An extended self-organisation principle for modelling and calculating the dissipation of 2D confined vortices

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Abstract. Steady Euler flows in a periodic square that, for positive vorticity distributions, minimise the entropy at given values of the energy and the circulations are non-confined vortices for optimal values of the circulation, and are confined vortices for certain non-optimal values. An extension of the self-organisation principle for the 2D Navier–Stokes equations is presented which models the dissipative evolution of such confined vortices by changing the values of the constraints in time in accordance with the effect of dissipation.

An efficient numerical procedure for calculating these confined states is described and calculations of the dissipative evolution are presented.

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1. Introduction

Numerical simulations of two-dimensional decaying turbulence show a remarkable discrepancy with rigorous mathematical statements about the time asymptotics of solutions of the Navier–Stokes equations.

The asymptotic results specialise the physical understanding of the self-organisation process and describe the asymptotics as a quasistationary evolution through a family of exact, steady solutions of the Euler equations. For flows in a bounded domain, these exact solutions are described most simply in terms of the vorticity ω of the flow and are proportional to the fundamental eigenfunction of the Laplace operator on the domain. Writing $\Omega(e)$ for the principal eigenfunction when normalised such that the total energy of the flow is e , the asymptotics are given, up to terms that decrease exponentially faster, by the planar vortex

$$t \rightarrow \Omega(e(t)) \quad \text{with } e(t) = e_0 \exp(-2\nu\lambda_1 t). \quad (1.1)$$

Here, ν is the kinematic viscosity and λ_1 is the smallest eigenvalue. The function $\Omega(e(t))$ in (1.1) is an exact solution of the Navier–Stokes equations (Taylor 1923) and represents a vorticity distribution with a fixed spatial pattern $\tilde{\Omega} = \Omega(1)$ decaying

exponentially in time: $\Omega(e) = \sqrt{e} \hat{\Omega}$. Note that, at each instant, for each e , the quasisteady state $\Omega(e)$ is not confined, i.e. the support of the vorticity is the whole fluid domain.

Numerical simulations for slightly viscous fluids, on the other hand, show clearly a tendency of the flow to confine the vorticity ω to relatively small, strictly separated regions of positive and negative vorticity, surrounded by potential flow (for which $\omega = 0$). These regions can be viewed as 'extended vortex points' and their dynamical behaviour, including interactions, is to a large extent well described as in inviscid (Eulerian) fluids.

The rigorous asymptotic results were obtained by Foais and Saut [2]. The treatment by van Groesen [3] relates the analysis to the arguments used in the physical literature about self-organisation and normal-mode cascades. (See also [4–8] and the references therein.) The approach taken in this paper follows general ideas described in [9]. Numerical results for decaying 2D turbulence which show the appearance of confined vortices were first obtained by McWilliams [10]; see also Benzi, Patarnello and Santangelo [11] for more recent high-resolution simulations.

Of course, the discrepancy between the analytical asymptotic results and the (reliable) numerical simulations could be explained by the fact that for slightly viscous fluid (ν small, in most calculations $\nu \approx 10^{-4}$ – 10^{-8}) the asymptotic time regime is never reached. In other words, the numerical simulations show the evolution at *intermediate* times rather than the asymptotic evolution.

The question then remains as to how the observed confined regions can be described properly and if and how they can evolve (smoothly) towards the quasisteady states of the asymptotic regime. This problem is tackled in this paper in the simplest possible setting. The simplification is obtained by restricting the analysis to only one confined vorticity region, thereby circumventing the difficulties connected with the interactions between several vortex regions.

Specifically, it will be shown that the one-parameter family of quasisteady states (abbreviated qss in the following) can be embedded in a two-parameter family of sign-definite vorticity distributions $\Omega(e, \gamma)$. The additional parameter γ is nothing but the total circulation of the flow. When for γ some optimal value $\gamma^* = \gamma^*(e)$ is taken, the qss $\Omega(e)$ is obtained:

$$\Omega(e) = \Omega(e, \gamma^*(e)) \quad (1.2)$$

while for values $\gamma < \gamma^*(e)$ the function $\Omega(e, \gamma)$ is an exact, steady solution of the Euler equations which is confined. In this way, the introduction of the parameter γ can be seen as an unfolding of the one-parameter family of qss.

