot}(D^{\cdot\cdot}(f \cdot g) \cdot D^{\cdot\cdot}(f \cdot g)). \quad (6)$$

This problem will be solved in the next section. The answer is slightly surprising: equation (5) already contains all non-trivial identities!

3. Finding all identities

To study the identities of the form (6) more closely, we introduce two subspaces of $\mathcal{A} \otimes \mathcal{A}$. The first one, denoted by $\mathcal{B}$, is the linear space spanned by the elements $\{f^{(i)} \otimes g^{(j)}\}$, $(i, j \in J)$. Clearly these elements form a basis. Note that $\pi|_{\mathcal{B}}$ is injective. This allows us to view $\mathcal{B}$ as a subspace of $\mathcal{A}$. We introduce

$$e_\alpha^{(\beta)} := \partial^\beta \mathcal{D}^\alpha(f \otimes g)$$

which are again elements of $\mathcal{B}$. For these elements one can prove the following lemma.

Lemma 2. $\{e_\alpha^{(\beta)}\}$ is a basis for $\mathcal{B}$.

Proof. The proof (by induction on the number of independent variables n) is based on the observation that $\mathcal{D}^\alpha(f \otimes g) \notin \sum_{r=1}^n \text{Im}(\partial_r)$.

Identifying $\mathcal{B}$ with a subspace of $\mathcal{A}$, we can define $\mathcal{D}_r$ (and ∂_r) : $\mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$. In $\mathcal{B} \otimes \mathcal{B}$ we define the subspace $\mathcal{C}$, spanned by the elements

$$\mathcal{D}^k(\mathcal{D}^l(f \cdot g) \otimes \mathcal{D}^m(f \cdot g)) \quad (k, l, m \in J).$$

Thanks to lemma 2, these elements are linearly independent, i.e. they form a basis for $\mathcal{C}$. This shows that finding identities of the form (6) is equivalent to finding $\ker(\pi) \mid_{\mathcal{C}}$. Before turning to $\ker(\pi)$ we mention:

Lemma 3. Let $\mathcal{B}$ and $\mathcal{C}$ be as above. Then $\mathcal{B} \otimes \mathcal{B} = \mathcal{C} \oplus \text{Im}(\partial)$, where $\text{Im}(\partial) := \sum_{r=1}^n \text{Im}(\partial_r)$.

Expressed in normal derivatives, $\ker(\pi) \mid_{\mathcal{B} \otimes \mathcal{B}}$ is easily described.

Lemma 4. $\ker(\pi) \mid_{\mathcal{B} \otimes \mathcal{B}}$ is generated by

1. $f^{(i)}g^{(j)} \otimes f^{(k)}g^{(l)} - f^{(k)}g^{(l)} \otimes f^{(i)}g^{(j)}$
2. $f^{(i)}g^{(j)} \otimes f^{(k)}g^{(l)} - f^{(i)}g^{(l)} \otimes f^{(k)}g^{(j)}.$

The elements mentioned under 1 give rise to trivial identities, namely the identities

$$D^k(D^l(f \cdot g) \otimes D^m(f \cdot g)) = (-1)^{|k|} D^k(D^m(f \cdot g) \otimes D^l(f \cdot g)).$$

To analyse the elements under 2, we introduce d_r and $\bar{d}_r$: $\mathcal{B} \rightarrow \mathcal{B}$, defined by

$$d_r(f^{(i)}g^{(j)}) = f^{(i+1,r)}g^{(j)}, \quad \bar{d}_r(f^{(i)}g^{(j)}) = f^{(i)}g^{(j+1,r)}.$$

