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DUAL AND INVERSE FORMULATIONS OF CONSTRAINED EXTREMUM PROBLEMS

E. W. C. VAN GROESEN

Mathematical Institute
Catholic University
Toernooiveld, Nijmegen
The Netherlands

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1. INTRODUCTION

Let V be a reflexive Banach space and f and t two real-valued functionals defined on V. For $p \in t(V)$ [$t(V)$ denotes the range of the functional t] we consider the constrained minimization problem

$$\mathcal{P}_p : \inf_{u \in t^{-1}(p)} f(u) . \tag{1.1}$$

Here $t^{-1}(p)$ denotes the level set of t:

$$t^{-1}(p) = \{u \in V \mid t(u) = p\},$$

which is a nonempty subset of V for $p \in t(V)$. A solution $\hat{u}$ of (1.1) is an element from V which satisfies

$$t(\hat{u}) = p \quad \text{and} \quad f(\hat{u}) = h(p),$$

wherein h is the function defined by

$$h : t(V) \to \mathbf{R} : h(p) := \inf_{u \in t^{-1}(p)} f(u). \tag{1.2}$$

To get an idea of the main types of functionals we want to consider we list the conditions which will occasionally be assumed to hold

(f1) f is weakly lower semicontinuous on V, and coercive on V [i.e., $f(u) \to \infty$ if $\|u\|_V \to \infty$];

(t1) t is weakly continuous on V;

(f,t2) $f \in C^1(V,\mathbf{R})$, $t \in C^1(V, \mathbf{R})$.

If f and t satisfy these assumptions, then the existence of at least one solution u of $\mathcal{P}_p$

Editor's note: Abstract appears at the end of the article.

is guaranteed (cf. Proposition 2.1) and if $t'(u) \neq 0$, this solution satisfies the equation

$$f'(u) = \mu t'(u) \tag{1.3}$$

for some unique multiplier $\mu \in \mathbf{R}$ (cf. Proposition 2.3).

The operator equation (1.3) can be viewed at as a nonlinear eigenvalue problem in the dual space V^* (with μ as the eigenvalue) for the operators $f', t' : V \to V^*$. Our aim is to derive properties of those solutions of (1.3) which are also solutions of (1.1). In particular it is tempting to describe solution branches of (1.3) [i.e., connected sets of pairs $(u,\mu) \in V \times \mathbf{R}$ which satisfy (1.3)] with the aid of the parameter p as it enters in (1.1). In many applications a parametrization of such a branch with p : $\{(u(p),\mu(p)) \mid p \in t(V)\} \subset V \times \mathbf{R}$, is possible and may be particularly fruitful if a continuation of a solution branch described with the eigenvalue μ as parameter (as is usually done) is not possible (e.g., if there is a "bending of the solution branch": cf. Crandall & Rabinowitz [2] and Example 5.5. A study of such a continuation process requires both a global investigation and a more precise local description of the solution sets.

In this paper we shall deal with the global aspects of such a continuation. In a subsequent paper [5] we shall give a more detailed analysis of the local properties of such a global solution branch.

The general results to be derived are applicable to problems of semilinear elliptic type: Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and L a uniformly elliptic operator of the form

$$L = -\sum_{i,j=1}^{n} \partial_{x_i}[a_{ij}(x)\partial_{x_j}] + c(x), \tag{1.4}$$

where the coefficients of L are real, $a_{ij}(x) = a_{ji}(x)$ is twice continuously differentiable in $\bar{\Omega}$ for $i,j = 1,\ldots,n$, and $c(x)$ is nonnegative and once continuously differentiable in $\bar{\Omega}$. Then the nonlinear eigenvalue problem

$$\begin{aligned} Lu &= \mu t'(u) \quad \text{in } \Omega \\ u &= 0 \qquad\quad \text{on } \partial\Omega \end{aligned} \tag{1.5}$$

can be described as an operator equation as (1.3), with $V = \mathring{H}^1(\Omega) = \mathring{W}^{1,2}(\Omega)$ the usual Sobolev space, if f is defined to be the quadratic functional

$$f(u) = \frac{1}{2}\langle u, Lu\rangle = \frac{1}{2}\int_\Omega u(x)\,Lu(x)\mathrm{d}x. \tag{1.6}$$

Then f satisfies (f1,2), and in this case f is equivalent to the square of the norm of $\mathring{H}^1(\Omega)$:

$$\frac{1}{\gamma}\,\|u\|^2_{\mathring{H}^1} \leq f(u) \leq \gamma\,\|u\|^2_{\mathring{H}^1} \tag{1.7}$$

for some $\gamma > 0$.

Then, for the class of functionals t on $\mathring{H}^1(\Omega)$ which satisfy (t1,2), the general theory will enable us to describe a global solution-branch of the nonlinear eigenvalue problem (1.5).

To describe the contents of this paper: in Sec. 2 we state some preliminary results, especially a simple relation between the function h, given by (1.2), and the multiplier of a solution of problem $\mathcal{P}_p$ is derived, and some important consequences are noticed. In Sec. 3 we first investigate when a solution of $\mathcal{P}_p$ also gives an extreme value for the functional t on a level set of the functional f (the *inverse* extremum problem for $\mathcal{P}_p$).

Furthermore, we show how useful qualitative information about the behaviour of the function h can be obtained from a study of such an inverse extremum problem. In Sec. 4 we show how ideas from "convex analysis" (as can be found, e.g., in Rockafellar [9] and Ekeland and Temam [3]) can be applied to obtain a dual formulation for the constrained extremum problems $\mathscr{P}_p$. Therefore it is not necessary to require the functionals f and t to be convex: (local) convexity of the function h suffices to obtain (local) duality, and this on its turn implies that a solution of $\mathscr{P}_p$ with multiplier μ is in fact a (local) minimum of the functional f-μt on V. The results derived in Secs. 2, 3, and 4 enable us to describe a global solution branch of (1.3) with the parameter p. For a class of semilinear eigenvalue problems as described above, this will be shown in Sec. 5. There some well known results are derived in a new way, and especially for problems for which "bending" occurs, this description gives new insights into these delicate problems.

To conclude we emphasize that for the global result to be derived here, no nondegeneracy condition for the (constrained) extreme points are required, although conditions of this kind turn up in a more detailed local description of the continuation process.

2. PRELIMINARIES

The first results deal with the solution set of problem $\mathscr{P}_p$.

Proposition 2.1. Assume that the functionals f and t satisfy conditions $(f,t1)$. Then for every $p \in t(V)$, $\mathscr{P}_p$ has at least one solution.

Proof. Because of condition $(f1)$ the functional f is bounded from below on all of V, and hence certainly on $t^{-1}(p)$. Put

$$\alpha := \inf_{u\in t^{-1}(p)} f(u),$$

and let $\{u_n\}$ be a minimizing sequence: $f(u_n) \downarrow \alpha$ for $n \to \infty$, $t(u_n) = p$ for all $n \in \mathbf{N}$. As f is coercive this minimizing sequence is uniformly bounded in V and hence has a weakly convergent subsequence, say $u_{n'} \rightharpoonup \hat{u}$ in V. Because of condition $(t1)$

$$: t(\hat{u}) = \lim_{n\to\infty} t(u_n) = p,$$

hence $\hat{u} \in t^{-1}(p)$. [The level set $t^{-1}(p)$ is weakly closed.] As f is weakly lower semicontinuous,

$$f(\hat{u}) \leq \liminf_{n\to\infty} f(u_{n'}) = \alpha.$$

By definition of α we also have $\alpha \leq f(\hat{u})$, hence $f(\hat{u}) = \alpha$. This shows that $\hat{u}$ is a solution of $\mathscr{P}_p$.

For simplicity denote the solution set of $\mathscr{P}_p$ by P_p:

$$P_p := \{u \in V | u \text{ is a solution of } \mathscr{P}_p\}.$$

Proposition 2.2. Assume conditions $(f,t1)$ to hold. Then P_p is a weakly compact subset of V. Moreover, if f satisfies the extra condition

$(f3)$ for every sequence u_n for which $u_n \rightharpoonup \hat{u}$ (weakly) in V and $f(u_n) \to f(\hat{u})$, it follows that $u_n \to \hat{u}$ (strongly) in V,

then P_p is a compact subset of V.