The functions $\Omega(e, \gamma)$ will be characterised in a variational way. Just as the qss can be described as the functions that minimise the quadratic enstrophy $W(\omega) = \int \frac{1}{2} \omega^2$ at given value of the energy $E = \int \frac{1}{2} v^2$, i.e. realising

$$\text{Min}\{W(\omega) \mid E = e\} \quad (1.3)$$

the elements of the unfolded family minimise W on non-negative vorticity distributions with given energy E and given circulation $C(\omega) = \int \omega$:

$$\text{Min}\{W(\omega) \mid E(\omega) = e, C(\omega) = \gamma, \omega \geq 0\}. \quad (1.4)$$

This extremal characterisation makes it possible to propose a model for the viscous dissipation of such confined vortices. Resembling the arguments of the self-organisation process which lead to the asymptotic behaviour described by (1.1), it is

argued that the selective dissipation of the functionals E , W and C leads to certain evolution equations for the parameters $e(t)$ and $\gamma(t)$ for which

$$t \rightarrow \Omega(e(t), \gamma(t)) \quad (1.5)$$

approximates the dissipative decay of a single confined vortex. The evolution is such that $\gamma(t)$ tends to its optimal value of the QSS, i.e.

$$\gamma(t) \rightarrow \gamma^*(e(t)) \quad \text{as } t \rightarrow \infty \quad (1.6)$$

in such a way that the correct asymptotic result (1.1) is satisfied.

For this reason the description (1.5) can be seen as a specification of the asymptotic behaviour (1.1), valid for initially confined vortices at intermediate and asymptotic timescales.

The paper also contains a description of a numerical procedure for calculating confined vortices from the extremal characterisation (1.4). With this procedure, the proposed viscous dissipation is calculated and results of these calculations are presented.

The contents of the paper are as follows. In §2 the precise fluid domain is described, and the arguments that are needed later about the self-organisation process are presented in a simple way. Section 3 investigates the unfolding into the two-parameter family of exact Euler solutions, while §4 provides arguments for the choice of the dynamic description of the dissipation of confined vortices. Section 5 deals with the numerical procedure.

2. Preliminaries

The Navier–Stokes equations for flows in a plane are written in terms of the fluid velocity $\mathbf{v}$ and the vorticity (the component of $\text{curl } \mathbf{v}$ perpendicular to the plane) as

$$\text{div } \mathbf{v} = 0 \quad \omega_t + \mathbf{v} \cdot \nabla \omega = \nu \Delta \omega. \quad (2.1)$$

Here, ν is the kinematic viscosity; for inviscid fluids $\nu = 0$ and then (2.1) are equivalent to the Euler equations.

To be specific, we will consider flow on a periodic square $[-1, 1]^2$ and, for ease of presentation, only those flows that have a vorticity distribution that is odd in both variables (a property that is conserved and inherited from initial distributions with this property). Attention may then be restricted to solutions on a quarter cell $[0, 1]^2 =: R$ with boundary conditions

$$\psi = 0 \quad \text{and} \quad \omega = 0 \text{ on } \partial R \quad (2.2)$$

where ψ is the Stokes stream function, related to ω by $-\Delta \psi = \omega$.

The basic functionals that will be used in the following are the energy E , the (quadratic) enstrophy W and the circulation (total vorticity) C . All these functionals are considered (without restriction) as functionals of ω :

$$E(\omega) = \int \frac{1}{2} v^2 \quad W(\omega) = \int \frac{1}{2} \omega^2 \quad C(\omega) = \int \omega$$

(here and in the following, integrals are over the square R).

The evolution of these integrals, at solutions of (2.1), is given by

$$\dot{E}(\omega) = -\nu \int \omega^2 = -2\nu W(\omega) \quad (2.3)$$

$$\dot{W}(\omega) = \nu \int \omega \Delta \omega = -\omega \int |\nabla \omega|^2 \quad (2.4)$$

$$\dot{C}(\omega) = \nu \int \Delta \omega. \quad (2.5)$$

The main element that underlies the self-organisation process, and likewise the asymptotic behaviour, is that the selective dissipation of the integrals E and W implies that W decreases 'faster' than E . This will be shown below to follow from (2.3) and (2.4). This selective dissipation agrees with the extremal characterisation of the functions $\Omega(e)$ by (1.3), for which W is minimised at given E . Since some of the arguments will play a crucial role in the following, we will briefly describe the main ingredients.