The elements $f^{(i)}g^{(j)} \otimes f^{(k)}g^{(l)} - f^{(i)}g^{(l)} \otimes f^{(k)}g^{(j)}$ are precisely the homogeneous terms in

$$\begin{aligned} & \exp\left(\sum_{r=1}^n (\alpha_r d_r + \beta_r \bar{d}_r)\right) f \cdot g \otimes \exp\left(\sum_{r=1}^n (\gamma_r d_r + \delta_r \bar{d}_r)\right) f \cdot g \\ & - \exp\left(\sum_{r=1}^n (\alpha_r d_r + \delta_r \bar{d}_r)\right) f \cdot g \otimes \exp\left(\sum_{r=1}^n (\gamma_r d_r + \beta_r \bar{d}_r)\right) f \cdot g. \end{aligned} \quad (7)$$

Using $\partial_r = d_r + \bar{d}_r$ and $\mathcal{D}_r = d_r - \bar{d}_r$, we can rewrite (7) in terms of ∂_r and $\mathcal{D}_r$: (7) turns into

$$\begin{aligned} & \exp\left(\frac{1}{4} \sum (\alpha_r + \beta_r + \gamma_r + \delta_r) \partial_r\right) \left\{ \exp\left(\frac{1}{4} \sum (\alpha_r + \beta_r - \gamma_r - \delta_r) \mathcal{D}_r\right) \right. \\ & \quad \left(\exp\left(\frac{1}{2} \sum (\alpha_r - \beta_r) \mathcal{D}_r\right) (f \cdot g) \otimes \exp\left(\frac{1}{2} \sum (\gamma_r - \delta_r) \mathcal{D}_r\right) (f \cdot g) \right) \\ & \quad - \exp\left(\frac{1}{4} \sum (\alpha_r + \delta_r - \gamma_r - \beta_r) \mathcal{D}_r\right) \left(\exp\left(\frac{1}{2} \sum (\alpha_r - \delta_r) \mathcal{D}_r\right) (f \cdot g) \right. \\ & \quad \left. \left. \otimes \exp\left(\frac{1}{2} \sum (\gamma_r - \beta_r) \mathcal{D}_r\right) (f \cdot g) \right) \right\}. \end{aligned} \quad (8)$$

Changing variables

$$\begin{aligned}\bar{\delta}_r &= \frac{1}{4}(\alpha_r + \beta_r + \gamma_r + \delta_r) & \bar{\alpha}_r &= \frac{1}{4}(\alpha_r + \beta_r - \gamma_r - \delta_r) \\ \bar{\beta}_r &= \frac{1}{2}(\alpha_r - \beta_r) & \bar{\gamma}_r &= \frac{1}{2}(\gamma_r - \delta_r)\end{aligned}\tag{9}$$

expression (8) transforms into

$$\begin{aligned}\exp\left(\sum \bar{\delta}_r \partial_r\right) & \left\{ \exp\left(\sum \bar{\alpha}_r \mathcal{D}_r\right) \left(\exp\left(\sum \bar{\beta}_r \mathcal{D}_r\right) (f \cdot g) \otimes \exp\left(\sum \bar{\gamma}_r \mathcal{D}_r\right) (f \cdot g) \right) \right. \\ & - \exp\left(\frac{1}{2} \sum (\bar{\beta}_r - \bar{\gamma}_r) \mathcal{D}_r\right) \left(\exp\left(\frac{1}{2} \sum (2\bar{\alpha}_r + \bar{\beta}_r + \bar{\gamma}_r) \mathcal{D}_r\right) (f \cdot g) \right. \\ & \left. \left. \otimes \exp\left(\frac{1}{2} \sum (-2\bar{\alpha}_r + \bar{\beta}_r + \bar{\gamma}_r) \mathcal{D}_r\right) (f \cdot g) \right) \right\}.\end{aligned}\tag{10}$$

Since (9) is an invertible transformation, the homogeneous terms in (10) span the same space as in (7). For $\ker(\pi) \mid_{\mathcal{C}}$ we only need to consider the coefficients in which $\bar{\delta}$ does not appear (see lemma 3). Hence we are left with the expression between braces, which is identical to theorem 1 for $a = c = f$ and $b = d = g$. So these (and the trivial ones) are all identities.

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References

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