Proof. As f is coercive on V, P_p is a bounded subset of V. Let $\{u_n\}$ be any sequence from P_p; then $t(u_n) = p$ and $f(u_n) = h(p)$ for all $n \in \mathbf{N}$, where $h(p)$ is defined in (1.2). As $\{u_n\}$ is bounded it contains a weakly convergent subsequence, say $u_{n'} \rightharpoonup \hat{u}$ in V. As in the proof of Proposition 2.1 it easily follows that $t(\hat{u}) = p$ and $f(\hat{u}) = h(p)$, i.e., that $\hat{u} \in P_p$. Hence P_p is weakly closed. If f satisfies the extra condition (f3), the subsequence $u_{n'}$ converges strongly to $\hat{u}$, which implies that P_p is compact in this case.

We now recall the Euler equation which must be satisfied by a solution of $\mathcal{P}_p$. This result, a generalization of Lagranges multiplier rule to infinite dimensions, is originally due to Lusternik [7]. See, e.g., Vainberg [11, Theorem 9.11] as a convenient reference.

Proposition 2.3. Assume f and t satisfy condition (f,t2). Then, if $u \in P_p$ with $t'(u) \neq 0$, there exists a unique multiplier $\mu \in \mathbf{R}$ such that u satisfies

$$f'(u) = \mu t'(u). \tag{2.1}$$

The next lemma relates the multiplier μ of a solution of $\mathcal{P}_p$ to the function h defined in (1.2). Note that, if f satisfies (f1), f is bounded from below on V, and hence h is bounded from below on $t(V)$. Moreover, the range of the functional t, $t(V) \subset \mathbf{R}$, is a connected interval if, e.g., t is continuous. Let $\mathring{t}(V)$ denote the interior of this range. Assume for the following conditions (f,t2) to hold.

Lemma 2.4. For $p \in \mathring{t}(V)$, let u be a solution of $\mathcal{P}_p$ with $t'(u) \neq 0$ and with multiplier μ. Then we have

$$h'_+(p) \leq \mu \leq h'_-(p), \tag{2.2}$$

where $h'_+(p)$ and $h'_-(p)$ denote the right- and left-hand-side derivative of the function h, respectively.

Proof. As $t'(u) \neq 0$ it is possible to take $v \in V$ such that $\langle t'(u), v\rangle = 1$. With $f'(u) = \mu t'(u)$ it follows that $\langle f'(u), v\rangle = \mu$. Furthermore, the function $\epsilon : \mathbf{R} \to \mathbf{R}$ defined by

$$t(u + \alpha v) = p + \epsilon(\alpha)$$

is continuously differentiable and satisfies

$$\epsilon(0) = 0, \frac{d\epsilon}{d\alpha}(0) = 1.$$

By definition of the function h we have for $\alpha \in \mathbf{R}$

$$h(p + \epsilon(\alpha)) \leq f(u + \alpha v).$$

Now, consider the expression

$$\frac{1}{\epsilon(\alpha)}\{h(p + \epsilon(\alpha)) - h(p)\}.$$

If $\alpha \downarrow 0$, we have $\epsilon(\alpha) \downarrow 0$ and

$$h'_+(p) \leftarrow \frac{h(p + \epsilon(\alpha)) - h(p)}{\epsilon(\alpha)} \leq \frac{f(u + \alpha v) - f(u)}{t(u + \alpha v) - t(u)} \to \frac{\langle f'(u), v\rangle}{\langle t'(u), v\rangle} = \mu.$$

In the same way, for $\alpha \uparrow 0$, it is found that $h'_-(p) \geq \mu$, which proves the lemma.

The foregoing lemma has two immediate corollaries which turn out to be useful.

Corollary 2.5. If h is differentiable at p, then *all* solutions of $\mathscr{P}_p$ have the same multiplier, which may therefore be denoted by $\mu(p)$, and which is given by

$$\mu(p) = h'(p). \tag{2.3}$$

Corollary 2.6. Suppose that h is locally convex at $p \in \mathring{t}(V)$. Then h is differentiable at $p : h'_+(p) \equiv h'(p)$, and hence the conclusion of Corollary 2.5 holds.

3. INVERSE EXTREMUM PROBLEMS

Related to the extremum problem $\mathscr{P}_p$ we consider as "inverse extremum problems" the two families of constrained extremum problems

$$\mathscr{S}_r : \sup_{u \in f^{-1}(r)} t(u) \tag{3.1}$$

$$\mathscr{Q}_r : \inf_{u \in f^{-1}(r)} t(u) \tag{3.2}$$

for $r \in f(V)$. Corresponding to $\mathscr{S}_r$ and $\mathscr{Q}_r$ we define solution sets S_r and Q_r and functions s and q in the same way as was done for problem $\mathscr{P}_p$:

$$S_r := \{u \in V | u \text{ is a solution of } \mathscr{S}_r\}$$

$$Q_r = \{u \in V | u \text{ is a solution of } \mathscr{Q}_r\},$$

$$s(r) := \sup_{u \in f^{-1}(r)} t(u) \tag{3.3}$$

$$\text{for } r \in f(V).$$

$$q(r) := \inf_{u \in f^{-1}(r)} t(u) \tag{3.4}$$

In this section we study the relation between problem $\mathscr{P}_p$ and these inverse extremum problems. The results of this section will be important for the rest of this paper: it will be shown how, for functionals which satisfy $(f, t1)$, qualitative behaviour of the function h, defined in (1.2), can be obtained from the functions s and q defined above (which are simpler to investigate as shall be shown).

To start, we investigate when a solution of $\mathscr{P}_p$ is also a solution of $\mathscr{S}_r$ or $\mathscr{Q}_r$ for $r = h(p)$. In general this will not be the case: a solution u of $\mathscr{P}_p$ will usually only give *locally* (i.e., in a sufficiently small neighbourhood of u) an extreme value for the constrained functional t on the level set $f^{-1}(h(p))$. The first lemma gives the usual relation between the functions h, s, and q.

Lemma 3.1. Suppose $P_p \neq \varnothing$. Then we have

$$(a) \quad s(h(p)) \geq p. \tag{3.5}$$

$$(b) \quad q(h(p)) \leq p. \tag{3.6}$$

Proof. Let $\bar{u} \in P_p$. Then $f(\bar{u}) = h(p)$ and $t(\bar{u}) = $ p. Hence $\bar{u} \in f^{-1}(h(p))$ and by definition of the functions s and q

$$s(h(p)) \geq t(\bar{u}) = p \text{ and } q(h(p)) \leq t(\bar{u}) = p,$$

which proves the lemma.

The following result gives in principle the complete answer to the question formulated above.

Proposition 3.2. Suppose $P_p \neq \emptyset$. Then we have

$$\text{(a)} \quad \text{if } s(h(p)) = p, \text{ then } P_p = S_{h(p)}$$

$$\text{(b)} \quad \text{if } q(h(p)) = p, \text{ then } P_p = Q_{h(p)}.$$

Proof. We shall prove (a). Then (b) can be obtained analogously. Let $\bar{u} \in P_p$: then $f(\bar{u}) = h(p)$ and $t(\bar{u}) = p$; thus, as $\bar{u} \in f^{-1}(h(p))$ and $s(h(p)) = p = t(\bar{u})$, $\bar{u}$ is clearly a solution of $\mathcal{S}_{h(p)}$. On the other hand, if $\hat{u} \in S_{h(p)}$, then $s(h(p)) = t(\hat{u})$ and $h(p) = f(\hat{u})$, and thus, if $s(h(p)) = p$, $\hat{u}$ satisfies $t(\hat{u}) = p$ and $f(\hat{u}) = h(p)$ which shows $\hat{u} \in P_p$.

The next lemma gives a criterion to decide whether the conditions of the foregoing proposition are satisfied.