Consider an 'integral diagram', i.e. a diagram in which the values of W and E are presented along the vertical and horizontal axes respectively. The oss i.e. the solutions of (1.3) for different values of e , determine a straight line in this diagram, the oss curve $W = \lambda_1 E$. Each point on this line corresponds to the oss of that particular value of E . Since the value $W(e)$ of the minimisation problem (1.3) is given by $W(e) = \lambda_1 e$, no points below this line correspond to physical vorticity distributions. The time-dependent solution (1.1) moves along this line towards the origin with a well defined speed.

It turns out that the self-organisation process can be described simply as the statement that the integral curves

$$t \rightarrow (E(\omega(t)), W(\omega(t))) \quad (2.6)$$

are tangent to the oss curve at the origin for 'most' solutions of (2.1) (we will be more specific in the following); see figure 1.

More analytically, this can be seen (and these arguments form the basis of a rigorous proof) by introducing the following functionals:

$$Q(\omega) := W(\omega)/E(\omega) \quad \Lambda(\omega) := \dot{W}(\omega)/\dot{E}(\omega). \quad (2.7)$$

Geometrically, these functionals can be interpreted as the direction vector and the tangent vector to the integral curve respectively. Physically, the functional Λ has the important interpretation of being the dissipation-rate quotient of W and E .

A simple application of the Cauchy-Schwarz inequality shows that

$$\Lambda(\omega) - Q(\omega) \geq 0 \quad \text{for all functions } \omega \quad (2.8)$$

and that equality holds only if ω is an eigenfunction of the Laplace operator. From the time evolution of Q , which is given by

$$\dot{Q}(\omega) = -2\nu Q[\Lambda - Q] \quad (2.9)$$

it follows that Q is monotonically decreasing. An independent argument shows that the limiting value of W must necessarily be an eigenvalue.

This leads to the following conclusion. Let I_1 be the set of all initial data ω_0 for which Q tends to the smallest eigenvalue λ_1 . (This set is a smooth manifold and is a nonlinear variant of the set of functions with non-vanishing Fourier component in

Figure 1. The integral diagram indicating the dynamical self-organisation principle. The quasi-stationary-state (QSS) curve $W = \lambda_1 E$ consists of solutions of the minimisation problem (1.3) and supports the fundamental vortex (1.1). The broken curve represents a typical integral curve: tangency to the QSS curve at the origin is equivalent to the self-organisation principle, i.e. to the time asymptotics (1.1).

Figure 2. The extended integral diagram. The QSS curve is now given by $E > 0$, $W = \lambda_1 E$ and $C = \gamma^*(E)$. Confined vortices, solutions of (1.4), are obtained for $0 < \gamma < \gamma^*(E)$.

the direction of the first eigenfunction, so containing 'almost all' initial data (see [1, 2]). Then the foregoing arguments show that any solution with initial value in I_1 has an integral curve that is tangent to the QSS curve and the asymptotic behaviour is given by (1.1).

We shall see in the following that the tangency condition that implies the self-organisation will play a decisive role in the selection of the required dynamics for the extended formulation. In that respect, it is useful to observe that the governing constraint dynamics for $e = e(t)$ as given in (1.1) could also have been obtained with the following arguments. From the evolution of E as given by (2.3), i.e. $\dot{E} = -2\nu W$, and from the fact that $W(e) = \lambda_1 e$ for the one-parameter family $t \rightarrow \Omega(e(t))$, it follows that the decrease is given by

$$\dot{e}(t) = -2\nu W(e(t)) = -2\nu \lambda_1 e(t)$$

which agrees with the evolution of e as given in (1.1).

3. The two-parameter family of confined vortices

In this section the two-parameter minimisation problem (1.4) will be considered. The value of the minimum will be denoted by $W(e, \gamma)$:

$$W(e, \gamma) = \text{Min}\{W(\omega) \mid E(\omega) = e, C(\omega) = \gamma, \omega \geq 0\}. \quad (3.1)$$

Then, if λ_1 and $\hat{\Omega}$ denote, as before, the fundamental eigenvalue and corresponding eigenfunction with energy $e = 1$ respectively, the optimal value of γ for which the elements of the one-parameter family $\Omega(e)$ are also solutions of (1.4) is readily found:

$$\lambda_1 e = W(e) = \min_{\gamma} W(e, \gamma) = W(e, \gamma^*) \quad \text{for } \gamma^* = \gamma^*(e) = \sqrt{e} C(\hat{\Omega}). \quad (3.2)$$