Lemma 3.3. Suppose $P_p \neq \emptyset$. Then,

$$\text{(a)} \quad \text{if } h(\zeta) > h(p) \text{ for } \zeta \in (p,\infty) \cap t(V), \text{ then } s(h(p)) = p;$$

$$\text{(b)} \quad \text{if } h(\zeta) > h(p) \text{ for } \zeta \in (-\infty,p) \cap t(V), \text{ then } q(h(p)) = p.$$

Proof. Again we shall prove only (a). Suppose $s(h(p)) = p + \alpha$ for some $\alpha > 0$ [because of (3.5) we need not to investigate the possibility $\alpha < 0$]. Then

$$\sup_{u \in f^{-1}(h(p))} t(u) = p + \alpha,$$

which implies that there exists an element $\hat{u} \in V$ and a number $\zeta \in \mathbf{R}$ with $p < \zeta < p + \alpha$ such that $t(\hat{u}) = \zeta$ and $f(\hat{u}) = h(p)$. From this it follows that

$$h(\zeta) := \inf_{u \in t^{-1}(\zeta)} f(u) \leq f(\hat{u}) = h(p).$$

Hence $h(\zeta) \leq h(p)$ for $\zeta > p$ if $s(h(p)) = p + \alpha$ for some $\alpha > 0$. This proves statement (a).

If we let $\partial h(p)$ denote the subdifferential of the function h at p (cf. also Sec. 4) we get a special case of the foregoing lemma.

Corollary 3.4. Suppose $P_p \neq \emptyset$. Then

$$\text{(a)} \quad \text{if } \partial h(p) \cap \mathbf{R}^+ \neq \emptyset, \text{ then } s(h(p)) = p;$$

$$\text{(b)} \quad \text{if } \partial h(p) \cap \mathbf{R}^- \neq \emptyset, \text{ then } q(h(p)) = p.$$

The foregoing results imply

Proposition 3.5.

(a) Suppose there exists $p_+ \in \mathbf{R}$ such that [writing $J_+ := (p_+,\infty) \cap t(V)$]

(i) $P_p \neq \emptyset$ for $p \in J_+$;

(ii) h is monotonically increasing on J_+.

Then $s(h(p)) = p$ for all $p \in J_+$; in other words: the function $s : \{h(p|p \in J_+\} \to \mathbf{R}$ is the inverse of the function $h : J_+ \to \mathbf{R}$.

(b) Suppose there exists p_- such that [writing $J_- := (-\infty,p_-) \cap t(V)$]

(i) $\mathcal{P}_p \neq \emptyset$ for $p \in J_-$;

(ii) h is monotonically decreasing for $p \in J_-$.

Then $q(h(p)) = p$ for all $p \in J_-$: the function $q : \{h(p)|p \in J_-\} \to \mathbf{R}$ is the inverse of the function $h : J_- \to \mathbf{R}$. Because of these results it will be clear why we have called problems $\mathcal{S}_r$ and $\mathcal{Q}_r$ *inverse* extremum problems of $\mathcal{P}_p$.

Now suppose that the functionals f and t satisfy condition $(f,t1)$. Then $P_p \neq \varnothing$ for $p \in t(V)$ (Proposition 2.1), and the foregoing results show how from qualitative behaviour of the function h we can deduce results concerning the coincidence of the solution sets P_p and S_r or Q_r and concerning the functions s and q for suitable values of r.

However, this qualitative behaviour of the function h is usually difficult to obtain directly from a study of the constrained extremum problem $\mathcal{P}_p$. It turns out, as we shall see below, that it is much simpler to obtain such qualitative information for the functions s and q on $f(V)$. Assuming this to be the case for the moment, we can "construct" the function h [at least on a subset of its domain $t(V)$] from the functions s and q in much the same way as was described above for the reversed problem. For convenience we shall list the main results.

Lemma 3.6.
(a) If $S_r \neq \varnothing$, then $h(s(r)) \leq r$.
(b) If $Q_r \neq \varnothing$, then $h(q(r)) \geq r$.

Proposition 3.7.
(a) If $S_r \neq \varnothing$ and $h(s(r)) = r$, then $P_{s(r)} = S_r$.
(b) If $Q_r \neq \varnothing$ and $h(q(r)) = r$, then $P_{q(r)} = Q_r$.

Lemma 3.8.
(a) If $S_r \neq \varnothing$ and if $s(\rho) > s(r)$ for $\rho \in (r,\infty) \cap f(V)$, then $h(s(r)) = r$.
(b) If $Q_r \neq \varnothing$ an if $q(\rho) > q(r)$ for $\rho \in (r,\infty) \cap f(V)$ then $h(q(r)) = r$.

Proposition 3.9.
(a) Suppose there is $\hat{r} \in \mathbf{R}$ such that [writing $J := (\hat{r},\infty) \cap f(V)$]
 (i) $S_r \neq \varnothing$ for $r \in J$;
 (ii) s is monotonically increasing for $r \in J$.
 Then $h(s(r)) = r$ for $r \in J$: the function $h : \{s(r|r \in J\} \to \mathbf{R}$ is the inverse of the function $s : J \to \mathbf{R}$.
(b) Suppose there is $\hat{r} \in \mathbf{R}$ such that [writing $J := (\hat{r},\infty) \cap f(V)$]
 (i) $Q_r \neq \varnothing$ for $r \in J$;
 (ii) q is monotonically decreasing on J.
 Then $h(q(r)) = r$ for $r \in J$: the function $h : \{q(r)|r \in J\} \to \mathbf{R}$ is the inverse of the function $q : J \to \mathbf{R}$.

For the applicability of these last results it is necessary to study the existence of solutions of $\mathcal{S}_r$ and $\mathcal{Q}_r$ and to obtain qualitative information of the functions s and q for $r \in f(V)$. To that end we consider the extremum problems

$$\overline{\mathcal{S}}_r : \sup_{u \in \bar{B}_r} t(u)$$

$$\overline{\mathcal{Q}}_r : \inf_{u \in \bar{B}_r} t(u)$$

where

$$\bar{B}_r := \{u \in V | f(u) \leq r\}.$$

The idea is that the level set $f^{-1}(r)$ is the boundary of the set $\bar{B}_r$, such that if it is known that t attains its maximum (or minimum) on $\bar{B}_r$ at a point which is not in the interior of $\bar{B}_r$, then $\mathcal{S}_r$ (or $\mathcal{Q}_r$) has a solution.

To make any progress in this direction we assume that f and t satisfy $(f,t1)$. Because of $(f1)$, f is bounded from below on V and attains its minimum, so that it is no restriction to assume that f satisfies

$$(f1^*) \qquad f(0) = 0, f(u) \geq 0 \qquad \forall u \in V.$$

Then we have the following standard result (see, e.g., Vainberg [11, Theorem 9.2], Berger [1, Theorem (6.1.1)]).

Proposition 3.10. Suppose f and t satisfy $(f,t1)$ and $(f1^*)$. Then, for every $r > 0$, the set $\overline{B}_r$ is bounded and weakly closed in V. Furthermore, t is bounded from above and from below on $\overline{B}_r$ and attains its maximum and minimum value at points of $\overline{B}_r$.

Of course it is possible that both the maximum and minimum value of t are attained at interior points of $\overline{B}_r$. Then these points are solutions of $t'(u) = 0$, and S_r and Q_r may be empty. But if it is known that t has at most one stationary point, at least one of the two extremal elements lies on the boundary of $\overline{B}_r$. In this way we obtain the following results (mononicity is a simple consequence of the fact that $\overline{B}_r \subset \overline{B}_{r_0}$ for $0 < r < r_0$).

Proposition 3.11. Suppose f and t satisfy $(f,t1,2)$ and $(f1^*)$.

(a) If $t'(u) \neq 0 \; \forall u \in V$, then $S_r \neq \emptyset$ and $Q_r \neq \emptyset$ for every $r \in \mathbf{R}^+$. Moreover, the functions $s : \mathbf{R}^+ \to \mathbf{R}$ and $q : \mathbf{R}^+ \to \mathbf{R}$ are monotonically increasing and monotonically decreasing, respectively.