For further reference, define λ and σ by

$$\lambda = \frac{\partial W}{\partial e}(e, \gamma) \quad \sigma = -\frac{\partial W}{\partial \gamma}(e, \gamma) \quad (3.3)$$

and note that

$$\sigma > 0 \text{ for } \gamma \in (0, \gamma^*(e)) \quad \sigma = 0 \text{ for } \gamma = \gamma^*(e). \quad (3.4)$$

In a three-dimensional C - E - W integral diagram the qss now form a space curve, lying in the plane $W = \lambda_1 E$ and with projections that are parabolas in the C - E and C - W planes (see figure 2).

We will now consider non-optimal values of γ , in particular $\gamma \in (0, \gamma^*(e))$.

Proposition 3.1. For $\gamma \in (0, \gamma^*(e))$ the minimisation problem (3.1) has at least one solution $\Omega(e, \gamma)$. Such a solution is a confined vortex, i.e. the closure of the vortex core $\{x \in R \mid \Omega(e, \gamma)(x) > 0\}$ is strictly contained in R .

Remark 3.2. It is to be expected that the solution of (3.1) is unique, but a formal proof of the uniqueness is lacking. As in [12] the core of such an extremising solution will be a simply connected domain.

Proof. The existence of a solution of (3.1) follows with standard techniques from functional analysis. On the set of $L_2(R)$ functions ω , equipped with the weak topology, the functional W is lower semicontinuous and the constrained set is closed. A generalisation of the Weierstrass theorem (see e.g. [13]) provides the existence of a minimiser.

Standard results from the calculus of variations show that such a minimiser $\Omega = \Omega(e, \gamma)$ satisfies the equation

$$\Omega = (\lambda\psi - \sigma)_+ \quad \text{in } R. \quad (3.5)$$

Here ψ is the stream function, ξ_+ denotes the positive part of ξ (i.e. $\xi_+(x) = \max(\xi(x), 0)$), while λ and $-\sigma$ are the Lagrange multipliers due to the constraints E and C respectively. These multipliers are related to the value function W according to (3.3). From (3.5) it follows that Ω satisfies

$$-\Delta\Omega = \lambda\Omega \quad \text{in the core } \{x \mid \omega(x) > 0\} \quad (3.6)$$

hence λ is positive. Since $\psi = 0$ on ∂R , it follows from (3.4) that $\lambda\psi - \sigma$ is negative on ∂R . This implies the confinement.

Figure 3. Calculated streamlines of a confined vortex on the quarter-cell with data $\gamma = 0.4$, $e = 0.3646$ ($w = 0.85$) [$\gamma^*(e) = 0.4$ for $e = 0.1217$]. The thick curve is the boundary of the core, i.e. of the set of points where ω is positive. The maximum value of ω is $\omega_{\max} = 6.81$.

Figure 4. (a) 3D representation of the vorticity distribution on the whole periodic cell for the vortex with data as in figure 3.

(b), (c) and (d) depict evolution of the vortex calculated according to the dynamics described in §4, represented at times $\nu t = 3 \times 10^{-3}$, 2×10^{-2} and 5×10^{-2} respectively. (See also figure 5(b).)

Figure 4. (continued)

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This result shows that, even for values of γ arbitrarily close to $\gamma^*(e)$, there exists a confined vortex as an exact steady solution of the Euler equations. The core of $\Omega(e, \gamma)$ is expanding for increasing $\gamma \uparrow \gamma^*(e)$ at fixed e , and for decreasing $e \downarrow \gamma^{*-1}(\gamma)$ at fixed γ .

The limiting function is the QSS $\Omega(e) = \sqrt{e} \hat{\Omega}$, which has all of R as the closure of its core, and is thus a non-confined vortex.

Using the numerical method described in §5, the streamlines of a typical confined vortex $\Omega(e, \gamma)$ are as depicted in figure 3. Figure 4 shows, for various values of the parameters, three-dimensional pictures of the corresponding vorticity distribution on the original square $[-1, 1]^2$.