(b) Suppose that t satisfies
  (i) $t(0) = 0$
  (ii) $t'(u) = 0 \Leftrightarrow u = 0$
  (iii) t takes positive values at every neighbourhood of $u = 0$. Then $S_r \neq \emptyset$ for every $r \in \mathbf{R}^+$ and $s : \mathbf{R}^+ \to \mathbf{R}$ is monotonically increasing.
  Moreover, if for some $\bar{r} > 0$:
  (iv) t has negative values at $f^{-1}(\bar{r})$, then we also have $Q_r \neq \emptyset$ for $r \geq \bar{r}$ and $q : \{r | r \geq \bar{r}\} \to \mathbf{R}$ is monotonically decreasing.

4. DUALITY

For what follows it is convenient to define the function h on all of $\mathbf{R}$ as a function into $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\} \cup \{-\infty\}$:

$$h : \mathbf{R} \to \overline{\mathbf{R}}, \; h(p) := \begin{cases} \inf\limits_{u \in t^{-1}(p)} f(u) & \text{if } p \in t(V) \\ \infty & \text{else} \end{cases} \tag{4.1}$$

Lemma 4.1. An equivalent formulation of problem $\mathcal{P}_p$ is

$$\mathcal{P}_p : \inf_{u \in V} \sup_{\mu \in R} \{f(u) - \mu(t(u) - p)\}. \tag{4.2}$$

Proof. Immediate from

$$\sup_{\mu \in R} \{f(u) - \mu(t(u) - p)\} = \begin{cases} f(u) \text{ if } t(u) = p \\ \infty \quad \text{if } t(u) \neq p \end{cases}$$

Now we define a dual variational problem for $\mathcal{P}_p$ and a local version thereof.

Definition 4.2. The *dual problem* $\mathcal{P}_p^*$ of $\mathcal{P}_p$ is defined to be the extremum problem

$$\mathcal{P}_p^* : \sup_{\mu \in R} \inf_{u \in V} \{f(u) - \mu(t(u) - p)\}, \tag{4.3}$$

and the *ε-local dual problem* $\text{loc}_\epsilon \mathcal{P}_p^*$ of $\mathcal{P}_p$ is defined for $\epsilon > 0$ as

$$\text{loc}_\epsilon \mathcal{P}_p^* : \sup_{\mu \in \mathbf{R}} \inf_{u \in t^{-1}(p-\epsilon, p+\epsilon)} \{f(u) - \mu(t(u) - p)\}. \tag{4.4}$$

Any number $\mu \in \mathbf{R}$ for which the supremum is achieved in (4.3) [or (4.4)] will be called a solution of $\mathcal{P}_p^*$ (of $\text{loc}_\epsilon \mathcal{P}_p^*$, respectively). The solution sets of $\mathcal{P}_p^*$ and $\text{loc}_\epsilon \mathcal{P}_p^*$ will be denoted by P_p^* and $\text{loc}_\epsilon P_p^*$, respectively.

Remark 4.3. Note that $\mathcal{P}_p$ and $\mathcal{P}_p^*$ can be considered to be limiting cases of $\text{loc}_\epsilon \mathcal{P}_p^*$

$$\mathcal{P}_p = \text{loc}_0 \mathcal{P}_p^*$$

$$\mathcal{P}_p^* = \text{loc}_\infty \mathcal{P}_p^*.$$

Let $h^* : \mathbf{R} \to \overline{\mathbf{R}}$ and $h^{**} : \mathbf{R} \to \overline{\mathbf{R}}$ denote the dual (= conjugate) and the bidual function of h. Then we have

Lemma 4.4

$$(i)\ h^*(\mu) = -\inf_{u \in V} \{f(u) - \mu t(u)\} \text{ for } \mu \in \mathbf{R} \tag{4.5}$$

$$(ii)\ h^{**}(p) = \sup_{\mu \in \mathbf{R}} \inf_{u \in V} \{f(u) - \mu(t(u) - p)\} \text{ for } p \in \mathbf{R}. \tag{4.6}$$

Because of Remark 4.3 this lemma is a special case (for $\epsilon = \infty$) of the following result which gives (for $\epsilon = \infty$) analogous statements for the local dual problem. Therefore define the function

$$h_{\epsilon,p} : \mathbf{R} \to \overline{\mathbf{R}} \text{ for } \epsilon > 0,\ p \in \mathbf{R} : h_{\epsilon,p}(q) := \begin{cases} h(q) \text{ if } q \in (p-\epsilon, p+\epsilon) \\ \infty \quad \text{else} \end{cases} \tag{4.7}$$

Lemma 4.5.

$$(i)\ h_{\epsilon,p}^*(\mu) = -\inf_{u \in t^{-1}(p-\epsilon,p+\epsilon)} \{f(u) - \mu t(u)\} \text{ for } \mu \in \mathbf{R} \tag{4.8}$$

$$(ii)\ h_{\epsilon,p}^{**}(p) = \sup_{\mu \in \mathbf{R}} \inf_{u \in t^{-1}(p-\epsilon,p+\epsilon)} \{f(u) - \mu(t(u) - p\} \text{ for } p \in \mathbf{R}, \tag{4.9}$$

where the infimum of a functional taken over an empty set is defined to be $+\infty$.

Proof.

(i) Substituting (4.7) and (4.1) in the definition of dual function, we obtain for $\mu \in \mathbf{R}$

$$h_{\epsilon,p}^*(\mu) = \sup_{q \in \mathbf{R}} \{\mu q - h_{\epsilon,p}(q)\} = \sup_{q \in (p-\epsilon,p+\epsilon)} \{\mu q - h(p)\}$$

$$= \sup_{q \in (p-\epsilon,p+\epsilon)} \{\mu q - \inf_{u \in t^{-1}(q)} f(u)\}$$

$$= -\inf_{q \in (p-\epsilon,p+\epsilon)} \inf_{u \in t^{-1}(q)} \{f(u) - \mu t(u)\}$$

$$= -\inf_{u \in t^{-1}(p-\epsilon,p+\epsilon)} \{f(u) - \mu t(u)\}.$$

(ii) Using $h^{**}_{\epsilon,p}(q) := \sup_{\mu \in \mathbf{R}} \{\mu q - h^*_{\epsilon,p}(\mu)\}$ it follows with (i):

$$h^{**}_{\epsilon,p}(q) = \sup_{\mu \in \mathbf{R}} \inf_{u \in t^{-1}(p-\epsilon,p+\epsilon)} \{f(u) - \mu(t(u) - q)\} \ \forall q \in \mathbf{R}.$$

For $q = p$ this is the desired result.

Lemma 4.6. For arbitrary $\epsilon > 0$, $p \in \mathbf{R}$ we have

$$h^{**}(p) \leq h^{**}_\epsilon(p) \leq h(p). \tag{4.10}$$

Proof. As $t^{-1}(p-\epsilon,p+\epsilon) \subset V$, it follows from (4.6) and (4.9) that $h^{**}(p) \leq h^{**}_{\epsilon,p}(p)$. Furthermore, as for arbitrary $f : \mathbf{R} \to \overline{\mathbf{R}}$ the inequality $f^{**}(x) \leq f(x)$ $\forall x \in \mathbf{R}$ holds, we have $h^{**}_{\epsilon,p}(q) \leq h_{\epsilon,p}(q)$ $\forall q \in \mathbf{R}$ and thus the second inequality because $h_{\epsilon,p}(p) = h(p)$ for every $p \in \mathbf{R}$.

We now come to the definition of (local) stability of problem $\mathcal{P}_p$. The importance of this notion will become clear in the following.

Definition 4.7. Problem $\mathcal{P}_p$ is said to be

(a) *stable* if (i) $P_p \neq \varnothing$
(ii) $P^*_p \neq \varnothing$
(iii) $h^{**}(p) = h(p)$;

(b) *locally stable* if (i) $P_p \neq \varnothing$
and for some $\epsilon > 0$: (ii) $\text{loc}_\epsilon P^*_p \neq \varnothing$
(iii) $h^{**}_{\epsilon,p}(p)$.

Proposition 4.8. Suppose $\mathcal{P}_p$ is locally stable. Let $\bar{u} \in P_p$ and $\bar{\mu} \in \text{loc}_\epsilon P^*_p$. Then $\bar{u} \in t^{-1}(p)$ is a solution of the extremum problem

$$\mathcal{H}^\epsilon_{\bar{\mu}} : \inf_{u \in t^{-1}(p-\epsilon,p+\epsilon)} \{f(u) - \bar{\mu} t(u)\}. \tag{4.11}$$

Consequently, if t is continuous, then $\bar{u}$ is a local minimal point of the functional $f - \mu t$ on all of V, and if $(f,t2)$ holds, then $(\bar{u},\bar{\mu}) \in V \times \mathbf{R}$ satisfies

$$f'(\bar{u}) = \bar{\mu} t'(u)$$
$$t(\bar{u}) = p.$$

Proof. For $\bar{u} \in P_p$ we have $f(\bar{u}) = h(p)$ and $t(\bar{u}) = p$, thus also

$$h(p) = f(\bar{u}) - \bar{\mu}(t(\bar{u}) - p).$$

For $\bar{\mu} \in \text{loc}_\epsilon P^*_p$ it follows from (4.9) that

$$h^{**}_{\epsilon,p}(p) = \inf_{u \in t^{-1}(p-\epsilon,p+\epsilon)} \{f(u) - \bar{\mu}(t(u) - p)\}.$$

As $h^{**}_{\epsilon,p}(\text{p}) = h(p)$ and $t(\bar{u}) = p$, it follows from these results that $\bar{u}$ is a solution of problem $\mathcal{H}^\epsilon_{\bar{\mu}}$. Furthermore, if t is continuous, $t^{-1}(p-\epsilon,p+\epsilon)$ is a neighbourhood of the level set $t^{-1}(p)$; in particular, for $\bar{u} \in t^{-1}(p)$ there exists $\delta > 0$ such that $B(\bar{u};\delta) = \{u \in V | \; \| u - \bar{u} \| < \delta\}$ is contained in $t^{-1}(p-\epsilon, p+\epsilon)$. Consequently, the functional $f - \bar{\mu}t$ achieves its infimum on $B(\bar{u},\delta)$ in the interior point $\bar{u}$, which is then a stationary point of this functional and thus, if $(f,t2)$ holds, we have $f'(\bar{u}) = \bar{\mu}t'(\bar{u})$.

Taking $\epsilon = \infty$ in Proposition 4.8 there results

Corollary 4.9. Suppose $\mathcal{P}_p$ is stable. Let $\bar{u} \in P_p$ and $\bar{\mu} \in P_p^*$. Then $\bar{u} \in t^{-1}(p)$ is a solution of the unconstrained extremum problem

$$\mathcal{H}_{\bar{\mu}} : \inf_{u \in V} \{f(u) - \bar{\mu}t(u)\}, \tag{4.12}$$

i.e., $\bar{u}$ is a global minimal point of $f - \bar{\mu}t$ on all of V.

Some other simple consequences of Proposition 4.8 can be stated:

Corollary 4.10. Suppose $\mathcal{P}_p$ is locally stable and f and t satisfy $(f,t2)$. Then we have

(i) If $\text{loc}_\epsilon \mathcal{P}_p^*$ has more than one solution, then for every $u \in P_p$ we have

$$f'(u) = 0 \text{ and } t'(u) = 0.$$

(ii) If $\bar{\mu} \in \text{loc}_\epsilon P_p^*$ is the only solution of $\text{loc}_\epsilon \mathcal{P}_p^*$, then for every $u \in P_p$ we have $t'(u) \neq 0$, and all solutions of $\mathcal{P}_p$ have the same multiplier $\bar{\mu}$.

(iii) If $f'(u) \neq 0$ for $u \in t^{-1}(p)$, or $t'(u) \neq 0$ for $u \in t^{-1}(p)$, then $\text{loc}_\epsilon \mathcal{P}_p^*$ has a unique solution.

With the notion of subdifferentiability it is possible to give an equivalent definition of stability of problem $\mathcal{P}_p$. To obtain the same results for local stability we introduce the concept of (ϵ−) local subdifferentiability in the following way: the function $h : \mathbf{R} \to \overline{\mathbf{R}}$ is said to be ϵ-locally subdifferentiable at p if $\partial_\epsilon h(p) \neq \emptyset$, where the ϵ-local subdifferential $\partial_\epsilon h(p)$ is defined with the aid of the function $h_{\epsilon,p} : \mathbf{R} \to \overline{\mathbf{R}}$ introduced in (4.7) as

$$\partial_\epsilon h(p) := \partial h_{\epsilon,p}(p). \tag{4.13}$$

Noticing that $\partial_\epsilon h(p) \supset \partial_{\epsilon_0} h(p)$ for every $0 < \epsilon < \epsilon_0$, we can define

$$\partial_{\text{loc}} h(p) := \bigcup_{\epsilon > 0} \partial_\epsilon h(p) \tag{4.14}$$

and call h locally subdifferentiable at p if its local subdifferential $\partial_{\text{loc}} h(p)$ is nonempty: $\partial_{\text{loc}} h(p) \neq \emptyset$. It is a simple matter to verify that h is continuous and locally subdifferentiable at p if and only if f is locally convex at p.

Lemma 4.11.

(i) $\mu \in P_p^* \Leftrightarrow \mu \in \partial h^{**}(p)$

(ii) $\mu \in \text{loc}_\epsilon P_p^* \Leftrightarrow \mu \in \partial h_{\epsilon,p}^{**}(p)$.

Proof. Writing h^{***} for the dual function of h^{**} we have by definition of subdifferential:

$$\mu \in \partial h^{**}(p) \Leftrightarrow h^{**}(p) + h^{***}(\mu) = \mu p.$$

Using the well known fact that $h^{***} \equiv h^*$ we have

$$\mu \in \partial h^{**}(p) \Leftrightarrow h^{**}(p) + h^*(\mu) = \mu p$$

from which statement (i) follows with the aid of Lemma 4.4. In the same way, using Lemma 4.5, (ii) is proved.

Proposition 4.12. Problem $\mathscr{P}_p$ is stable (resp. locally stable) if and only if

(i) $P_p \neq \emptyset$

(ii) h is subdifferentiable at p: $\partial h(p) \neq \emptyset$

[h is locally subdifferentiable at p: $\partial_{\text{loc}} h(p) \neq \emptyset$, respectively.]

Proof. Suppose $\mathscr{P}_p$ is locally stable. Then (i) $P_p \neq \emptyset$. Furthermore for some $\epsilon > 0$, $h^{**}_{\epsilon,p}(p) = h(p)$, which implies $\partial h^{**}_{\epsilon,p}(p) = \partial h(p)$, and as $\text{loc}_\epsilon P^*_p \neq \emptyset$, it follows from Lemma 4.11 that $\partial h^{**}_{\epsilon,p} \neq \emptyset$. Hence (ii) $\partial_\epsilon h(p) = \partial h_{\epsilon,p}(p) \neq \emptyset$.

On the other hand, if $\partial_\epsilon h(p) \neq \emptyset$, then $h_{\epsilon,p}(p) = h^{**}_{\epsilon,p}(p)$ and then $\partial h_{\epsilon,p}(p) = \partial h^{**}_{\epsilon,p}(p) \neq 0$. Hence $h(p) = h^{**}_\epsilon(p)$ and, with Lemma 4.11, $\text{loc}_\epsilon P^*_p \neq \emptyset$. Together with $P_p \neq \emptyset$ this means that $\mathscr{P}_p$ is locally stable. In the same way, or taking $\epsilon = \infty$, the equivalence of the statements for the stability case is proved.

Corollary 4.13. Let $\mathscr{P}_p$ be locally stable and suppose that $t'(u) \neq 0$ or $f'(u) \neq 0$ for every $u \in t^{-1}(p)$. Then h is differentiable at p with

$$\left\{ \frac{dh}{dp}(p) \right\} = \partial_{\text{loc}} h(p). \tag{4.15}$$

Proof. From Corollary 4.10 (iii) it follows that $\text{loc}_\epsilon \mathscr{P}_p$ has a unique solution, say $\bar{\mu}$. Then, according to Lemma 4.11, $\{\mu\} = \partial h^{**}_{\epsilon,p}(p)$, and then $\{\mu\} = \partial_{\text{loc}} h(p)$, which implies that h is differentiable at p, and that (4.15) holds.