To conclude this section, define a quantity $\beta = \sigma/\lambda$. Then (3.5) can be written as

$$\Omega = \lambda(\psi - \beta)_+ \quad (3.7)$$

and it follows that $\beta > 0$ is the value of ψ at the (free) boundary of the core. Inserting (3.7) into the energy functional yields

$$E = \int \frac{1}{2} v^2 = \frac{1}{2} \int \Omega \psi = \frac{1}{2} \int \left(\frac{1}{\lambda} \Omega^2 + \beta \Omega \right) = \frac{1}{\lambda} W(\Omega) + \frac{1}{2} \beta C(\Omega)$$

from which it is seen that

$$W(e, \gamma) = \lambda(e - \frac{1}{2}\beta\gamma) = \lambda e^* \quad \text{with } e^* := e - \frac{1}{2}\beta\gamma. \quad (3.8)$$

Note that e^* can be interpreted as the energy of the core of the vortex.

4. The extended self-organisation principle

In the foregoing section we studied (confined) vortices for fixed values of the parameters e and γ . These functions are exact, steady solutions of the Euler equations. In the presence of viscosity the value of the energy e and circulation γ will change according to (2.3) and (2.5). In this section we investigate how to describe the viscous evolution of a vorticity distribution that is initially such a confined vortex, say $\Omega(e_0, \gamma_0)$ for given e_0 and γ_0 with $\gamma_0 < \gamma^*(e_0)$.

We shall show that the viscous evolution can be approximated at each instant by a confined vortex from the same family $\Omega(e, \gamma)$ for suitably adapted values e and γ . Stated differently, we shall present dynamical equations for the constraints such that $t \rightarrow \Omega(e(t), \gamma(t))$ approximates the viscous evolution.

It must be remarked here that at each instant $\Omega(e(t), \gamma(t))$ is confined as long as $\gamma(t) < \gamma^*(e(t))$. Therefore this approximation is not an exact solution of the Navier–Stokes equations, unlike the corresponding case of the vortices given by (1.1). We shall present preliminary analytical arguments that this is a good approximation for the proposed evolution of $e(t)$ and $\gamma(t)$. Numerical simulations of the dynamics will also show that the description is qualitatively correct. A detailed investigation of the approximation will be presented in another paper.

If the value of the enstrophy $W(\Omega(e(t), \gamma(t)))$ is denoted by $w(t)$, the functions $e(t)$, $w(t)$ and $\gamma(t)$ are related according to (3.8) by

$$w(t) = W(e(t), \gamma(t)) = \lambda(e - \frac{1}{2}\beta\gamma) = \lambda e^*. \quad (4.1)$$

The compatibility condition

$$\dot{w} = \lambda \dot{e} - \sigma \dot{\gamma} = \lambda(\dot{e} - \beta \dot{\gamma}) \quad (4.2)$$

results from differentiating with respect to time and from (3.3). The evolution of E , W and C given by (2.3)–(2.5) for exact solutions of the NS equation is the starting point for the derivation of the constraint dynamics. If in these expressions the integrals are evaluated at a function $\Omega(e, \gamma)$ satisfying (3.7), while $\dot{E}$, $\dot{W}$ and $\dot{C}$ are equated to $\dot{e}$, $\dot{w}$ and $\dot{\gamma}$ respectively, there results

$$\dot{\gamma} = -\nu\lambda\gamma \quad (4.3a)$$

$$\dot{e} = -2\nu w = -2\nu\lambda e^* \quad (4.3b)$$

$$\dot{w} = -2\nu\lambda w = -2\nu\lambda^2 e^*. \quad (4.3c)$$

It is to be noted that, except for the exact solution (1.1), the conditions (4.3b) and (4.3c) are not compatible with (4.3a). This is related to the fact that $\Omega(e(t), \gamma(t))$ is not an exact solution if $\gamma(t) < \gamma^*(e(t))$.