Corollary 4.14. Suppose t is continuous and assume that $\mathscr{P}_p$ has a solution u with multiplier μ which is *not* a local minimum of the functional $f - \mu t$ on V. Then f is not locally subdifferentiable at p; in particular, h is not locally convex.

Proof. From Proposition 4.8 it follows that $\mathscr{P}_p$ is not locally stable. As $P_p \neq \emptyset$ by assumption, Proposition 4.12 implies that $\partial_{\text{loc}} h(p) = \emptyset$.

We shall now derive a *stability criterion*, i.e., we shall derive a criterion to determine the complete set of numbers $p \in \mathbf{R}$ for which $\mathscr{P}_p$ is stable. This is possible through an investigation of the family of unconstrained extremum problems $\mathscr{K}_\mu$ introduced in (4.12). Let K_μ denote the solution set of $\mathscr{K}_\mu$

$$K_\mu := \{u \in V | u \text{ is a solution of } \mathscr{K}_\mu\},$$

and let

$$k : \mathbf{R} \to \overline{\mathbf{R}},\ k(\mu) := \inf_{u \in V} \{f(u) - \mu t(u)\}. \tag{4.16}$$

Note that $k(\mu) = -h^*(\mu)$ and thus $\text{dom} k \subset \mathbf{R}$ is a simply connected interval of $\mathbf{R}$ on which k is a finite concave function. Corollary 4.9 may be formulated: if $\mathscr{P}_p$ is stable, then $u \in K_\mu$ for every $u \in P_p$ and every $\mu \in P^*_p$. We shall now prove the "converse" of this result.

Proposition 4.15. If $K\mu \neq \emptyset$ then $\mathscr{P}_p$ is stable for $p \in \{t(u) | u \in K_\mu\}$.

Proof. Let $\bar{u} \in K_\mu$ and put $\bar{p} = t(\bar{u})$. As $\bar{u} \in t^{-1}(\bar{p})$ is a global minimal point of the functional $f - \mu t$ on all of V, $\bar{u}$ is certainly a minimal point of $f - \mu t$ on the level set $t^{-1}(\bar{p})$. Hence $\bar{u} \in P_{\bar{p}}$, $f(\bar{u}) = h(\bar{p})$ and

$$-h^*(\mu) = \inf_{u \in V} \{f(u) - \mu t(u)\} = f(\bar{u}) - \mu t(\bar{u}) = h(\bar{p}) - \mu \bar{p}.$$

Hence $h(\bar{p}) + h^*(\mu) = \mu \bar{p}$ which means that $\mu \in \partial h(\bar{p})$. This shows that $\mathscr{P}_{\bar{p}}$ is stable.

The last lemma characterizes the values $\mu \in \mathbf{R}$ for which $K_\mu \neq \emptyset$ in a simple situation.

Lemma 4.16. Suppose that the functionals f and t satisfy conditions $(f,t1)$.
Then we have

$$K_\mu \neq \emptyset \Leftrightarrow \mu \in \operatorname{dom} k \; [\text{i.e., } k(\mu) < \infty].$$

Moreover, if $f - \mu t : V \to \mathbf{R}$ is coercive on V, then $\mu \in \operatorname{dom}k$. In particular:

(i) if t is bounded from below on V, then $\operatorname{dom}k \supset \overline{\mathbf{R}}^-$

(ii) if t is bounded from above on V, then $\operatorname{dom}k \supset \overline{\mathbf{R}}^+$

(iii) if t is bounded from above and from below on V, then $\operatorname{dom}k = \mathbf{R}$.

Proof. If f and t satisfy $(f,t1)$, then for every $\mu \in IR$, the functional $f - \mu t$ is weakly lower semicontinuous. The results then follow immediately from the fact that for such functionals the infimum, if finite, is actually attained.

Remark 4.17. If for $\bar{p} \in t(V)$ it is known that $h(p)$ is finite and locally a *concave*, differentiable function, then it is not difficult to show [4] that for sufficiently small $\epsilon > 0$:

$$h(\bar{p}) = \inf_{u \in t^{-1}(\bar{p})} f(u) = \inf_{\mu \in \mathbf{R}} \sup_{q \in (p-\epsilon, p+\epsilon)} \inf_{u \in t^{-1}(q)} \{f(u) - \mu(t(u) - p)\} \tag{4.17}$$

where $\inf\limits_{\mu \in \mathbf{R}}$ is attained for $\mu = \bar{\mu} = \dfrac{dh}{dp}(\bar{p})$, $\sup\limits_{q \in (p-\epsilon, p+\epsilon)}$ is attained for $q = \bar{p}$ and $\inf\limits_{u \in t^{-1}(\bar{p})}$ is attained for any solution of $\mathcal{P}_p$ (which has necessarily multiplier $\bar{\mu}$). In this case of locally concave functions, the extremum problem defined in the right hand side of (4.17) is not easier to deal with than the original problem $\mathcal{P}_p$ and seems to be of no use to extract information from it for problem $\mathcal{P}_p$, in contradistinction to the case of a locally convex function h which gave rise to the study of the ϵ-local dual problem $\operatorname{loc}_\epsilon \mathcal{P}_p^*$.

5. APPLICATIONS

We shall now demonstrate some of the foregoing results to functionals f and t which lead to semilinear eigenvalue problems of elliptic-type. Therefore, let Ω be a bounded domain in $\mathbf{R}^n$ and let L be given by (1.4). Then, as is well known, with $V = \mathring{H}^1(\Omega) = \mathring{W}^{1,2}(\Omega)$ the usual Sobolev space, the functional f defined by (1.6) satisfies conditions $(f1,2)$ and condition $(f3)$ of Proposition 2.2

We consider functionals t of the form

$$t(u) = \int_\Omega \left[\int_0^{u(x)} \gamma(x,z) dz \right] dx, \tag{5.1}$$

where the function $\gamma \in C^3(\Omega \times \mathbf{R}, \mathbf{R})$ satisfies the following growth condition:

$(t3)$: if $n > 2$ then for some constants, $b_1 \geq 0$, $b_2 \geq 0$:

$$|\gamma(x,z)| \leq b_1 + b_2 |z|^s \text{ for } z \in \mathbf{R}, x \in \Omega,$$

$$\text{where } s < \frac{n+2}{n-2}; \text{ if } n = 2, \text{ then } |\gamma(x,z)| \leq \exp\chi(z), \tag{5.2}$$

$$\text{where } \lim_{|z| \to \infty} \frac{\chi(z)}{z^2} = 0.$$

From standard embedding results for $\mathring{H}^1(\Omega)$, it follows that such functionals t are defined and finite on V and satisfy conditions (t1,2).

From Propositions 2.1 and 2.3 it follows that for every $p \in t(V)$, $\mathscr{P}_p$ has at least one solution, and solutions u for which $t'(u) \neq 0$ satisfy for some $\mu \in \mathbf{R}$ the equation

$$Lu = \mu\gamma(x,u) \qquad x \in \Omega. \tag{5.3}$$

Example 5.1. The simplest case is obtained when the functional t is given by

$$t(u) = \int_\Omega \tfrac{1}{2} u^2 \, dx. \tag{5.4}$$

This leads to the linear eigenvalue problem for the operator L:

$$Lu = \mu u. \tag{5.5}$$

For $p > 0$, problem $\mathscr{P}_p$ (and $\mathscr{S}_r$ for $r > 0$) characterize the eigenfunction u_1 of (5.5) corresponding to the principal, i.e., smallest, eigenvalue μ_1 (>0), normalized in such a way that $t(u_1) = p$ (or $\frac{1}{2}\langle u, Lu_1\rangle = r$, respectively). Apart from sign, the solution of $\mathscr{P}_p$ is unique and positive on Ω. Note that in this case we have $h(p) = \mu_1 \cdot p$ for $p > 0$, $\partial h(0) = (-\infty, \mu_1]$, $\partial h(p) = \{\mu_1\}$ for $p > 0$. The function $s(r)$ is given by $s(r) = (1/\mu_1) \cdot r$ for $r \geq 0$, and for the function $q(r)$ we have $q(r) = 0$ for $r > 0$: Q_r has no solution, but $\overline{Q}_r$ has $u \equiv 0$ as unique solution, for which $t'(0) = 0$.

Example 5.2. Equation (5.3) is *sublinear* if γ satisfies the estimate (5.2) with $0 < s < 1$. Then the functional t satisfies for some constants $c_1 \geq 0$, $c_2 \geq 0$.

$$|t(u)| \leq c_1 + c_2 \, \| u \|^{1+s}.$$

From Example 5.1 it follows that the principal eigenvalue μ_1 satisfies

$$\langle u, Lu\rangle \geq \mu_1 \| u \|^2 \qquad \forall u \in V, \tag{5.6}$$

for which we can conclude that the functional $\frac{1}{2}\langle u, Lu\rangle - \mu t(u)$ is bounded from below and coercive on V for every $\mu \in \mathbf{R}$. Hence, Lemma 4.16, dom$k = \mathbf{R}$ and $K_\mu \neq \emptyset$ for $\mu \in \mathbf{R}$. Consequently (Corollary 4.13), at points where h is continuous it is differentiable and subdifferentiable. As a specific example, consider the bounded functional

$$t(u) = \int_\Omega (1 - \cos u) dx.$$

Then $t(V) = [0, p_0)$ where $p_0 = 2\int_\Omega dx$. For $n = 1$, $\Omega = (0, l)$ and $L = -(d^2/dx^2)$, the equation

$$Lu = \mu \sin u \tag{5.7}$$

describes the plane steady-states of an elastic, flexible rod with constant mass-density: if in the (y,z) plane the rod is described with an arclength coordinate $x \in [0, l]$ as $(y(x), z(x))$, then $u = u(x)$ denotes the tangent of the rod with the positive y-axis:

$$\frac{dy}{dx}(x) = \cos u(x), \frac{dz}{dx} = \sin u(x).$$

In this case, problem $\mathcal{P}_p$ can be interpreted as the principle of minimal potential energy, the requirement $t(u) = p$ being the constraint that the potential energy has to be compared for configurations, described by $u(x)$, which [are horizontally inclined at the endpoints: $u(0) = u(l) = 0$ and which] have prescribed horizontal distance $y(l) - y(0)$ between the endpoints:

$$y(l) - y(0) = \int_0^l \cos u(x)\mathrm{d}x = l - p.$$

The inverse extremum problem $\mathcal{S}_r$ determines among all configurations which have r as value of the potential energy, that configuration which has the least horizontal distance $y(l) - y(0)$ between its endpoints. In this case, the multiplier μ also has a physical interpretation: it is proportional to the horizontal component of the compressive load necessary to maintain the rod in the required position. Concerning the unconstrained extremum problem $\mathcal{K}_\mu$, it is easily seen that $k(\mu) = 0$ and $K_\mu = \{0\}$ for $\mu \leq 0$, whereas for $\mu > 0$ this problem is well known in the literature [8, 10]. In fact, the solution can be explicitly expressed in terms of Jacobi elliptic functions. From the available information, or in a direct way, one obtains the following results. For $r \in (0,\infty)$ the function $s(r)$ monotonically increases from 0 to p_0, such that the inverse function $h(p)$ monotonically increases on $(0,p_0)$ with $h(p) \to \infty$ if $p \uparrow p_0$. Moreover, the function h is differentiable and subdifferentiable on $(0,p_0)$, and thus there is a one-to-one correspondence between $p \in (0,p_0)$ and the multiplier μ of the solutions of $\mathcal{P}_p$. From this, together with $K_\mu = \{0\}$ if $0 < \mu \leq \mu_1$, and $K\mu = \{\pm U(\mu)\}$ if $\mu > \mu_1$ [where $U(\mu)$ is the unique positive solution of $\mathcal{K}_\mu$] this implies that the "first buckling mode" $\{U(\mu)\}_{\mu>\mu_1}$ can also be parameterized with the parameter $p \in (0,p_0)$ [and also with the parameter $r \in (0,\infty)$]:

$$\{\mu(p)\} = \left\{\frac{\mathrm{d}h}{\mathrm{d}p}(p)\right\} = \partial h(p) \text{ for } p \in (0,p_0).$$

For this problem, $q(r) = 0$ and $Q_r = \emptyset$ for $r > 0$.

Let us now consider some problems which give rise to equations of *superlinear* type.

Example 5.3. As a first specific example of this kind, let

$$t(u) = \int_\Omega \left[\tfrac{1}{2}u^2 - \tfrac{1}{4}g^2(x)u^4\right]\mathrm{d}x, \tag{5.8}$$

where $g \in C^0(\overline{\Omega},\mathbf{R})$ is a given function which satisfies $g(x) \geq g_0 > 0$ for $x \in \overline{\Omega}$. The corresponding equation (5.3) reads

$$Lu = \mu[u - g^2(x)u^3]. \tag{5.9}$$

The functional t is bounded from above but not from below:

$$t(V) = (-\infty,p_0), \text{ where } p_0 = \int_\Omega \tfrac{1}{4}g^{-2}(x)\mathrm{d}x.$$

Noticing that $t'(u) = 0$ if and only if $u = 0$ (for $u \in V$), problem $\mathcal{S}_r$ has for every $r > 0$ a solution and on $[0,\infty)$, $s(r)$ is a continuous, monotonically increasing function from 0 to p_0. Furthermore, for $\mu \geq 0$, $k(\mu)$ is finite and $\mathcal{K}_\mu$ has a solution [hence h : $(0,p_0) \to \mathbf{R}$ is a finite (sub-) differentiable convex function]: $K_\mu = \{0\}$ if $0 \leq \mu \leq \mu_1$, and for $\mu > \mu_1$, $K_\mu = \{\pm U(\mu)\}$, where $U(\mu)$ is the unique nonnegative solution of (5.9) [1, p. 312].

By the extremal characterization and the maximum principle for L, every solution of $\mathcal{P}_p$, $p > 0$, must be of the same sign on Ω, from which it follows that $P_p = \{\pm U(\mu(p))\}$ for $p > 0$, where

$$\{\mu(p)\} = \left\{\frac{dh}{dp}(p)\right\} = \partial h(p).$$

As was shown by Berger, the mapping $(\mu_1,\infty) \ni \mu \to U(\mu) \in C_0^0(\Omega)$ is continuous. Because the mapping $(0,p_0) \ni p \to \mu(p) \in (\mu_1,\infty)$ is also continuous, the solution branch $U(\mu(p))$ depends continuously on p, $p \in (0, p_0)$. To investigate $\mathcal{P}_p$ for $p < 0$, first note that in this case every solution has a negative multiplier: this follows by multiplying Eq. (5.9) with u and integrating over Ω:

$$0 \le \langle u,Lu\rangle = \mu \int_\Omega (u^2 - g^2u^4)dx = \mu\left(-\int_\Omega u^2dx + 4p\right).$$

As $k(\mu) = -\infty$ for $\mu < 0$, the curve $h(p)$ is not subdifferentiable for any $p < 0$, and $[0,\mu_1] = \partial h(0)$. Moreover, as t can take negative values, the solution of Q_r is nontrivial ($u \not\equiv 0$) if r is sufficiently large: for some $r_0 > 0$, Q_r has a solution and $q(r)$ is a continuous, monotonically decreasing function for $r > r_0$. It is not difficult to see that h is in fact a concave function of p for $p < 0$: therefore it suffices to show that if $\bar{u}$ is a solution of $\mathcal{P}_p$, with $p < 0$ and then necessarily with multiplier $\bar{\mu} < 0$, then $\bar{u}$ is not a local minimum of the functional $f - \bar{\mu}t$ (Corollary 4.14). To that end consider the function

$$\begin{aligned}\chi(\rho) &:= f(\rho\bar{u}) - \bar{\mu}t(\rho\bar{u})\\ &= \rho^2\{\tfrac{1}{2}\langle\bar{u},L\bar{u}\rangle - \tfrac{1}{2}\bar{\mu}\langle\bar{u},\bar{u}\rangle\} + \tfrac{1}{4}\bar{\mu}\rho^4\int_\Omega u^4dx\end{aligned}$$

in a neighbourhood of $\rho = 1$. Using $(d\chi/d\rho)(1) = 0$ in the expression for $(d^2\chi/d\rho^2)(1)$, we find

$$\frac{d^2\chi}{d\rho^2}(1) = 2\bar{\mu}\int_\Omega \bar{u}^4dx < 0,$$

which result contradicts the condition $(d^2\chi/d\rho^2)(1) \ge 0$ which is necessary in order that $\bar{u}$ be a local minimum of the functional $f - \bar{\mu}t$.