Since we have only two degrees of freedom, it is appealing to choose two conditions from (4.3); the third one could then be found from the compatibility condition (4.2). The combination (4.3b) and (4.3c) is excluded above. The choice of (4.3a) and (4.3c) results in

$$\frac{dw}{d\gamma} = \frac{\dot{w}}{\dot{\gamma}} = \frac{2w}{\gamma} \quad (4.4)$$

while the choice of (4.3a) and (4.3b) leads to

$$\frac{de}{d\gamma} = \frac{\dot{e}}{\dot{\gamma}} = \frac{2e^*}{\gamma} < \frac{2e}{\gamma}. \quad (4.5)$$

However, both possibilities must be rejected as a consequence of the self-organisation principle. Indeed, from §2 it follows that the integral curve of the solution of the NS equation must be tangent to the oss curve in the C – E – W diagram, and in particular second-order tangent in the C – W and C – E planes. This condition is not satisfied if (4.4) or (4.5) holds. In fact, since the self-organisation process for a solution of the NS equation is based on the inequality $\Lambda(\omega) - Q(\omega) > 0$ (see (2.8)), it is natural to require this condition to hold also for the constraint dynamics, i.e. to require

$$\frac{\dot{w}}{\dot{e}} - \frac{w}{e} > 0.$$

This condition is satisfied only if

$$\frac{de}{d\gamma} = \frac{\dot{e}}{\dot{\gamma}} > \frac{2e}{\gamma} \quad (4.6)$$

and can be met by taking

$$\dot{\gamma} = -\nu\lambda\gamma \quad (4.7a)$$

$$\dot{w} = -2\nu\lambda^2 e \quad (4.7b)$$

after which $\dot{e}$ is evaluated from the compatibility condition (4.2):

$$\dot{e} = -2\nu\lambda(e + \frac{1}{2}\beta\gamma). \quad (4.7c)$$

Note that the enstrophy dissipation according to (4.3c) is proportional to the energy of the core only, while in the model (4.7b) it is proportional to the total energy of the vortex.

Figure 5. Calculated evolution of the initial vortex of figure 3. (a) Boundary at the core at times $\nu t = 0, 3 \times 10^{-3}$ and 2×10^{-2} . (b) Graphs of the amplitude $\omega_{\max}$, of the normalised value $Q/2\pi^2$ and of the diameter d of the core (defined at the horizontal insection for $y = 0.5$) as a function of time $\nu t \times 10^3$.

The dynamics described by (4.7) satisfies (4.6) and therefore models the self-organisation property of the exact solution. Numerical calculations (see figure 5) show that for the dynamics (4.7) the core of the vortices $\Omega(e(t), \gamma(t))$ is expanding monotonically in time, which should be expected from a model that describes vortex dissipation. Figure 5(b) depicts the increase in the diameter of the core, and the decrease of Q and of the maximal vorticity.

Essentially qualitative arguments were used above to introduce the model. A quantitative justification of the model must be based on an error analysis. As a first step towards such an analysis, let r denote the residual when $\Omega = \Omega(e(t), \gamma(t))$ is substituted in the NS equation:

$$r := \Omega_t + V \cdot \nabla \Omega - \nu \Delta \Omega$$

(here V is the divergence-free velocity field corresponding to Ω). Since $V \cdot \nabla \Omega = 0$ and $-\Delta \Omega = \lambda \Omega$ hold at each instant, r is given by

$$r = \Omega_t + \gamma \lambda \Omega.$$

The L_2 projections of r onto constant functions and onto Ω are given by

$$\int r = \dot{\gamma} + \nu \lambda \gamma \quad \int r \Omega = \dot{w} + 2\nu \lambda w.$$

Specifically for the dynamics (4.7), this amounts to

$$\int r = 0 \quad \int r \Omega = 2\nu(w - \lambda e) = 2\nu\lambda(Q - \lambda)e. \quad (4.8)$$

For the evolution considered it can be shown that $Q - \lambda \rightarrow 0$ as $t \rightarrow \infty$. In figure 6 this property is shown for the numerical calculations. Therefore it follows from (4.8) that asymptotically the component of r in the direction of the fundamental eigenfunction $\hat{\Omega} = \lim_{t \rightarrow \infty} \Omega / \sqrt{e}$ vanishes to higher order in $\sqrt{e}$ (or $\sqrt{w}$):

$$\int \frac{r}{\sqrt{w}} \hat{\Omega} \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (4.9)$$

Figure 6. Graph of $Q - \lambda$ against e (cf formula (4.8)).

Figure 7. The self-organisation process for the calculated evolution illustrated by the tangency of the $\gamma \rightarrow w(\gamma)$ and $\gamma \rightarrow e(\gamma)$ curves to the QSS curve. Note that E is normalised with λ_1 so that the projections of the 3D QSS curve (see figure 2) to the E - C and W - C planes are represented by the same parabola in this figure.

These results to a large extent justify the choice of the dynamics. In particular, the importance of imposing the self-organisation behaviour $Q - \lambda \rightarrow 0$ should be noted. Figure 7, is another illustration of these results.

Another way to envisage the residual is to parametrise the dynamical trajectory with the value of γ ($\gamma \approx \exp(-\nu\lambda_1 t)$ as $t \rightarrow \infty$) and to note that

$$r = \Omega_t - \dot{\gamma}/\gamma \Omega = \gamma \frac{d}{d\gamma} (\Omega/\gamma) = \gamma \dot{\gamma} \frac{d}{d\gamma} (\Omega/\gamma) = -\nu\lambda_1 \gamma^2 \frac{d}{d\gamma} (\Omega/\gamma) \quad (4.10)$$

where in $\Omega(e, \gamma)$ the parameter $e = e(\gamma)$ is taken according to (4.7). The result (4.10) specifies the residual to be of order $\exp(-2\nu\lambda_1 t)$ and relates its spatial dependence to the change of the normalised and confined function Ω/γ . In particular, (4.10) implies that the residual has its largest value at the boundary of the confined vortex. This is the place for which it is known that the Lagrangian flow is most unstable [11], and where most dissipation will take place.

5. Numerical procedure

In this section we describe a simple but very efficient numerical procedure for calculating the qss. This procedure is an adapted version of a method described in [14].

It is possible to describe a procedure for the qss by using the extremal characterisation (3.1) (see [15]). However, to keep as close as possible to [14] we shall change the variational formulation slightly and consider instead of (3.1), for relevant values of the constraints, the equivalent family of maximisation problems

$$\max\{E(\omega) \mid W(\omega) = w, C(\omega) = \gamma, \omega \geq 0\}. \quad (5.1)$$

This formulation is equivalent to (3.1) in the sense that the same solutions are obtained for appropriate values of the constraints (see [16]).

For ease of notation we denote the constrained set by $M(w, \gamma)$

$$M(w, \gamma) = \{\omega \in L_2(R) \mid W(\omega) = w, C(\omega) = \gamma, \omega \geq 0\}.$$

By virtue of the inequality $(\int \omega)^2 \leq \int \omega^2$, a solution of (5.1) can exist only if the prescribed values of the constraints satisfy $\gamma^2 \leq 2w$. In the case when the constraints are compatible, i.e. when w and γ are such that $M(w, \gamma)$ is not empty, the existence of a solution of (5.1) can be proved as for (3.1). Again, uniqueness cannot yet be proved rigorously although it can be expected. For the following we note that the energy functional can also be written in terms of the Stokes stream function ψ :

$$E(\omega) = \frac{1}{2} \int \psi \omega$$

and we introduce the energy norm $\|\cdot\|_E$ according to

$$E(\omega) = \frac{1}{2} \|\omega\|_E^2.$$

Solutions of (5.1) are found by the following variational *iterative procedure*.

Take an arbitrary initial guess $\omega^{(0)}$; then $\omega^{(k)}$, $k = 1, 2, \dots$ are defined successively: if $\omega^{(k)}$ has been calculated, the next iterate $\omega^{(k+1)}$ is found as the solution of the maximisation problem

$$\max \left\{ \int \psi^{(k)} \omega \mid \omega \in M(w, \gamma) \right\} \quad (5.2)$$

where $\psi^{(k)}$ is the stream function corresponding to $\omega^{(k)}$:

$$-\Delta \psi^{(k)} = \omega^{(k)} \text{ in } R \quad \psi^{(k)} = 0 \text{ on } \partial R. \quad (5.3)$$

The *implementation* of this iterative procedure turns out to be quite simple in the present case. Having found $\psi^{(k)}$ from $\omega^{(k)}$ by solving (5.3) with a standard Poisson solver, write

$$\omega^{(k+1)} = (\lambda^{(k+1)} \psi^{(k)} - \sigma^{(k+1)})_+ = \lambda^{(k+1)} (\psi^{(k)} - \beta^{(k+1)})_+. \quad (5.4)$$