Example 5.4. As another specific example, consider the functional

$$t(u) = \int_\Omega (\tfrac{1}{2}u^2 + \tfrac{1}{4}u^4)dx.$$

Then $t(V) = [0,\infty)$ and $t'(u) = 0$ if and only if $u = 0$ on V. The function $s(r)$ is continuous and monotonically increasing on $\mathbf{R}^+$. Hence the same applies for the function h on $\mathbf{R}^+$, and thus $\mu \ge 0$ for any solution of $\mathcal{P}_p$. It is easily seen that dom$k = (-\infty,0]$, and thus h is *not* subdifferentiable for any $p > 0$. In much the same way as was done in the previous example (for $p < 0$), it can be shown that h is a concave function on $\mathbf{R}^+$. Hence $\mu(p)$ is a monotonically decreasing function of p, and in fact $\mu(0) = \mu_1$, $\mu(p) \to 0$ for $p \to \infty$.

Example 5.5. Another class of problems is obtained if we impose on the function γ the extra condition

$$\gamma(x,0) > 0 \quad \text{for } x \in \overline{\Omega}. \tag{5.10}$$

Then $S_r \neq 0$ for every $r > 0$ and $s : (0,\infty) \to \mathbf{R}^+$ is continuous and monotonically increasing with $s(0) = 0$. Hence $h(0) = 0$ and $h : t(V) \cap \mathbf{R}^+ \to \mathbf{R}^+$ is continuous and monotonically increasing. Because of condition (5.10), the multiplier μ of any solution of $\mathcal{S}_p$ tends to zero if $p \downarrow 0$. Together with $h(0) = 0$ and $h(p) > 0$ for $p > 0$, this implies that there exists a (maximum) value p^*, $0 < p^* \leq \infty$, such that h is finite and convex on the interval $(0,p^*)$. Hence h is differentiable and locally subdifferentiable for $p \in (0, p^*)$ and $\mathcal{P}_p$ is locally stable there. Further information of the function γ determines the value of p^* and the behaviour of μ and h in a neighbourhood of p^*. For instance it is possible that $p^* = \infty$, such as for the simple functional

$$t(u) = \int_\Omega u \, \mathrm{d}x$$

in which case $\mathcal{P}_p$ is stable for $p > 0$, or that p^* is the finite least upper bound of the functional t, such as for

$$t(u) = \int_\Omega (u - \tfrac{1}{2}u^2)\mathrm{d}x : p^* = \tfrac{1}{2}\int_\Omega \mathrm{d}x$$

in which case $\mathcal{P}_p$ is stable for $p \in (0,p^*)$ and $h(p) \to \infty$, $\mu(p) \to \infty$ for $p \uparrow p^*$.

Let us now describe a situation for which p^* is finite but not an endpoint of $t(V)$. Therefore suppose that γ satisfies the extra conditions

$$\begin{aligned} &\frac{\mathrm{d}\gamma}{\mathrm{d}z}(x,0) > 0 \quad \text{for } x \in \overline{\Omega} \\ &\frac{\mathrm{d}^2\gamma}{\mathrm{d}z^2}(x,z) > 0 \quad \text{for } x \in \Omega,\ z > 0. \end{aligned} \tag{3.11}$$

It was shown by Crandall and Rabinowitz [2] (cf. also Keener & Keller [6]) that in this case there exists a $\overline{\lambda} > 0$ such that (5.3) has positive solutions if and only if $\mu < \overline{\lambda}$, and that there exists a continuous solution curve $[0,\overline{\lambda}) \ni \mu \to U(\mu) \in C^{2,\alpha}(\overline{\Omega})$ with the properties: (i) $U(\mu)$ is a minimal solution of (5.3) [i.e., if v is another positive solution of (5.3), then $U(\mu)(x) < v(x)$ for $x \in \Omega$], (ii) $U(\mu)$ is the only positive solution of (5.3) which is a local minimum of the functional $\frac{1}{2}\langle u,Lu\rangle - \mu t(u)$.

These results may be interpreted in terms of constrained extremum problems if we impose an extra condition which assures that solutions of $\mathcal{P}_p$ for $p > 0$ are necessarily nonnegative [e.g., $\gamma(x,z) > 0$ for $z \in \mathbf{R}$]. Then we have the following situation: there exists a connected interval $(0,p^*)$ such that $\mathcal{P}_p$ is locally stable for $0 < p < \underline{p}^*$ and h is convex, differentiable on $(0,p^*)$ with $\mu(p) = (\mathrm{d}h/\mathrm{d}p)(p)$ increasing from 0 to $\overline{\lambda}$. If $p^* < \infty$, for $p \in (p^*,\infty)$, $h(p)$ is a continuous, concave function [because of (ii) and Corollary 4.14] at which $\mu(p)$ ($< \overline{\lambda}$) is monotonically decreasing, say $\mu(p) \downarrow \mu_\infty$ for $p \to \infty$. Then necessarily $\mu_\infty \geq 0$. Note that if in a specific situation ($\overline{\lambda} >$) $\mu_\infty > 0$, then $\mathcal{P}_p$ is in fact stable for those values of $p < p^*$ for which $\mu(p) < \mu_\infty$, in which case the solutions

$U(\mu(p))$ are in fact global minimum points of the functional $\frac{1}{2}\langle u,Lu\rangle - \mu(p)t(u)$. With the extra conditions

$$\lim_{z\to\infty} \frac{\gamma(x,z)}{z} = \infty \qquad\qquad (5.12)$$
$$\int_0^z \gamma(x,t)\mathrm{d}t \le \theta z\gamma(x,z) \text{ for } z > \bar{z}, \text{ for some } \bar{z} > 0, \theta \in [0,\tfrac{1}{2}),$$

it was shown in Ref. 2 that for every $\mu \in (0,\bar{\lambda})$ there exists at least a second nonnegative solution of (5.3). As this second solution has a larger value of t than $t(U(\mu))$ [by the minimality of $U(\mu)$], this implies that $\mu_\infty = 0$ and thus $\mu(p) \downarrow 0$ for $p \to \infty$. [Note that this can also immediately be concluded from the fact that $k(\mu) = -\infty$ for every $\mu > 0$ if γ satisfies (5.12), which also shows the existence of a finite value p^*.] In the following paper [5] we shall describe the continuation of the solution branch with the parameter p instead of with the parameter μ in a more detailed way.

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Abstract—In this paper we consider constrained extremum problems of the form

$$\mathscr{P}_p: \inf_{u\in t^{-1}(p)} f(u),$$

where f and t are continuously differentiable functionals on a reflexive Banach space V and where $t^{-1}(p)$ denotes the level set of the functional t with value $p \in \mathbf{R}$.

Related to problems $\mathscr{P}_p$ we investigate inverse extremum problems, which are extremum problems for the functional t on level sets of the functional f. Under conditions that guarantee the existence of solutions of $\mathscr{P}_p$, let $h(p)$ denote the value of f at such a solution. If h is a (locally) convex function at some $\bar{p} \in \mathbf{R}$, we show that it is possible to define a dual problem of $\mathscr{P}_{\bar{p}}$. This dual problem is a saddle-point formulation for the functional $V \times \mathbf{R} \ni (u,\mu) \to f(u)-\mu[t(u)-\bar{p}]$: for some extreme value $\bar{\mu}$ (which is the Lagrange multiplier of a solution of $\mathscr{P}_{\bar{p}}$) the solutions of $\mathscr{P}_{\bar{p}}$ are precisely the (local) minimal points of the functional $f-\bar{\mu}t$ on V.

It is shown how these results can be used to describe solution branches of nonlinear eigenvalue problems (of semilinear elliptic type) with a global parameter, such as $p \in \mathbf{R}$, instead of with the eigenvalue as parameter.