Here $\lambda^{(k+1)}$ and $\sigma^{(k+1)}$ or $\beta^{(k+1)}$ are multipliers to be determined from the requirement $\omega \in M(w, \gamma)$. Note that (5.4) is the Euler–Lagrange equation satisfied by the solution $\omega^{(k+1)}$ of (5.2) (compare with (3.7) and (3.9)). To determine $\lambda^{(k+1)}$ and $\beta^{(k+1)}$ from the constraints $W(\omega^{(k+1)}) = w$ and $C(\omega^{(k+1)}) = \gamma$, it is simpler first to eliminate $\lambda^{(k+1)}$ from these two equations. Then one equation for $\beta^{(k+1)}$ remains: $\beta^{(k+1)}$ has to be a root of the function F , where

$$F(\beta) = \int (\psi^{(k)} - \beta)_+^2 \left[\int (\psi^{(k)} - \beta)_+ \right]^{-2} - \frac{2w}{\gamma^2}.$$

A simple calculation shows that F is a monotonically increasing function of β , and therefore its (unique) root $\beta^{(k+1)}$ can be found by a simple bisection method. Once $\beta^{(k+1)}$ is computed, $\lambda^{(k+1)}$ follows at once from the constraint $C(\omega^{(k+1)}) = \gamma$:

$$\lambda^{(k+1)} = \gamma \left(\int (\psi^{(k)} - \beta^{(k+1)})_+ \right)^{-1}.$$

We will now show that this procedure serves its purpose. Therefore assume that the constraints are compatible: $M(w, \gamma) \neq \emptyset$. Then we have the following proposition.

Proposition 5.1. For the sequence of iterates $\omega^{(k)}$ constructed above, it holds that
(i) the energy is monotonically increasing:

$$E(\omega^{(k+1)}) \geq E(\omega^{(k)})$$

(ii) $E(\omega^{(k+1)} - \omega^{(k)}) = \frac{1}{2} \|\omega^{(k+1)} - \omega^{(k)}\|_E^2 \leq E(\omega^{(k+1)}) - E(\omega^{(k)})$
where $\|\cdot\|_E$ is the energy norm.

Proof. Since $\omega^{(k+1)}$ is a solution of (5.2) it follows that $\int \psi^{(k)} \omega^{(k+1)} \geq \int \psi^{(k)} \omega^{(k)}$, i.e.

$$\int \psi^{(k)} (\omega^{(k+1)} - \omega^{(k)}) \geq 0. \quad (5.5)$$

Then $(d/d\tau)E(\omega^{(k)} + \tau(\omega^{(k+1)} - \omega^{(k)}))|_{\tau=0} \geq 0$, which by the convexity of E implies that $E(\omega^{(k+1)}) \geq E(\omega^{(k)})$. This is the monotonicity property (i). From the identity

$$E(\omega^{(k+1)}) = E(\omega^{(k)}) + \int \psi^{(k)} (\omega^{(k+1)} - \omega^{(k)}) + E(\omega^{(k+1)} - \omega^{(k)})$$

the inequality (ii) follows by virtue of (5.5).

From the first part of the proposition it follows that $E(\omega^{(k+1)})$ monotonically increases towards some value E^* . It can be shown that E^* is necessarily a critical value, i.e. there is a critical point of E on the constrained set which has value E^* [17]. In principle, E^* could be a local maximum, different from the global maximum value of (5.1). Of course, this is due to the fact that the numerical procedure uses only the (approximate) Euler-Lagrange equations (5.4).

Therefore, let $O = O(w, \gamma)$ denote the complete set of solutions of the Euler-Lagrange equation (3.7) that satisfy the constraints. Then the results of [15, 17] show that inequality (ii) of the above proposition implies that the following convergence results are true.

Proposition 5.2. For the sequence of iterates $\omega^{(k)}$ it holds that

- (i) $\lim_{k \rightarrow \infty} \inf_{\omega^* \in O} \|\omega^{(k)} - \omega^*\| = 0$
- (ii) either the set of limit points is a single point, or this set cannot have isolated points.

The property (ii) means that either the sequence $\omega^{(k)}$ converges in a standard sense to a certain solution $\omega^* \in O$ or that any $\omega^{(k)}$ is an approximation of some (possibly different) $\omega^* \in O$ from a set without isolated points.

Despite this possible ambiguity, all the numerical results that we obtained show a convergence to the global maximiser of (5.1).

To calculate the time-dependent evolution we simply used an Euler-forward scheme for the equations (4.9a) and (4.9b) with a sufficiently small time step ($\nu \cdot \Delta t$ running from 10^{-4} to 5×10^{-3}). The results depicted in the figures were found in this way.

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